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Univalence of Integral Operators Involving Mittag-Leffler Functions

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Abstract: In this paper, we first introduce a new family of integral operators involving Mittag-Leffler functions. We then find sufficient conditions for univalence of these integral operators and present the generalized versions of the well-known Ahlfors' and Becker's univalence criteria. Finally, we derive several inequalities for the normalized Mittag-Leffler functions.

Keywords: Analytic functions; Univalent functions; Integral operators; Mittag-Leffler functions; Univalent criteria; Fractional differential and fractional integro-differential equations.

1 Introduction, Definitions and Preliminaries

The familiar Mittag-Leffler function $E_\alpha(z)$ introduced by Mittag-Leffler [20] and its generalization $E_{\alpha,\beta}(z)$ introduced by Wiman (see [29] and [30]) are defined by

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)} =: E_{\alpha,1}(z)$$

and

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)} \tag{1}$$

$$\big(z,\alpha,\beta \in \mathbb{C};\ \Re(\alpha) > 0\big).$$

The above-defined functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$, as well as their various further generalizations, arise naturally in the solution of fractional differential equations and fractional integro-differential equations which are associated with (for example) the kinetic equation, random walks, Lévy flights, super-diffusive transport problems and in the study of complex systems. Several properties of the Mittag-Leffler functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$, together with their generalizations, can be found in a number of recent works [1], [13] to [19] and [25] to [27].

Let $\mathscr{A}$ denote the class of the normalized functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{2}$$

which are analytic in the open unit disk

$$\mathbb{D} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

We also denote by $\mathscr{S}$ the subclass of the normalized analytic function class $\mathscr{A}$ which are univalent (or schlicht) in $\mathbb{D}$.

Since the Mittag-Leffler function $E_{\alpha,\beta}$ in (1) does not belong to the class $\mathscr{A}$, we choose to consider here the following normalization of the Mittag-Leffler function:

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$$\mathbb{E}_{\alpha,\beta}(z) := \Gamma(\beta) z E_{\alpha,\beta}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} z^n \tag{3}$$

$$\left(z, \alpha, \beta \in \mathbb{C};\ \Re(\alpha) > 0;\ \beta \neq 0, -1, -2, \cdots\right).$$

Whilst the definition (3) holds true for complex-valued parameters α and β and $z \in \mathbb{C}$, yet (for the purpose of this paper) we shall restrict our attention to the case of real-valued parameters α and β and $z \in \mathbb{D}$. We observe that the normalized Mittag-Leffler function $\mathbb{E}_{\alpha,\beta}$ in (3) contains such well-known functions as its special cases as given below:

$$\mathbb{E}_{2,1} = z\cosh(\sqrt{z}), \quad \mathbb{E}_{2,2} = \sqrt{z}\sinh(\sqrt{z}),$$

$$\mathbb{E}_{2,3} = 2\left[\cosh(\sqrt{z}) - 1\right]$$

and

$$\mathbb{E}_{2,4} = \frac{6\left[\sinh(\sqrt{z}) - \sqrt{z}\right]}{\sqrt{z}}.$$

Geometric properties including starlikeness, convexity and close-to-convexity for the Mittag-Leffler function $E_{\alpha,\beta}$ were recently investigated by Bansal and Prajapat [2]. Răducanu [24], on the other hand, investigated the ratio of the normalized Mittag-Leffler function $\mathbb{E}_{\alpha,\beta}$ defined by (3) to its sequence of partial sums given by

$$(\mathbb{E}_{\alpha,\beta})_m(z) = z + \sum_{n=2}^{m} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} z^n \tag{4}$$

$$(\alpha > 0;\ \beta > 0;\ m \in \mathbb{N}),$$

where $\mathbb{N}$ denotes the set of *positive* integers.

Recently, many authors derived the univalence criteria of several integral operators which preserve the normalized univalent function class $\mathscr{S}$ (see, for example, [3] to [12]). In particular, Breaz and Güney [5] obtained various sufficient conditions for the univalence of the following families of integral operators:

$$F_{\lambda_1,\cdots,\lambda_n,\zeta}(z) = \left[\zeta \int_0^z t^{\zeta-1} \prod_{j=1}^{n} \left(\frac{f_j(t)}{t}\right)^{1/\lambda_j} \mathrm{d}t\right]^{1/\zeta} \tag{5}$$

and

$$F_\gamma(z) = \left[\gamma \int_0^z t^{\gamma-1} \left(e^{f(t)}\right)^{\gamma} \mathrm{d}t\right]^{1/\gamma}, \tag{6}$$

where the functions $f_1, \cdots, f_n$ and f belong to the normalized analytic function class $\mathscr{A}$ and the parameters $\lambda_1, \cdots, \lambda_n$, ζ and γ are complex numbers such that the integrals in (5) and (6) exist. Here, and throughout in this paper, every many-valued function is taken with the principal branch.

In this paper, we are mainly interested in some integral operators of the type (5) and (6) which involve the normalized Mittag-Leffler function $\mathbb{E}_{\alpha,\beta}$. More precisely, we propose to show that, by using some inequalities for the normalized Mittag-Leffler function, the univalence of some integral operators involving Mittag-Leffler functions can be derived easily via some well-known univalence criteria. By appropriately specializing our results, we obtain simple sufficient conditions for some integral operators which involve the hyperbolic sine and the hyperbolic cosine functions.

In the proofs of our main results, we need each of the following univalence criteria.

Lemma 1. (see Pescar [22]) *Let ζ and c be complex numbers such that*

$$\Re(\zeta) > 0 \quad \textit{and} \quad |c| \leqq 1 \ (c \neq -1).$$

If the function $h \in \mathscr{A}$ satisfies the following inequality:

$$\left| c|z|^{2\zeta} + \left(1 - |z|^{2\zeta}\right) \frac{zh''(z)}{\zeta h'(z)} \right| \leqq 1$$

for all $z \in \mathbb{D}$, then the function $F_\zeta \in \mathscr{A}$ defined by

$$F_\zeta(z) := \left[\zeta \int_0^z t^{\zeta-1} h'(t)\, \mathrm{d}t\right]^{1/\zeta} \tag{7}$$

is in the normalized univalent function class $\mathscr{S}$ in $\mathbb{D}$.

Lemma 2. (see Pascu [21]) *Let $\lambda \in \mathbb{C}$ such that*

$$\Re(\lambda) > 0.$$

Also let $h \in \mathscr{A}$ satisfy the following inequality:

$$\frac{1 - |z|^{2\Re(\lambda)}}{\Re(\lambda)} \left| \frac{zh''(z)}{h'(z)} \right| \leqq 1$$

for all $z \in \mathbb{D}$. Then, for all $\zeta \in \mathbb{C}$ such that $\Re(\zeta) \geqq \Re(\lambda)$, the function $F_\zeta(z)$ defined by (7) is in the normalized univalent function class $\mathscr{S}$ in $\mathbb{D}$.

Lemma 3. (see Pescar [23]) *Let $\gamma \in \mathbb{C}$ and $\lambda \in \mathbb{R}$ such that*

$$\Re(\gamma) \geqq 1, \ \ \lambda > 1 \quad \textit{and} \quad 2\lambda|\gamma| \leqq 3\sqrt{3}.$$

If $h \in \mathscr{A}$ satisfies the the following inequality:

$$\left|zh'(z)\right| \leqq \lambda$$

for all $z \in \mathbb{D}$, then the function $F_\gamma : \mathbb{D} \to \mathbb{C}$, defined by

$$F_\gamma(z) = \left[\gamma \int_0^z t^{\gamma-1} \left(e^{h(t)}\right)^{\gamma} \mathrm{d}t\right]^{1/\gamma}$$

is in the normalized univalent function class $\mathscr{S}$ in $\mathbb{D}$.

In our present investigation, we also need the following result which is based mainly upon the recent work [2].

Lemma 4. *Let $\alpha \geqq 1$ and $\beta \geqq 1$. Then the following inequalities hold true for all $z \in \mathbb{D}$:*

$$\left| \frac{z\mathbb{E}'_{\alpha,\beta}(z)}{\mathbb{E}_{\alpha,\beta}(z)} - 1 \right| \leqq \frac{2\beta+1}{\beta^2-\beta-1} \tag{8}$$

and

$$\left| z\mathbb{E}'_{\alpha,\beta}(z) \right| \leqq \frac{\beta^2+3\beta+2}{\beta^2}. \tag{9}$$

Proof. It is known from [2] that, for all $\alpha \geqq 1$, $\beta \geqq 1$ and $z \in \mathbb{D}$, we have

$$\left| \mathbb{E}'_{\alpha,\beta}(z) - \frac{\mathbb{E}_{\alpha,\beta}(z)}{z} \right| \leqq \frac{2\beta+1}{\beta^2}$$

and

$$\left| \frac{\mathbb{E}_{\alpha,\beta}(z)}{z} \right| \geqq \frac{\beta^2-\beta-1}{\beta^2}.$$

Combining these last inequalities, we immediately find that (8) holds true.

Next, it is known from [24] that, for all $\alpha \geqq 1$, $\beta \geqq 1$ and $z \in \mathbb{D}$, we have

$$\left| \mathbb{E}'_{\alpha,\beta}(z) \right| \leqq \frac{\beta^2+3\beta+2}{\beta^2},$$

which obviously implies the inequality (9).

2 Univalence of Integral Operators Involving Mittag-Leffler Functions

Our first main result (Theorem 2 below) is an application of Lemma 1 and contains sufficient conditions for an integral operator of the type (5) when the functions f_j are the normalized Mittag-Leffler functions with various parameters.

Theorem 1. *Let*

$$\alpha_1, \cdots, \alpha_n \geqq 1 \quad \text{and} \quad \beta_1, \cdots, \beta_n \geqq \frac{1}{2}(1+\sqrt{5})$$

and consider the normalized Mittag-Leffler functions $\mathbb{E}_{\alpha_j,\beta_j}$ defined by

$$\mathbb{E}_{\alpha_j,\beta_j}(z) = \Gamma(\beta_j) z E_{\alpha_j,\beta_j}(z). \tag{10}$$

Let

$$\beta = \min\{\beta_1, \cdots, \beta_n\} \quad \text{and} \quad \zeta \in \mathbb{C}$$

with

$$\Re(\zeta) > 0 \quad \text{and} \quad c \in \mathbb{C} \ \ (c \neq -1).$$

Suppose also that $\lambda_1, \cdots, \lambda_n$ are nonzero complex numbers and that all these and the aforementioned numbers satisfy the following inequality:

$$|c| + \frac{2\beta+1}{\beta^2-\beta-1} \sum_{j=1}^{n} \frac{n}{|\zeta\lambda_j|} \leqq 1.$$

Then the function $\mathbb{F}_{\alpha_j,\beta_j,\lambda_j,\zeta}$ defined by

$$\mathbb{F}_{\alpha_j,\beta_j,\lambda_j,\zeta}(z) := \left[\zeta \int_0^z t^{\zeta-1} \prod_{j=1}^{n} \left(\frac{\mathbb{E}_{\alpha_j,\beta_j}(t)}{t} \right)^{1/\lambda_j} \mathrm{d}t \right]^{1/\zeta}$$

is in the normalized univalent function $\mathscr{S}$ in $\mathbb{D}$.

Proof. Define the function $\mathbb{F}_{\alpha_j,\beta_j,\lambda_j}$ by

$$\mathbb{F}_{\alpha_j,\beta_j,\lambda_j}(z) = \int_0^z \prod_{j=1}^{n} \left(\frac{\mathbb{E}_{\alpha_j,\beta_j}(t)}{t} \right)^{1/\lambda_j} \mathrm{d}t.$$

Since, for all $j \in \{1, \cdots, n\}$, we have $\mathbb{E}_{\alpha_j,\beta_j} \in \mathscr{A}$, that is,

$$\mathbb{E}_{\alpha_j,\beta_j}(0) = \mathbb{E}'_{\alpha_j,\beta_j}(0) - 1 = 0,$$

it follows readily that $\mathbb{F}_{\alpha_j,\beta_j,\lambda_j} \in \mathscr{A}$, that is,

$$\mathbb{F}_{\alpha_j,\beta_j,\lambda_j}(0) = \mathbb{F}'_{\alpha_j,\beta_j,\lambda_j}(0) - 1 = 0.$$

On the other hand, it is easy to see that

$$\mathbb{F}'_{\alpha_j,\beta_j,\lambda_j}(z) = \prod_{j=1}^{n} \left(\frac{\mathbb{E}_{\alpha_j,\beta_j}(t)}{t} \right)^{1/\lambda_j}$$

and

$$\frac{z\mathbb{F}''_{\alpha_j,\beta_j,\lambda_j}(z)}{\mathbb{F}'_{\alpha_j,\beta_j,\lambda_j}(z)} = \sum_{j=1}^{n} \frac{1}{\lambda_j} \left(\frac{z\mathbb{E}'_{\alpha_j,\beta_j}(z)}{\mathbb{E}_{\alpha_j,\beta_j}(z)} - 1 \right).$$

Now, by using the inequality (8) for each β_j $(j \in \{1, \cdots, n\})$, we obtain

$$\begin{aligned} \left| \frac{z\mathbb{F}''_{\alpha_j,\beta_j,\lambda_j}(z)}{\mathbb{F}'_{\alpha_j,\beta_j,\lambda_j}(z)} \right| &= \sum_{j=1}^{n} \frac{1}{|\lambda_j|} \left| \frac{z\mathbb{E}'_{\alpha_j,\beta_j}(z)}{\mathbb{E}_{\alpha_j,\beta_j}(z)} - 1 \right| \\ &\leqq \sum_{j=1}^{n} \frac{1}{|\lambda_j|} \frac{2\beta_j+1}{\beta_j^2-\beta_j-1} \\ &\leqq \sum_{j=1}^{n} \frac{n}{|\lambda_j|} \frac{2\beta+1}{\beta^2-\beta-1} \end{aligned}$$

for all $z \in \mathbb{D}$ and $\beta_1, \cdots, \beta_n \geqq \frac{1}{2}(1+\sqrt{5})$. Here we have used the easily verifiable fact that the function

$$\varphi : \left(\frac{1+\sqrt{5}}{2}, \infty \right) \to \mathbb{R},$$

defined by

$$\varphi(x) = \frac{2x+1}{x^2-x-1},$$

is decreasing. Therefore, for all $j \in \{1,\cdots,n\}$, we have

$$\frac{2\beta_j+1}{\beta_j^2-\beta_j-1} \leqq \frac{2\beta+1}{\beta^2-\beta-1}.$$

Thus, by using the triangle inequality and the hypothesis of Theorem 1, we obtain

$$\left| c\,|z|^{2\zeta} + (1-|z|^{2\zeta}) \frac{z\mathbb{F}''_{\alpha_j,\beta_j,\lambda_j}(z)}{\zeta\mathbb{F}'_{\alpha_j,\beta_j,\lambda_j}(z)} \right| \leqq |c| + \frac{2\beta+1}{\beta^2-\beta-1} \sum_{j=1}^{n} \frac{n}{|\zeta\lambda_j|} \leqq 1,$$

which, in view of Lemma 1, implies that $\mathbb{F}_{\alpha_j,\beta_j,\lambda_j,\zeta} \in \mathscr{S}$. This evidently completes the proof of Theorem 1.

By setting $\lambda_1 = \cdots = \lambda_n = \lambda$ in Theorem 1, we deduce the following result.

Corollary 1. *Let the numbers*

$$\zeta,\ c,\ \alpha_1,\cdots,\alpha_n,\ \beta \quad \textit{and} \quad \beta_1,\cdots,\beta_n$$

be as in Theorem 1 and let λ *be a nonzero complex number. Suppose also that the functions*

$$\mathbb{E}_{\alpha_j,\beta_j} \in \mathscr{A} \quad (j \in \{1,\cdots,n\})$$

are as in Theorem 1 and that the following inequality:

$$|c| + \frac{1}{|\lambda\zeta|} \frac{(2\beta+1)n^2}{\beta^2-\beta-1} \leqq 1$$

is valid. Then the function $\mathbb{F}_{\alpha_j,\beta_j,\lambda_j,\zeta}$ *defined by*

$$\mathbb{F}_{\alpha_j,\beta_j,\lambda_j,\zeta}(z) = \left[\zeta \int_0^z t^{\zeta-1} \prod_{j=1}^{n} \left(\frac{\mathbb{E}_{\alpha_j,\beta_j}(t)}{t} \right)^{1/\lambda} \mathrm{d}t \right]^{1/\zeta}$$

is in the normalized univalent function class $\mathscr{S}$ *in* $\mathbb{D}$.

If we put $n = 1$ in Theorem 1 or in Corollary 1, we immediately obtain the following result.

Corollary 2. *Let*

$$\alpha \geqq 1,\ \beta \geqq \frac{1}{2}(1+\sqrt{5}) \quad \textit{and} \quad \zeta \in \mathbb{C}$$

with

$$\Re(\zeta) > 0,\ c \in \mathbb{C}\ (c \neq -1) \quad \textit{and} \quad \lambda \in \mathbb{C}\ (\lambda \neq 0).$$

Suppose also that these numbers satisfy the following inequality:

$$|c| + \frac{1}{|\lambda\zeta|} \frac{2\beta+1}{\beta^2-\beta-1} \leqq 1.$$

Then the function $\mathbb{F}_{\alpha,\beta,\lambda,\zeta}$ *defined by*

$$\mathbb{F}_{\alpha,\beta,\lambda,\zeta}(z) = \left[\zeta \int_0^z t^{\zeta-1} \left(\frac{\mathbb{E}_{\alpha,\beta}(t)}{t} \right)^{1/\lambda} \mathrm{d}t \right]^{1/\zeta}$$

is in the normalized univalent function class $\mathscr{S}$ *in* $\mathbb{D}$.

The following illustrative example provides several univalent functions in $\mathbb{D}$:

Example 1. (i) If

$$|c| + \frac{5}{|\lambda\zeta|} \leqq 1,$$

then the function $\mathbb{F}_{2,2,\lambda,\zeta}$ defined by

$$\mathbb{F}_{2,2,\lambda,\zeta}(z) = \left[\zeta \int_0^z t^{\zeta-1} \left(\frac{\sinh(\sqrt{t})}{\sqrt{t}} \right)^{1/\lambda} \mathrm{d}t \right]^{1/\zeta}$$

is univalent in $\mathbb{D}$.

(ii) If

$$|c| + \frac{7}{5|\lambda\zeta|} \leqq 1,$$

then the function $\mathbb{F}_{2,3,\lambda,\zeta}$ defined by

$$\mathbb{F}_{2,3,\lambda,\zeta}(z) = \left[\zeta \int_0^z t^{\zeta-1} \left(\frac{2[\cosh(\sqrt{t})-1]}{t} \right)^{1/\lambda} \mathrm{d}t \right]^{1/\zeta}$$

is univalent in $\mathbb{D}$.

(iii) If

$$|c| + \frac{9}{11|\lambda\zeta|} \leqq 1,$$

then the function $\mathbb{F}_{2,4,\lambda,\zeta}$ defined by

$$\mathbb{F}_{2,4,\lambda,\zeta}(z) = \left[\zeta \int_0^z t^{\zeta-1} \left(\frac{6[\sinh(\sqrt{t})-\sqrt{t}]}{t^{3/2}} \right)^{1/\lambda} \mathrm{d}t \right]^{1/\zeta}$$

is univalent in $\mathbb{D}$.

The following result contains another set of sufficient conditions for integrals of the type (5) to be univalent in the unit disk $\mathbb{D}$. The key tools in the proof of Theorem 2 are Lemma 2 and the inequality (8).

Theorem 2. *Let*

$$\alpha_1, \cdots, \alpha_n \geqq 1 \quad \text{and} \quad \beta_1, \cdots, \beta_n \geqq \frac{1}{2}(1+\sqrt{5}).$$

Consider the normalized Mittag-Leffler functions $\mathbb{E}_{\alpha_i,\beta_i}$ *defined by* (10). *Let*

$$\beta = \min\{\beta_1, \cdots, \beta_n\} \qquad \text{and} \qquad \lambda \in \mathbb{C} \quad (\Re(\lambda) > 0)$$

and suppose that these numbers satisfy the following inequality:

$$|\lambda| \leqq \frac{1}{n}\left(\frac{\beta^2-\beta-1}{2\beta+1}\right)\Re(\lambda).$$

Then the function $\mathbb{F}_{\alpha_i,\beta_i,\lambda,n}$ *defined by*

$$\mathbb{F}_{\alpha_i,\beta_i,\lambda,n}(z) = \left[(n\lambda+1)\int_0^z \prod_{i=1}^n \left(\mathbb{E}_{\alpha_i,\beta_i}(t)\right)^\lambda \, \mathrm{d}t\right]^{1/(n\lambda+1)}$$

is in the normalized univalent function class $\mathscr{S}$ *in* $\mathbb{D}$.

Proof. Let us consider the auxiliary function $\mathbb{F}_{\alpha_i,\beta_i,\lambda}$ defined by

$$\mathbb{F}_{\alpha_i,\beta_i,\lambda}(z) = \int_0^z \prod_{i=1}^n \left(\frac{\mathbb{E}_{\alpha_i,\beta_i}(t)}{t}\right)^\lambda \, \mathrm{d}t.$$

We then observe that

$$\mathbb{F}_{\alpha_i,\beta_i,\lambda} \in \mathscr{A},$$

that is, that

$$\mathbb{F}_{\alpha_i,\beta_i,\lambda}(0) = \mathbb{F}'_{\alpha_i,\beta_i,\lambda}(0) - 1 = 0.$$

On the other hand, by using (8) and the fact that, for all $i \in \{1, \cdots, n\}$,

$$\frac{2\beta_i+1}{\beta_i^2-\beta_i-1} \leqq \frac{2\beta+1}{\beta^2-\beta-1},$$

we find, for all $z \in \mathbb{D}$, that

$$\begin{aligned}
&\frac{1-|z|^{2\Re(\lambda)}}{\Re(\lambda)}\left|\frac{z\mathbb{F}''_{\alpha_i,\beta_i,\lambda}(z)}{\mathbb{F}'_{\alpha_i,\beta_i,\lambda}(z)}\right| \\
&\qquad \leqq \frac{|\lambda|}{\Re(\lambda)}\sum_{i=1}^n \left|\frac{z\mathbb{E}'_{\alpha_i,\beta_i}(z)}{\mathbb{E}_{\alpha_i,\beta_i}(z)} - 1\right| \\
&\qquad \leqq \frac{|n\lambda|}{\Re(\lambda)} \cdot \frac{2\beta+1}{\beta^2-\beta-1} \leqq 1.
\end{aligned}$$

Now, since $\Re(n\lambda+1) > \Re(\lambda)$ and since the function $\mathbb{F}_{\alpha_i,\beta_i,\lambda,n}$ can be rewritten in the form:

$$\mathbb{F}_{\alpha_i,\beta_i,\lambda,n}(z) = \left[(n\lambda+1)\int_0^z t^{n\lambda}\prod_{i=1}^n \left(\frac{\mathbb{E}_{\alpha_i,\beta_i}(t)}{t}\right)^\lambda \mathrm{d}t\right]^{1/(n\lambda+1)},$$

by applying Lemma 2, we have $\mathbb{F}_{\alpha_i,\beta_i,\lambda,n}(z) \in \mathscr{S}$, which completes the proof of Theorem 2.

By choosing $n = 1$ in Theorem 2, we obtain the following result.

Corollary 3. *Let*

$$\alpha \geqq 1 \quad \text{and} \quad \beta \geqq \frac{1}{2}(1+\sqrt{5})$$

and consider the normalized Mittag-Leffler functions $\mathbb{E}_{\alpha_i,\beta_i}$ *defined by* (10). *Also let the parameter* $\lambda \in \mathbb{C}$ *be so constrained that*

$$\Re(\lambda) > 0 \qquad \text{and} \qquad |\lambda| \leqq \left(\frac{\beta^2-\beta-1}{2\beta+1}\right)\Re(\lambda).$$

Then the function $\mathbb{F}_{\alpha,\beta,\lambda}$ *defined by*

$$\mathbb{F}_{\alpha,\beta,\lambda}(z) = \left[(\lambda+1)\int_0^z \left(\mathbb{E}_{\alpha,\beta}(t)\right)^\lambda \, \mathrm{d}t\right]^{1/(\lambda+1)}$$

is univalent in $\mathbb{D}$.

In particular, we are led to the following univalent functions in $\mathbb{D}$.

Example 2. (i) If

$$|\lambda| \leqq \frac{1}{5}\Re(\lambda),$$

then the function $\mathbb{F}_{2,2,\lambda}$ defined by

$$\mathbb{F}_{2,2,\lambda}(z) = \left[(\lambda+1)\int_0^z \left[\sqrt{t}\sinh(\sqrt{t})\right]^\lambda \, \mathrm{d}t\right]^{1/(\lambda+1)}$$

is univalent in $\mathbb{D}$.

(ii) If

$$|\lambda| \leqq \frac{5}{7}\Re(\lambda),$$

then the function $\mathbb{F}_{2,3,\lambda}$ defined by

$$\mathbb{F}_{2,3,\lambda}(z) = \left[(\lambda+1)\int_0^z \left(2[\cosh(\sqrt{t})-1]\right)^\lambda \, \mathrm{d}t\right]^{1/(\lambda+1)}$$

is univalent in $\mathbb{D}$.

(iii) If

$$|\lambda| \leqq \frac{11}{9}\Re(\lambda),$$

then the function $\mathbb{F}_{2,4,\lambda}$ defined by

$$\mathbb{F}_{2,4,\lambda}(z) = \left[(\lambda+1)\int_0^z \left(\frac{6[\sinh(\sqrt{t})-\sqrt{t}]}{\sqrt{t}}\right)^\lambda \, \mathrm{d}t\right]^{1/(\lambda+1)}$$

is univalent in $\mathbb{D}$.

Finally, by applying Lemma 3 and the inequality (9), we can easily prove Theorem 3 below.

Theorem 3. *Let $\gamma \in \mathbb{C}$, $\alpha \geqq 1$ and $\beta \geqq 1$. Consider the normalized Mittag-Leffler function $\mathbb{E}_{\alpha,\beta}$ defined by (10). If*

$$\Re(\gamma) \geqq 1 \qquad \text{and} \qquad |\gamma| \leqq \frac{3\sqrt{3}\beta^2}{2\beta^2+6\beta+4},$$

then the function $\mathbb{F}_{\alpha,\beta,\gamma}$ defined by

$$\mathbb{F}_{\alpha,\beta,\gamma}(z) = \left[\gamma \int_0^z t^{\gamma-1}\left(e^{\mathbb{E}_{\alpha,\beta}(t)}\right)^{\gamma} \mathrm{d}t\right]^{1/\gamma}$$

is univalent in $\mathbb{D}$.

Example 3. (i) If the parameter $\gamma \in \mathbb{C}$ is such that

$$\Re(\gamma) \geqq 1 \qquad \text{and} \qquad |\gamma| \leqq \frac{\sqrt{3}}{4},$$

then the function $\mathbb{F}_{2,1,\gamma}$ defined by

$$\mathbb{F}_{2,1,\gamma}(z) = \left[\gamma \int_0^z t^{\gamma-1}\left(e^{t\cosh(\sqrt{t})}\right)^{\gamma} \mathrm{d}t\right]^{1/\gamma}$$

is univalent in $\mathbb{D}$.

(ii) If the parameter $\gamma \in \mathbb{C}$ is such that

$$\Re(\gamma) \geqq 1 \qquad \text{and} \qquad |\gamma| \leqq \frac{\sqrt{3}}{2},$$

then the function $\mathbb{F}_{2,2,\gamma}$ defined by

$$\mathbb{F}_{2,2,\gamma}(z) = \left[\gamma \int_0^z t^{\gamma-1}\left(e^{\sqrt{t}\sinh(\sqrt{t})}\right)^{\gamma} \mathrm{d}t\right]^{1/\gamma}$$

is univalent in $\mathbb{D}$.

(iii) If the parameter $\gamma \in \mathbb{C}$ is such that

$$\Re(\gamma) \geqq 1 \qquad \text{and} \qquad |\gamma| \leqq \frac{27\sqrt{3}}{40},$$

then the function $\mathbb{F}_{2,3,\gamma}$ defined by

$$\mathbb{F}_{2,3,\gamma}(z) = \left[\gamma \int_0^z t^{\gamma-1}\left(e^{2[\cosh(\sqrt{t})-1]}\right)^{\gamma} \mathrm{d}t\right]^{1/\gamma}$$

is univalent in $\mathbb{D}$.

(iv) If the parameter $\gamma \in \mathbb{C}$ is such that

$$\Re(\gamma) \geqq 1 \qquad \text{and} \qquad |\gamma| \leqq \frac{4\sqrt{3}}{5},$$

then the function $\mathbb{F}_{2,4,\gamma}$ defined by

$$\mathbb{F}_{2,4,\gamma}(z) = \left[\gamma \int_0^z t^{\gamma-1}\left(e^{\frac{6[\sinh(\sqrt{t})-\sqrt{t}]}{\sqrt{t}}}\right)^{\gamma} \mathrm{d}t\right]^{1/\gamma}$$

is univalent in $\mathbb{D}$.

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