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Unification and Generalization of Two Product Theorems of Srinivasa Ramanujan Associated with Quadruple Hypergeometric Functions

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Abstract: The object of the present paper is to show that Ramanujan's two reduction formulas [Entries 24 and 22 in chapter 13 of Ramanujan's first notebook,see also Entries 18 and 16 in chapter 11 of Ramanujan's second notebook] for the products of two hypergeometric functions, can be unified and generalized for general quadruple Gaussian hypergeometric functions of Srivastava, Pathan and Saigo, using series rearrangement technique.Using the theory of bounded multiple sequences, further generalizations of two product theorems of Srinivasa Ramanujan, are also given.

Keywords: Ramanujan's theorems; Dixon's theorem; Multiple Gaussian hypergeometric functions; Series rearrangement technique.

1 Introduction

In the usual notation, let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers respectively. Also let

$$\mathbb{N}_0=\mathbb{N}\cup\{0\}\ ,\qquad \mathbb{N}=\{1,2,3,\dots\}=\mathbb{N}_0\backslash\{0\}$$

and

$$\mathbb{Z}_0^-=\{0,-1,-2,\dots\}=\mathbb{Z}^-\cup\{0\}\ ,\mathbb{Z}^-=\{-1,-2,-3,\dots\}$$

and $\mathbb{Z}=\mathbb{Z}_0^-\cup\mathbb{N}$ being the set of integers.

The *generalized hypergeometric function* ${}_pF_q$ with p numerator parameters $\alpha_1,\alpha_2,\dots,\alpha_p$ and q denominator parameters $\beta_1,\beta_2,\dots,\beta_q$, is defined by

$${}_pF_q\left[\begin{array}{c}\alpha_1,\alpha_2,\dots,\alpha_p;\\ \\ \beta_1,\beta_2,\dots,\beta_q;\end{array} z\right]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(\alpha_j)_n}{\prod_{j=1}^{q}(\beta_j)_n}\frac{z^n}{n!}, \tag{1}$$

$\big(p,q\in\mathbb{N}_0;\ \ p\leqq q+1\ ;\ p\leqq q$ and $|z|<\infty$,or $p=q+1$ and $|z|<1$,or $p=q+1,|z|=1$ and $\Re(\omega)>0$, or $p=q+1,|z|=1,z\neq1$ and $0\geq\Re(\omega)>-1\big)$,
where

$$\omega:=\sum_{j=1}^{q}\beta_j-\sum_{j=1}^{p}\alpha_j\ ,$$

$$\big(\alpha_j\in\mathbb{C}\ (j=1,2,\dots,p);\beta_j\in\mathbb{C}\backslash\mathbb{Z}_0^-(j=1,2,\dots,q)\big).$$

In terms of Gamma function $\Gamma(z)$, the widely-used Pochhammer symbol $(\lambda)_\nu\quad(\lambda,\nu\in\mathbb{C})$ is defined, in general, by

$$(\lambda)_\nu:=\frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}$$
$$=\begin{cases}1 & ,(\nu=0;\lambda\in\mathbb{C}\setminus\{0\})\\ \lambda(\lambda+1)\dots(\lambda+n-1) & ,(\nu=n\in\mathbb{N};\lambda\in\mathbb{C}),\end{cases} \tag{2}$$

it being understood *conventionally* that $(0)_0=1$ and assumed *tacitly* that the Γ quotient exists

$$\int_0^\infty e^{-st}t^{\alpha-1}dt=\frac{\Gamma(\alpha)}{s^\alpha}, \tag{3}$$

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$$\Big(\Re(s) > 0,\ 0 < \Re(\alpha) < \infty \text{ or } \Re(s) = 0,\ 0 < \Re(\alpha) < 1\Big).$$

The following reduction formulas for the product of two generalized hypergeometric functions are attributed to Ramanujan (see, for example, Hardy[6,p.503,equations (9.6) and (9.7)],Berndt[1]):

First Product Theorem:

$$ {}_1F_1\left[\begin{matrix}\rho;\\ \sigma;\end{matrix} y\right]\ {}_1F_1\left[\begin{matrix}\rho;\\ \sigma;\end{matrix} -y\right] = {}_2F_3\left[\begin{matrix}\rho,\sigma-\rho;\\ \sigma,\Delta(2;\sigma);\end{matrix} \frac{y^2}{4}\right]. \tag{4}$$

Second Product Theorem:

$$ {}_0F_2\left[\begin{matrix}-;\\ \nu,\zeta;\end{matrix} x\right]\ {}_0F_2\left[\begin{matrix}-;\\ \nu,\zeta;\end{matrix} -x\right]$$
$$= {}_3F_8\left[\begin{matrix}\Delta(3;\nu+\zeta-1);\\ \Delta(2;\nu+\zeta-1),\Delta(2;\nu),\Delta(2;\zeta),\nu,\zeta;\end{matrix} \frac{-27x^2}{64}\right] \tag{5}$$

where,for convenience, $\Delta(m;\zeta)$ abbreviates the array of m parameters given by $\zeta/m,(\zeta+1)/m,\dots,(\zeta+m-1)/m$ and m = 1,2,3,...

Formula (4) is the Entry 24 and formula (5) is the Entry 22, in chapter 13 of Ramanujan's first note book [9, pp.201-203].These formulas are also given as Entry 18 and Entry 16 respectively, in chapter 11 of Ramanujan's second note book [10,pp. 132-133; see also 8,p.106(Q.No. 5&7)].

The notation (a_A) denotes the set of A number of parameters given by $a_1,a_2,\dots,a_A$.
The notation $\Delta[N;(b_B)]$ denote the array of BN parameters[21,p.47(equation 8),pp.193-194] given by $\Delta(N;b_1),\Delta(N;b_2),\dots,\Delta(N;b_B)$.

If $j = 0,1,2,3,\dots,(N-1)$ then the asterisk in $\Delta^*(N;j+1)$ represents $(N-1)$ denominator parameters in such a way that the denominator parameter $\frac{N}{N}$ is always omitted[21,p.194(equation 12);see also 19,p.194].

$$[(a_A)]_n = \prod_{i=1}^{A}(a_i)_n = (a_1)_n(a_2)_n\dots(a_A)_n. \tag{6}$$

$$(b)_{mn} = m^{mn}\prod_{j=1}^{m}\left(\frac{b+j-1}{m}\right)_n, \tag{7}$$
$$(n = 0,1,2,\dots;\quad m = 1,2,3,\dots).$$

$$[(a_A)]_{mn} = m^{mnA}\left[\frac{(a_A)}{m}\right]_n\left[\frac{(a_A)+1}{m}\right]_n\dots\left[\frac{(a_A)+m-1}{m}\right]_n. \tag{8}$$

For the sake of convenience, we write

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Lambda(m,n,p,q) = \sum_{m,n,p,q=0}^{\infty}\Lambda(m,n,p,q)$$

$$\sum_{m,n=0}^{\infty}\Lambda(m,n) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Lambda(m,n)$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{m}\Lambda(m-n,n). \tag{9}$$

$$\sum_{p=0}^{\infty}\Upsilon(p) = \sum_{i=0}^{1}\sum_{p=0}^{\infty}\Upsilon(2p+i)$$
$$= \sum_{p=0}^{\infty}\Upsilon(2p) + \sum_{p=0}^{\infty}\Upsilon(2p+1). \tag{10}$$

$$\sum_{p,q=0}^{\infty}\Upsilon(p,q) = \sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{p,q=0}^{\infty}\Upsilon(2p+i,2q+j). \tag{11}$$

Dixon Theorem:[8,p.92(Theorem 33)]

$$ {}_3F_2\left[\begin{matrix}A,B,D;\\ 1+A-B,1+A-D;\end{matrix} 1\right]$$
$$= \frac{\Gamma(1+A-B)\Gamma(1+A-D)\Gamma(1+\frac{A}{2})\Gamma(1+\frac{A}{2}-B-D)}{\Gamma(1+\frac{A}{2}-B)\Gamma(1+\frac{A}{2}-D)\Gamma(1+A)\Gamma(1+A-B-D)}, \tag{12}$$

where $\Re(A-2B-2D) > -2$ and $1+A-B, 1+A-D \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Lemma 1: Using Dixon theorem, we get

$$ {}_3F_2\left[\begin{matrix}-2m,1-\nu-2m,1-\zeta-2m;\\ \nu,\zeta;\end{matrix} 1\right]$$
$$= \frac{(-1)^m(\frac{1}{2})_m 3^{3m}(\frac{\nu+\zeta-1}{3})_m(\frac{\nu+\zeta}{3})_m(\frac{\nu+\zeta+1}{3})_m}{(\nu)_m(\zeta)_m(\frac{\nu+\zeta-1}{2})_m(\frac{\nu+\zeta}{2})_m} \tag{13}$$

and

$$ {}_3F_2\left[\begin{matrix}-2m-1,-\nu-2m,-\zeta-2m;\\ \nu,\zeta;\end{matrix} 1\right] = 0, \tag{14}$$

where m is non-negative integer and $\nu,\zeta \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Lemma 2: Using Dixon theorem, we get

$$ {}_3F_2\left[\begin{matrix}-2p,\rho,1-\sigma-2p;\\ \sigma,1-\rho-2p;\end{matrix} 1\right] = \frac{(\rho)_p(\sigma-\rho)_p(\frac{1}{2})_p}{(\sigma)_p(\frac{\rho}{2})_p(\frac{\rho+1}{2})_p} \tag{15}$$

and

$$ {}_3F_2\left[\begin{matrix}-2p-1,\rho,-\sigma-2p;\\ \sigma,-\rho-2p;\end{matrix} 1\right] = 0, \tag{16}$$

where p is non-negative integer; $\sigma,-\rho-2p \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $(1-\rho) \neq 0,\pm1,\pm2,\pm3,\dots$

Kampé de Fériet's general double hypergeometric function:

In 1921, Appell's four double hypergeometric functions F_1,F_2,F_3,F_4 and Humbert's seven confluent forms $\Phi_1,\Phi_2,\Phi_3,\Psi_1,\Psi_2,\Xi_1,\Xi_2$ were unified and generalized by Kampé de Fériet. We recall the definition of general double hypergeometric function of Kampé de Fériet in the slightly modified notation of Srivastava and Panda[22,p.423(equation 26),p.424(equation 27); see also 23,p.23(equations 1.2,1.3)]

$$F^{A:B;D}_{E:G;H}\left[\begin{array}{l}(a_A):(b_B);(d_D);\\ \\(e_E):(g_G);(h_H);\end{array} x,y\right]$$
$$=\sum_{r,s=0}^{\infty}\frac{\prod_{i=1}^{A}(a_i)_{r+s}\prod_{i=1}^{B}(b_i)_r\prod_{i=1}^{D}(d_i)_s x^r y^s}{\prod_{i=1}^{E}(e_i)_{r+s}\prod_{i=1}^{G}(g_i)_r\prod_{i=1}^{H}(h_i)_s r!s!}$$
$$=\sum_{r,s=0}^{\infty}\frac{[(a_A)]_{r+s}[(b_B)]_r[(d_D)]_s x^r y^s}{[(e_E)]_{r+s}[(g_G)]_r[(h_H)]_s r!s!},\tag{17}$$

it being understood that $|x|$ and $|y|$ are sufficiently small to ensure the convergence of the concerned double series.

Srivastava's general triple hypergeometric function:
Triple hypergeometric function $F^{(3)}$ of Srivastava [15,p.428] is the unification and generalization of Lauricella's fourteen hypergeometric functions $F_A^{(3)}$, $F_B^{(3)}$, $F_C^{(3)}$, $F_D^{(3)}$ including Saran's ten triple hypergeometric functions F_E, F_F, F_G, F_K, F_M, F_N, F_P,F_R, F_S, F_T, three additional functions H_A,H_B,H_C of Srivastava, confluent triple hypergeometric functions of Jain and Exton, Erdélyi's function $\Phi_2^{(3)}$, Humbert's function $\Psi^{(3)}$, Exton's function $\Phi_3^{(3)}$, $\Xi_1^{(3)}$, Srivastava-Exton function $\Phi_D^{(3)}$, Exton's functions ${}^{(k)}_{(1)}E_D^{(3)}$, ${}^{(k)}_{(2)}E_D^{(3)}$, Chandel's function ${}^{(k)}_{(1)}E_C^{(3)}$ etcetera.Triple series $F^{(3)}$ of Srivastava is given by:

$$F^{(3)}\left[\begin{array}{l}(a_A)::(b_B);(d_D);(e_E):(g_G);\\ \\(m_M)::(n_N);(p_P);(q_Q):(r_R);\end{array}\right.$$
$$\left.\begin{array}{l}(h_H);(\ell_L);\\ \\(s_S);(t_T);\end{array} x,y,z\right]$$
$$=\sum_{i,j,k=0}^{\infty}\frac{[(a_A)]_{i+j+k}[(b_B)]_{i+j}[(d_D)]_{j+k}[(e_E)]_{k+i}}{[(m_M)]_{i+j+k}[(n_N)]_{i+j}[(p_P)]_{j+k}[(q_Q)]_{k+i}}\times$$
$$\times\frac{[(g_G)]_i[(h_H)]_j[(\ell_L)]_k x^i y^j z^k}{[(r_R)]_i[(s_S)]_j[(t_T)]_k i!j!k!},\tag{18}$$

it being understood that $|x|$,$|y|$ and $|z|$ are sufficiently small to ensure the convergence of the concerned triple series.

Srivastava's quadruple hypergeometric function:
During 1970-71, Srivastava gave the following quadruple hypergeometric function $F^{(4)}$ [16, p. 70(equation 2.5); 17, p. 229(equation 1.1); 18, pp. 35-36(equation 1.2), pp. 39-40(equations 2.3 ,2.4); 21, pp. 147-148(equations 36,37), p. 232(equations 43, 44)]

$$F^{(4)}\left[\begin{array}{l}a::b,\ c;d,e:\ f,c;g,e;\\ h::r,m;n,p:q,m;s,p;\end{array} x,y,z,t\right]$$
$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(a)_{i+j}(b)_i(d)_j(c)_i(e)_j x^i y^j}{(h)_{i+j}(r)_i(n)_j(m)_i(p)_j i!j!}\times$$
$$\times F^{1:2;2}_{1:2;2}\left[\begin{array}{l}a+i+j:f,\ c+i;g,e+j;\\ h+i+j:q,m+i;s,p+j;\end{array} z,t\right]$$
$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(a)_{i+j}(b)_i(d)_j(c)_i(e)_j x^i y^j}{(h)_{i+j}(r)_i(n)_j(m)_i(p)_j i!j!}\times$$
$$\times\sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a+i+j)_{\ell+k}(f)_\ell(c+i)_\ell(g)_k(e+j)_k z^\ell t^k}{(h+i+j)_{\ell+k}(q)_\ell(m+i)_\ell(s)_k(p+j)_k \ell!k!}$$
$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a)_{i+j+\ell+k}(c)_{i+\ell}(e)_{j+k}(b)_i(d)_j}{(h)_{i+j+\ell+k}(m)_{i+\ell}(p)_{j+k}(r)_i(n)_j}\times$$
$$\times\frac{(f)_\ell(g)_k x^i y^j z^\ell t^k}{(q)_\ell(s)_k i!j!\ell!k!},$$

Now we generalize above quadruple series by the introduction of an arbitrary number of numerator and denominator parameters. We recall the definition of general quadruple hypergeometric function $F_S^{(4)}$ of Srivastava[18, pp. 35-36(equation 1.2); 17, p. 229(equation 1.1); 16, p. 70(equation 2.5)] which is the generalization and unification of Srivastava's function $F^{(3)}$, Exton's quadruple hypergeometric function K_5, K_9, K_{10}, K_{12}, K_{13}, K_{20}, K_{21}, Chandel's function ${}^{(2)}_{(1)}E_C^{(4)}$, Exton's functions ${}^{(2)}_{(1)}E_D^{(4)}$, ${}^{(2)}_{(2)}E_D^{(4)}$, Lauricella's functions $F_A^{(4)}$, $F_B^{(4)}$, $F_C^{(4)}$, $F_D^{(4)}$, its confluent forms given by Exton, Erdélyi's function $\Phi_2^{(4)}$, Humbert's function $\Psi_2^{(4)}$, Exton's functions $\Phi_3^{(4)}$, $\Xi_1^{(4)}$, Srivastava-Exton function $\Phi_D^{(4)}$, etcetera. In our slightly modified notation, $F_S^{(4)}$ is given by:

$$F_S^{(4)}\left[\begin{array}{l}(a_A)::(b_B);(d_D);(e_E);(g_G):(h_H);\\ \\(n_N)::(p_P);(q_Q);(r_R);(s_S):(t_T);\end{array}\right.$$
$$\left.\begin{array}{l}(d_D);(m_M);(g_G);\\ \\(q_Q);(u_U);(s_S);\end{array} w,x,y,z\right]$$
$$=\sum_{i,j,k,\ell=0}^{\infty}\frac{[(a_A)]_{i+j+k+\ell}[(d_D)]_{i+k}[(g_G)]_{j+\ell}[(b_B)]_i[(e_E)]_j}{[(n_N)]_{i+j+k+\ell}[(q_Q)]_{i+k}[(s_S)]_{j+\ell}[(p_P)]_i[(r_R)]_j}\times$$
$$\times\frac{[(h_H)]_k[(m_M)]_\ell w^i x^j y^k z^\ell}{[(t_T)]_k[(u_U)]_\ell i!j!k!\ell!},\tag{19}$$

it being understood that $|w|$,$|x|$,$|y|$ and $|z|$ are sufficiently small to ensure the convergence of the concerned quadruple series.

Pathan's general quadruple hypergeometric function:

Pathan's quadruple hypergeometric function [7, p. 172 (equation 1.2)] $F_P^{(4)}$ is the generalization and unification of Srivastava's function $F^{(3)}$, Lauricella's functions $F_A^{(4)}$, $F_B^{(4)}$, $F_C^{(4)}$, $F_D^{(4)}$, Exton's functions K_2, K_{11}, K_{15}, confluent forms of Lauricella's functions given by Exton, Carlson's function of four variables R. Erdélyi's function $\Phi_2^{(4)}$, Humbert's function $\Psi_2^{(4)}$, Exton's functions $\Phi_3^{(4)}$, $\Xi_1^{(4)}$, Srivastava-Exton function $\Phi_D^{(4)}$, Exton's functions ${}^{(3)}_{(1)}E_D^{(4)}$, ${}^{(3)}_{(2)}E_D^{(4)}$, Chandel's function ${}^{(3)}_{(1)}E_C^{(4)}$ etcetera.

In 1979, a general quadruple hypergeometric series $F_P^{(4)}$ was considered by Pathan [7,p.172 (equation 1.2)] in the form

$$F_P^{(4)}\left[\begin{array}{l}(a_A)::(d_D);(g_G);(m_M);(q_Q):(s_S);(v_V);\\(b_B)::(e_E);(h_H);\ (n_N);\ (r_R):(t_T);(w_W);\end{array}\begin{array}{l}(x_X);(z_Z);\\(y_Y);(u_U);\end{array}C_1,C_2,C_3,C_4\right]$$
$$=\sum_{i,j,k,\ell=0}^{\infty}\frac{[(a_A)]_{i+j+k+\ell}[(d_D)]_{i+j+k}[(g_G)]_{j+k+\ell}[(m_M)]_{k+\ell+i}}{[(b_B)]_{i+j+k+\ell}[(e_E)]_{i+j+k}[(h_H)]_{j+k+\ell}[(n_N)]_{k+\ell+i}}\times$$
$$\times\frac{[(q_Q)]_{\ell+i+j}[(s_S)]_i[(v_V)]_j[(x_X)]_k[(z_Z)]_\ell C_1{}^iC_2{}^jC_3{}^kC_4{}^\ell}{[(r_R)]_{\ell+i+j}[(t_T)]_i[(w_W)]_j[(y_Y)]_k[(u_U)]_\ell i!j!k!\ell!},\qquad(20)$$

it being understood that $|C_1|$,$|C_2|$,$|C_3|$ and $|C_4|$ are sufficiently small to ensure the convergence of the concerned quadruple series.

Megumi Saigo's general quadruple hypergeometric function:

In 1988, M. Saigo defined a more general quadruple hypergeometric function[11, p. 15(equation 17); 12, pp. 455-456(equation 16)] $F_M^{(4)}$ (slightly modified notation) in the following form:

$$F_M^{(4)}\left[\begin{array}{l}(a_A):::(b_B);(d_D);(e_E);(g_G)::(h_H);\\(a'_{A'}):::(b'_{B'});(d'_{D'});(e'_{E'});(g'_{G'})::(h'_{H'});\end{array}\begin{array}{l}(m_M);(n_N);(p_P);(q_Q);(r_R):(s_S);(t_T);\\(m'_{M'});(n'_{N'});(p'_{P'});(q'_{Q'});(r'_{R'}):(s'_{S'});(t'_{T'});\end{array}\begin{array}{l}(u_U);(w_W);\\(u'_{U'});(w'_{W'});\end{array}x,y,z,c\right]$$
$$=\sum_{i,j,k,\ell=0}^{\infty}\frac{[(a_A)]_{i+j+k+\ell}[(b_B)]_{i+j+k}[(d_D)]_{j+k+\ell}[(e_E)]_{k+\ell+i}}{[(a'_{A'})]_{i+j+k+\ell}[(b'_{B'})]_{i+j+k}[(d'_{D'})]_{j+k+\ell}[(e'_{E'})]_{k+\ell+i}}\times$$
$$\times\frac{[(g_G)]_{\ell+i+j}[(h_H)]_{i+j}[(m_M)]_{i+k}[(n_N)]_{i+\ell}[(p_P)]_{j+k}}{[(g'_{G'})]_{\ell+i+j}[(h'_{H'})]_{i+j}[(m'_{M'})]_{i+k}[(n'_{N'})]_{i+\ell}[(p'_{P'})]_{j+k}}\times$$
$$\times\frac{[(q_Q)]_{j+\ell}[(r_R)]_{k+\ell}[(s_S)]_i[(t_T)]_j[(u_U)]_k[(w_W)]_\ell x^iy^jz^kc^\ell}{[(q'_{Q'})]_{j+\ell}[(r'_{R'})]_{k+\ell}[(s'_{S'})]_i[(t'_{T'})]_j[(u'_{U'})]_k[(w'_{W'})]_\ell i!j!k!\ell!},\qquad(21)$$

it being understood that $|x|$,$|y|$,$|z|$ and $|c|$ are sufficiently small to ensure the convergence of the concerned quadruple series.

It is the generalization and unification of Pathan's quadruple hypergeometric function $F_P^{(4)}$, Srivastava's quadruple hypergeometric function $F_S^{(4)}$, quadruple hypergeometric functions of Bhati-Purohit [2,3], Chandel-Agarwal-Kumar [4] and Sharma-Parihar [13,14] etcetera.

2 Three Generalization and unification of product theorems

If denominator parameters in multiple hypergeometric functions are neither zero nor negative integers and any values of arguments and numerator parameters leading to the results which do not make sense, are tacitly excluded.

(i) Main Generalization of First Product Theorem of Ramanujan

(Associated with quadruple hypergeometric function of Saigo and triple hypergeometric function of Srivastava)

$$F_M^{(4)}\left[\begin{array}{l}(e_E):::—;(a_A);(c_C);—::(h_H);—;—;\\ (g_G):::—;(b_B);(d_D);—::(k_K);—;—;\end{array}\right.$$
$$\left.\begin{array}{l}—;—;(\ell_L):(s_S);(u_U);\rho;\rho;\\ —;—;(r_R):(t_T);(w_W);\sigma;\sigma;\end{array} x,z,y,-y\right]$$
$$=\sum_{i=0}^{1}\sum_{j=0}^{1}\frac{[(e_E)]_{i+j}[(h_H)]_{i+j}[(c_C)]_i[(s_S)]_i[(a_A)]_j}{[(g_G)]_{i+j}[(k_K)]_{i+j}[(d_D)]_i[(t_T)]_i[(b_B)]_j}\times$$
$$\times\frac{[(u_U)]_j x^i z^j}{[(w_W)]_j\, i!\, j!}F^{(3)}\left[\begin{array}{l}\Delta[2;(e_E)+i+j]::\Delta[2;(h_H)+i+j];\\ \Delta[2;(g_G)+i+j]::\Delta[2;(k_K)+i+j];\end{array}\right.$$
$$\begin{array}{l}\Delta[2;(a_A)+j];\ \Delta[2;(c_C)+i]:\Delta[2;(s_S)+i],\\ \Delta[2;(b_B)+j];\ \Delta[2;(d_D)+i]:\Delta[2;(t_T)+i],\end{array}$$
$$\begin{array}{l};\Delta[2;(u_U)+j],\qquad\qquad ;\\ \Delta^*(2;1+i);\Delta[2;(w_W)+j],\ \Delta^*(2;1+j);\end{array}$$
$$\left.\begin{array}{l}\Delta[2;(\ell_L)],\rho,\sigma-\rho\ ;\\ \Delta[2;(r_R)],\sigma,\Delta(2;\sigma);\end{array}\ \frac{x^2}{4^\alpha},\frac{z^2}{4^\beta},\frac{y^2}{4^\gamma}\right], \tag{22}$$

provided that involved multiple series on both sides are absolutely convergent[5,20],
where
α = 1+G+D+K+T-E-C-H-S
β = 1+G+B+K+W-E-A-H-U
γ = 1+G+B+D+R-E-A-C-L.

Proof:
Suppose left hand side of the equation (22) is denoted by Θ and write Saigo quadruple hypergeometric function $F_M^{(4)}$ in power series form, we get

$$\Theta=\sum_{m,n,p,q=0}^{\infty}\frac{[(e_E)]_{m+n+p+q}[(a_A)]_{n+p+q}[(c_C)]_{p+q+m}}{[(g_G)]_{m+n+p+q}[(b_B)]_{n+p+q}[(d_D)]_{p+q+m}}\times$$

$$\times\frac{[(h_H)]_{m+n}[(\ell_L)]_{p+q}[(s_S)]_m[(u_U)]_n(\rho)_p(\rho)_q(-1)^q x^m y^{p+q} z^n}{[(k_K)]_{m+n}[(r_R)]_{p+q}[(t_T)]_m[(w_W)]_n(\sigma)_p(\sigma)_q m!n!p!q!}$$

$$=\sum_{m,n,p=0}^{\infty}\sum_{q=0}^{p}\frac{[(e_E)]_{m+n+p}[(a_A)]_{n+p}[(c_C)]_{p+m}[(h_H)]_{m+n}}{[(g_G)]_{m+n+p}[(b_B)]_{n+p}[(d_D)]_{p+m}[(k_K)]_{m+n}}\times$$

$$\times\frac{[(\ell_L)]_p[(s_S)]_m[(u_U)]_n(-1)^q x^m y^p z^n(\rho)_{p-q}(\rho)_q}{[(r_R)]_p[(t_T)]_m[(w_W)]_n(\sigma)_{p-q}(\sigma)_q m!n!(p-q)!q!}$$

$$=\sum_{m,n,p=0}^{\infty}\frac{[(e_E)]_{m+n+p}[(a_A)]_{n+p}[(c_C)]_{p+m}[(h_H)]_{m+n}}{[(g_G)]_{m+n+p}[(b_B)]_{n+p}[(d_D)]_{p+m}[(k_K)]_{m+n}}\times$$

$$\times\frac{[(\ell_L)]_p[(s_S)]_m[(u_U)]_n(\rho)_p x^m z^n y^p}{[(r_R)]_p[(t_T)]_m[(w_W)]_n(\sigma)_p m!n!p!}{}_3F_2\left[\begin{array}{r}-p,1-\sigma-p,\rho;\\ 1-\rho-p,\sigma;\end{array}1\right]$$

$$=\sum_{m,n,p=0}^{\infty}\Phi(m,n,p)\,{}_3F_2\left[\begin{array}{r}-p,1-\sigma-p,\rho;\\ 1-\rho-p,\sigma;\end{array}1\right]$$

where

$$\Phi(m,n,p)=\frac{[(e_E)]_{m+n+p}[(a_A)]_{n+p}[(c_C)]_{p+m}[(h_H)]_{m+n}}{[(g_G)]_{m+n+p}[(b_B)]_{n+p}[(d_D)]_{p+m}[(k_K)]_{m+n}}\times$$

$$\times\frac{[(\ell_L)]_p[(s_S)]_m[(u_U)]_n(\rho)_p x^m z^n y^p}{[(r_R)]_p[(t_T)]_m[(w_W)]_n(\sigma)_p m!n!p!}$$

Therefore

$$\Theta=\sum_{m,n,p=0}^{\infty}\Phi(m,n,2p)\,{}_3F_2\left[\begin{array}{r}-2p,1-\sigma-2p,\rho;\\ 1-\rho-2p,\sigma;\end{array}1\right]$$
$$+\sum_{m,n,p=0}^{\infty}\Phi(m,n,2p+1)\,{}_3F_2\left[\begin{array}{r}-2p-1,-\sigma-2p,\rho;\\ -\rho-2p,\sigma;\end{array}1\right]$$

Using Dixon Summation Theorems (15,16), we get

$$\Theta=\sum_{m,n,p=0}^{\infty}\frac{[(e_E)]_{m+n+2p}[(a_A)]_{n+2p}[(c_C)]_{2p+m}[(h_H)]_{m+n}}{[(g_G)]_{m+n+2p}[(b_B)]_{n+2p}[(d_D)]_{2p+m}[(k_K)]_{m+n}}\times$$
$$\times\frac{[(\ell_L)]_{2p}[(s_S)]_m[(u_U)]_n(\rho)_{2p}x^m z^n y^{2p}}{[(r_R)]_{2p}[(t_T)]_m[(w_W)]_n(\sigma)_{2p}m!n!(2p)!}\times$$
$$\times\,{}_3F_2\left[\begin{array}{r}-2p,1-\sigma-2p,\rho;\\ 1-\rho-2p,\sigma;\end{array}\ 1\right]+0$$

Now replacing m,n by $2m+i,2n+j$ respectively, we get

$$\Theta=\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{m,n,p=0}^{\infty}\frac{[(e_E)]_{2m+2n+2p+i+j}[(a_A)]_{2n+2p+j}}{[(g_G)]_{2m+2n+2p+i+j}[(b_B)]_{2n+2p+j}}\times$$

$$\times\frac{[(c_C)]_{2p+2m+i}[(h_H)]_{2m+2n+i+j}[(\ell_L)]_{2p}[(s_S)]_{2m+i}}{[(d_D)]_{2p+2m+i}[(k_K)]_{2m+2n+i+j}[(r_R)]_{2p}[(t_T)]_{2m+i}}\times$$

$$\times\frac{[(u_U)]_{2n+j}(\rho)_p(\sigma-\rho)_p x^{2m+i}z^{2n+j}y^{2p}}{[(w_W)]_{2n+j}2^{2p}(\sigma)_p(\frac{\sigma}{2})_p(\frac{\sigma+1}{2})_p(2m+i)!(2n+j)!p!}$$

$$=\sum_{i=0}^{1}\sum_{j=0}^{1}\frac{[(e_E)]_{i+j}[(h_H)]_{i+j}[(c_C)]_i[(s_S)]_i[(a_A)]_j[(u_U)]_j x^i z^j}{[(g_G)]_{i+j}[(k_K)]_{i+j}[(d_D)]_i[(t_T)]_i[(b_B)]_j[(w_W)]_j i!j!}\times$$

$$\times\sum_{m,n,p=0}^{\infty}\frac{[(e_E)+i+j]_{2(m+n+p)}[(a_A)+j]_{2(n+p)}}{[(g_G)+i+j]_{2(m+n+p)}[(b_B)+j]_{2(n+p)}}\times$$

$$\times\frac{[(c_C)+i]_{2(p+m)}[(h_H)+i+j]_{2(m+n)}[(\ell_L)]_{2p}[(s_S)+i]_{2m}}{[(d_D)+i]_{2(p+m)}[(k_K)+i+j]_{2(m+n)}[(r_R)]_{2p}[(t_T)+i]_{2m}}\times$$

$$\times\frac{[(u_U)+j]_{2n}(\rho)_p(\sigma-\rho)_p x^{2m}z^{2n}y^{2p}}{[(w_W)+j]_{2n}2^{2p}(\sigma)_p(\frac{\sigma}{2})_p(\frac{\sigma+1}{2})_p(1+i)_{2m}(1+j)_{2n}p!}$$

$$= \sum_{i=0}^{1}\sum_{j=0}^{1} \frac{[(e_E)]_{i+j}[(a_A)]_j[(c_C)]_i[(h_H)]_{i+j}[(s_S)]_i[(u_U)]_j}{[(g_G)]_{i+j}[(b_B)]_j[(d_D)]_i[(k_K)]_{i+j}[(t_T)]_i[(w_W)]_j}\times$$

$$\times\frac{x^i z^j}{i!\,j!}\sum_{m,n,p=0}^{\infty}\frac{[\frac{(e_E)+i+j}{2}]_{m+n+p}[\frac{1+(e_E)+i+j}{2}]_{m+n+p}}{[\frac{(g_G)+i+j}{2}]_{m+n+p}[\frac{1+(g_G)+i+j}{2}]_{m+n+p}}\times$$

$$\times\frac{[\frac{(a_A)+j}{2}]_{n+p}[\frac{1+(a_A)+j}{2}]_{n+p}[\frac{(\ell_L)}{2}]_p[\frac{1+(\ell_L)}{2}]_p[\frac{(c_C)+i}{2}]_{p+m}}{[\frac{(b_B)+j}{2}]_{n+p}[\frac{1+(b_B)+j}{2}]_{n+p}[\frac{(r_R)}{2}]_p[\frac{1+(r_R)}{2}]_p\frac{(d_D)+i}{2}]_{p+m}}\times$$

$$\times\frac{[\frac{1+(c_C)+i}{2}]_{p+m}[\frac{(h_H)+i+j}{2}]_{m+n}[\frac{1+(h_H)+i+j}{2}]_{m+n}(1)_m}{[[\frac{1+(d_D)+i}{2}]_{p+m}[\frac{(k_K)+i+j}{2}]_{m+n}[\frac{1+(k_K)+i+j}{2}]_{m+n}}\times$$

$$\times\frac{[\frac{(s_S)+i}{2}]_m[\frac{1+(s_S)+i}{2}]_m(1)_n[\frac{(u_U)+j}{2}]_n[\frac{1+(u_U)+j}{2}]_n}{[\frac{(t_T)+i}{2}]_m[\frac{1+(t_T)+i}{2}]_m[\frac{(w_W)+j}{2}]_n[\frac{1+(w_W)+j}{2}]_n(\frac{1+i}{2})_m(\frac{2+i}{2})_m}\times$$

$$\times\frac{(\rho)_p(\sigma-\rho)_p}{4^{\{1+(G-E)+(B-A)+(K-H)+(W-U)\}n}(\frac{1+j}{2})_n(\frac{2+j}{2})_n}\times$$

$$\times\frac{1}{4^{\{1+(G-E)+(D-C)+(K-H)+(T-S)\}m}(\sigma)_p(\frac{\sigma}{2})_p\times(\frac{\sigma+1}{2})_p}$$

$$\times\frac{x^{2m}y^{2p}z^{2n}}{4^{\{1+(G-E)+(B-A)+(D-C)+(R-L))\}p}\,m!n!p!},$$

using the definition of $\Delta^*(2;1+i), \Delta^*(2;1+j), \Delta[2;(b_B)]$ and hypergeometric notation of Srivastava's triple series $F^{(3)}$, we can obtain the right hand side of (22).

(ii) Main Generalization of Second Product Theorem of Ramanujan

(Associated with quadruple hypergeometric function of Saigo and triple hypergeometric function of Srivastava)

$$F_M^{(4)}\left[\begin{array}{l}(e_E):::(a_A);—;—;(c_C)::(h_H);—;—;—;\\(g_G):::(b_B);—;—;(d_D)::(k_K);—;—;—;\end{array}\right.$$

$$\left.\begin{array}{l}—;(\ell_L):—;\ —;\ (s_S);\ \ (u_U);\\—;(r_R):\nu,\zeta;\ \nu,\zeta;(t_T);(w_W);\end{array}\ x,-x,y,z\right]$$

$$= \sum_{i=0}^{1}\sum_{j=0}^{1}\frac{[(e_E)]_{i+j}[(\ell_L)]_{i+j}[(a_A)]_i[(s_S)]_i[(c_C)]_j[(u_U)]_j y^i z^j}{[(g_G)]_{i+j}[(r_R)]_{i+j}[(b_B)]_i[(t_T)]_i[(d_D)]_j[(w_W)]_j i!\,j!}\times$$

$$\times F^{(3)}\left[\begin{array}{l}\Delta[2;(e_E)+i+j]::\Delta[2;(a_A)+i];\Delta[2;(\ell_L)+i+j];\\\Delta[2;(g_G)+i+j]::\Delta[2;(b_B)+i];\Delta[2;(r_R)+i+j];\end{array}\right.$$

$$\begin{array}{l}\Delta[2;(c_C)+j]:\Delta[2;(h_H)],\ \Delta(3;\nu+\zeta-1)\\\Delta[2;(d_D)+j]:\Delta[2;(k_K)],\nu,\zeta,\Delta(2;\nu),\Delta(2;\zeta),\\\qquad;\Delta[2;(s_S)+i]\qquad;\\\Delta(2;\nu+\zeta-1);\Delta[2;(t_T)+i],\ \Delta^*(2;1+i);\end{array}$$

$$\left.\begin{array}{l}\Delta[2;(u_U)+j]\qquad;\\\Delta[2;(w_W)+j],\Delta^*(2;1+j);\end{array}\ \frac{-27x^2}{4^\theta},\frac{y^2}{4^\phi},\frac{z^2}{4^\psi}\right], \tag{23}$$

provided that involved multiple series on both sides are absolutely convergent[5,20],

where
θ = 3+G+B+D+K-E-A-C-H
ϕ = 1+G+B+R+T-E-A-L-S
ψ = 1+G+R+D+W-E-L-C-U.

(iii) Main Unification and Generalization of First and Second Product Theorems of Ramanujan
(Associated with quadruple hypergeometric function of Srivastava and double hypergeometric function of Kampé de Fériet)

$$F_S^{(4)}\left[\begin{array}{l}(e_E)::—;\ (h_H);\rho;\ \ (\ell_L):—\ ;(h_H);\rho;\\(g_G)::\nu,\zeta;(k_K);\sigma;(r_R):\nu,\zeta;(k_K);\sigma;\end{array}\right.$$

$$\left.\begin{array}{l}(\ell_L);\\(r_R);\end{array}\ x,y,-x,-y\right]$$

$$= F^{2E:3+2H;2+2L}_{2G:8+2K;3+2R}\left[\begin{array}{l}\Delta[2;(e_E)]:\Delta(3;\nu+\zeta-1),\\\Delta[2;(g_G)]:\Delta(2;\nu+\zeta-1),\nu,\zeta,\end{array}\right.$$

$$\begin{array}{l}\Delta[2;(h_H)];\quad\rho,\sigma-\rho,\Delta[2;(\ell_L)];\\\Delta(2;\nu),\Delta(2;\zeta),\Delta[2;(k_K)];\sigma,\Delta(2;\sigma),\Delta[2;(r_R)];\end{array}$$

$$\left.\frac{-27x^2}{4^\lambda},\frac{y^2}{4^\mu}\right], \tag{24}$$

provided that involved multiple series on both sides are absolutely convergent.[5,20],
where

$\lambda = (3+G+K-E-H)$
$\mu = (1+G+R-E-L)$.

3 Special Cases of Theorems (22) and (23)

(Associated with multiple Gaussian hypergeometric functions of Kampé de Fériet,Srivastava and Pathan.)

Making suitable adjustment of parameters we can derive a number of new reduction formulas associated with double, triple and quadruple hypergeometric functions scattered in the literature of special functions.

In equation (22) setting K=H=L=R=0 and using the notation of Pathan's quadruple hypergeometric function $F_P^{(4)}$, we get

$$F_P^{(4)}\left[\begin{array}{l}(e_E)::\,—\,;(a_A)\,;(c_C)\,;—:(s_S)\,;(u_U)\,;\rho\,;\rho\,;\\(g_G)::\,—\,;(b_B)\,;(d_D)\,;—:(t_T)\,;(w_W)\,;\sigma\,;\sigma\,;\end{array}\ x,z,y,-y\right]$$

$$=\sum_{i=0}^{1}\sum_{j=0}^{1}\frac{[(e_E)]_{i+j}[(c_C)]_i[(s_S)]_i[(a_A)]_j[(u_U)]_j x^i z^j}{[(g_G)]_{i+j}[(d_D)]_i[(t_T)]_i[(b_B)]_j[(w_W)]_j\, i!\,j!}\times$$

$$\times F^{(3)}\left[\begin{array}{l}\Delta[2;(e_E)+i+j]::\,—\,;\Delta[2;(a_A)+j]\,;\\\Delta[2;(g_G)+i+j]::\,—\,;\Delta[2;(b_B)+j]\,;\end{array}\right.$$

$$\begin{array}{l}\Delta[2;(c_C)+i]:\Delta[2;(s_S)+i]\qquad;\Delta[2;(u_U)+j]\\\Delta[2;(d_D)+i]:\Delta[2;(t_T)+i],\Delta^*(2;1+i)\,;\Delta[2;(w_W)+j],\end{array}$$

$$\left.\begin{array}{l};\rho,\sigma-\rho\quad;\\\Delta^*(2;1+j)\,;\sigma,\Delta(2;\sigma)\,;\end{array}\ \frac{x^2}{4^{\alpha^*}},\frac{z^2}{4^{\beta^*}},\frac{y^2}{4^{\gamma^*}}\right],\tag{25}$$

provided that involved multiple series on both sides are absolutely convergent,
where
α^* = 1+G+D+T-E-C-S
β^* = 1+G+B+W-E-A-U
γ^* = 1+G+B+D-E-A-C.

In equation (22) setting A=B=C=D=0 and using the notation of Srivastava's quadruple hypergeometric function $F_S^{(4)}$, we get

$$F_S^{(4)}\left[\begin{array}{l}(e_E)::\rho\,;(\ell_L)\,;(s_S)\,;(h_H):\rho\,;(\ell_L)\,;(u_U)\,;(h_H)\,;\\(g_G)::\sigma\,;(r_R)\,;(t_T)\,;(k_K):\sigma\,;(r_R)\,;(w_W)\,;(k_K)\,;\end{array}\ y,x,-y,z\right]$$

$$=\sum_{i=0}^{1}\sum_{j=0}^{1}\frac{[(e_E)]_{i+j}[(h_H)]_{i+j}[(s_S)]_i[(u_U)]_j x^i z^j}{[(g_G)]_{i+j}[(k_K)]_{i+j}[(t_T)]_i[(w_W)]_j\, i!\,j!}\times$$

$$\times F^{(3)}\left[\begin{array}{l}\Delta[2;(e_E)+i+j]::\Delta[2;(h_H)+i+j]\,;—\,;\,—:\\\Delta[2;(g_G)+i+j]::\Delta[2;(k_K)+i+j]\,;—\,;\,—:\end{array}\right.$$

$$\begin{array}{l}\Delta[2;(s_S)+i]\qquad;\Delta[2;(u_U)+j]\qquad;\\\Delta[2;(t_T)+i],\Delta^*(2;1+i)\,;\Delta[2;(w_W)+j],\ \Delta^*(2;1+j)\,;\end{array}$$

$$\left.\begin{array}{l}\Delta[2;(\ell_L)],\rho,\sigma-\rho\quad;\\\Delta[2;(r_R)],\sigma,\Delta(2;\sigma)\,;\end{array}\ \frac{x^2}{4^{\alpha^{**}}},\frac{z^2}{4^{\beta^{**}}},\frac{y^2}{4^{\gamma^{**}}}\right],\tag{26}$$

provided that involved multiple series on both sides are absolutely convergent,
where
α^{**} = 1+G+K+T-E-H-S
β^{**} = 1+G+K+W-E-H-U
γ^{**} = 1+G+R-E-L.

In equation (23) Setting K=H=L=R=0 and using the notation of Pathan's quadruple hypergeometric function $F_P^{(4)}$, we get

$$F_P^{(4)}\left[\begin{array}{l}(e_E)::(a_A)\,;—\,;—\,;(c_C):\ \ —\,;\ \ —\,;(s_S)\,;(u_U)\,;\\(g_G)::(b_B)\,;—\,;—\,;(d_D):\nu,\zeta\,;\nu,\zeta\,;(t_T)\,;(w_W)\,;\end{array}\ x,-x,y,z\right]$$

$$=\sum_{i=0}^{1}\sum_{j=0}^{1}\frac{[(e_E)]_{i+j}[(a_A)]_i[(s_S)]_i[(c_C)]_j\,[(u_U)]_j y^i z^j}{[(g_G)]_{i+j}[(b_B)]_i[(t_T)]_i[(d_D)]_j[(w_W)]_j\, i!\,j!}\times$$

$$\times F^{(3)}\left[\begin{array}{l}\Delta[2;(e_E)+i+j]::\Delta[2;(a_A)+i]\,;—\,;\\\Delta[2;(g_G)+i+j]::\Delta[2;(b_B)+i]\,;—\,;\end{array}\right.$$

$$\begin{array}{l}\Delta[2;(c_C)+j]:\Delta(3;\nu+\zeta-1)\qquad;\\\Delta[2;(d_D)+j]:\nu,\zeta,\Delta(2;\nu),\Delta(2;\zeta),\Delta(2;\nu+\zeta-1)\,;\end{array}$$

$$\begin{array}{l}\Delta[2;(s_S)+i]\qquad;\Delta[2;(u_U)+j]\qquad;\\\Delta[2;(t_T)+i],\ \Delta^*(2;1+i)\,;\Delta[2;(w_W)+j],\Delta^*(2;1+j)\,;\end{array}$$

$$\left.\frac{-27x^2}{4^{\theta^*}}, \frac{y^2}{4^{\phi^*}}, \frac{z^2}{4^{\psi^*}}\right], \tag{27}$$

provided that involved multiple series on both sides are absolutely convergent,
where
θ^* = 3+G+B+D-E-A-C
ϕ^* = 1+G+B+T-E-A-S
ψ^* = 1+G+D+W-E-C-U.

In equation (23) setting A=B=C=D=0 and using the notation of Srivastava's quadruple hypergeometric function $F_S^{(4)}$, we get

$$F_S^{(4)}\left[\begin{matrix}(e_E) :: \ —\ ; (h_H)\ ; (s_S)\ ; (\ell_L) : \ —\ ; (h_H)\ ; \ (u_U)\ ; (\ell_L)\ ; \\ (g_G) :: \nu, \zeta\ ; (k_K)\ ; (t_T)\ ; (r_R) : \nu, \zeta\ ; (k_K)\ ; (w_W)\ ; (r_R)\ ;\end{matrix}\ x, y, -x, z\right]$$

$$= \sum_{i=0}^{1}\sum_{j=0}^{1} \frac{[(e_E)]_{i+j}\,[(\ell_L)]_{i+j}\,[(s_S)]_i\,[(u_U)]_j\, y^i z^j}{[(g_G)]_{i+j}\,[(r_R)]_{i+j}\,[(t_T)]_i\,[(w_W)]_j\, i!\, j!}\times$$

$$\times F^{(3)}\left[\begin{matrix}\Delta[2;(e_E)+i+j] :: —\ ; \Delta[2;(\ell_L)+i+j]\ ; — : \Delta[2;(h_H)],\ \Delta(3;\nu+\zeta-1)\ ; \Delta[2;(s_S)+i]\ ; \Delta[2;(u_U)+j]\ ; \\ \Delta[2;(g_G)+i+j] :: —\ ; \Delta[2;(r_R)+i+j]\ ; — : \Delta[2;(k_K)], \nu, \zeta, \Delta(2;\nu), \Delta(2;\zeta), \Delta(2;\nu+\zeta-1)\ ; \Delta[2;(t_T)+i], \Delta^*(2;1+i)\ ; \Delta[2;(w_W)+j], \Delta^*(2;1+j)\ ;\end{matrix}\ \frac{-27x^2}{4^{\theta^{**}}}, \frac{y^2}{4^{\phi^{**}}}, \frac{z^2}{4^{\psi^{**}}}\right], \tag{28}$$

provided that involved multiple series on both sides are absolutely convergent,
where
θ^{**} = 3+G+K-E-H
ϕ^{**} = 1+G+R+T-E-L-S
ψ^{**} = 1+G+R+W-E-L-U.

In equation (25) put z=0 and C=D=U=W=0, we get

$$F^{(3)}\left[\begin{matrix}(e_E) :: —\ ; (a_A)\ ; — : (s_S)\ ; \rho\ ; \rho\ ; \\ (g_G) :: —\ ; (b_B)\ ; — : (t_T)\ ; \sigma\ ; \sigma\ ;\end{matrix}\ x, y, -y\right]$$

$$= \sum_{i=0}^{1} \frac{[(e_E)]_i\,[(s_S)]_i\, x^i}{[(g_G)]_i\,[(t_T)]_i\, i!} F^{2E:\ \ 2S;2A+2}_{2G:2T+1;2B+3}\left[\begin{matrix}\Delta[2;(e_E)+i] : \Delta[2;(s_S)+i]\ ; \ \rho, \sigma-\rho, \Delta[2;(a_A)]\ ; \\ \Delta[2;(g_G)+i] : \Delta[2;(t_T)+i], \Delta^*(2;1+i)\ ; \sigma, \Delta(2;\sigma), \Delta[2;(b_B)]\ ;\end{matrix}\ \frac{x^2}{4^{\alpha^{***}}}, \frac{y^2}{4^{\gamma^{***}}}\right], \tag{29}$$

provided that involved multiple series on both sides are absolutely convergent,
where
α^{***} = 1+G+T-E-S
γ^{***} = 1+G+B-E-A.

In equation (27) put z=0 and A=B=U=W=0, we get

$$F^{(3)}\left[\begin{matrix}(e_E) :: (c_C)\ ; —\ ; — : —\ ; —\ ; (s_S)\ ; \\ (g_G) :: (d_D)\ ; —\ ; — : \nu, \zeta\ ; \nu, \zeta\ ; (t_T)\ ;\end{matrix}\ x, -x, y\right]$$

$$= \sum_{i=0}^{1} \frac{[(e_E)]_i\,[(s_S)]_i\, y^i}{[(g_G)]_i\,[(t_T)]_i\, i!} F^{2E:\ \ 2S;2C+3}_{2G:2T+1;2D+8}\left[\begin{matrix}\Delta[2;(e_E)+i] : \Delta[2;(s_S)+i]\ ; \Delta(3;\nu+\zeta-1), \Delta[2;(c_C)]; \\ \Delta[2;(g_G)+i] : \Delta[2;(t_T)+i], \Delta^*(2;1+i)\ ; \nu, \zeta, \Delta(2;\nu), \Delta(2;\zeta), \Delta(2;\nu+\zeta-1), \Delta[2;(d_D)]\ ;\end{matrix}\ \frac{y^2}{4^{\phi^{***}}}, \frac{-27x^2}{4^{\theta^{***}}}\right], \tag{30}$$

provided that involved multiple series on both sides are absolutely convergent,
where
θ^{***} = 3+G+D-E-C
ϕ^{***} = 1+G+T-E-S.

4 Further Unification and Generalization

Unification and Generalization of equations (22), (25), (26) and (29):
Let $\{\Phi_1(n)\}_{n=0}^{\infty}$, $\{\Phi_2(n)\}_{n=0}^{\infty}$, $\{\Phi_3(n)\}_{n=0}^{\infty}$, $\{\Phi_4(n)\}_{n=0}^{\infty}$, $\{\Phi_5(n)\}_{n=0}^{\infty}$, $\{\Phi_6()n\}_{n=0}^{\infty}$, $\{\Phi_7(n)\}_{n=0}^{\infty}$ are the bounded multiple sequences of arbitrary real or complex numbers, $\sigma \neq 0, -1, -2, \ldots$ and $(1-\rho) \neq 0, \pm1, \pm2, \pm3, \ldots$.
Then

$$\sum_{m,n,p,q=0}^{\infty} \Phi_1(m+n+p+q)\Phi_2(n+p+q)\Phi_3(p+q+m)\times$$

$$\times \Phi_4(m+n)\Phi_5(p+q)\Phi_6(m)\Phi_7(n)\frac{(\rho)_p(\rho)_q(-1)^q x^m y^{p+q} z^n}{(\sigma)_p(\sigma)_q m!n!p!q!}$$

$$= \sum_{i=0}^{1}\sum_{j=0}^{1}\frac{x^i z^j}{i!j!}\sum_{m,n,p=0}^{\infty}\Phi_1(2m+2n+2p+i+j)\times$$

$$\times \Phi_2(2n+2p+j)\Phi_3(2p+2m+i)\Phi_4(2m+2n+i+j)\times$$

$$\times \Phi_5(2p)\Phi_6(2m+i)\Phi_7(2n+j)\times$$

$$\times \frac{(\rho)_p(\sigma-\rho)_p(\frac{x^2}{4})^m(\frac{y^2}{4})^p(\frac{z^2}{4})^n}{(\sigma)_p(\frac{\sigma}{2})_p(\frac{\sigma+1}{2})_p(\frac{1+i}{2})_m(\frac{2+i}{2})_m(\frac{1+j}{2})_n(\frac{2+j}{2})_n p!}, \quad (31)$$

provided that each of the series involved is absolutely convergent.

Unification and Generalization of equations (23), (27), (28) and (30):

Let $\{\Psi_1(n)\}_{n=0}^{\infty}$, $\{\Psi_2(n)\}_{n=0}^{\infty}$, $\{\Psi_3(n)\}_{n=0}^{\infty}$, $\{\Psi_4(n)\}_{n=0}^{\infty}$, $\{\Psi_5(n)\}_{n=0}^{\infty}$, $\{\Psi_6(n)\}_{n=0}^{\infty}$, $\{\Psi_7(n)\}_{n=0}^{\infty}$ are the bounded multiple sequences of arbitrary real or complex numbers and $\nu, \zeta, \frac{\nu+\zeta-1}{2}, \frac{\nu+\zeta}{2} \neq 0, -1, -2, \ldots$.
Then

$$\sum_{m,n,p,q=0}^{\infty}\Psi_1(m+n+p+q)\Psi_2(m+n+p)\Psi_3(q+m+n)\Psi_4(m+n)\times$$

$$\times \Psi_5(p+q)\Psi_6(p)\Psi_7(q)\frac{(-1)^n x^{m+n} y^p z^q}{(\nu)_m(\zeta)_m(\nu)_n(\zeta)_n m!n!p!q!}$$

$$= \sum_{i=0}^{1}\sum_{j=0}^{1}\frac{y^i z^j}{i!j!}\sum_{m,p,q=0}^{\infty}\Psi_1(2m+2p+2q+i+j)\Psi_2(2m+2p+i)\times$$

$$\times \Psi_3(2q+2m+j)\Psi_4(2m)\Psi_5(2p+2q+i+j)\Psi_6(2p+i)\Psi_7(2q+j)\times$$

$$\times \frac{(\frac{\nu+\zeta-1}{3})_m(\frac{\nu+\zeta}{3})_m(\frac{\nu+\zeta+1}{3})_m(\frac{-27x^2}{64})^m(\frac{y^2}{4})^p(\frac{z^2}{4})^q}{(\nu)_m(\frac{\nu}{2})_m(\frac{\nu+1}{2})_m(\zeta)_m(\frac{\zeta}{2})_m(\frac{\zeta+1}{2})_m(\frac{\nu+\zeta-1}{2})_m(\frac{\nu+\zeta}{2})_m}\times$$

$$\times \frac{1}{(\frac{i+1}{2})_p(\frac{2+i}{2})_p(\frac{1+j}{2})_q(\frac{2+j}{2})_q m!}, \quad (32)$$

provided that each of the series involved is absolutely convergent.

Generalization of equation (24):
Let $\{\Upsilon_1(n)\}_{n=0}^{\infty}$, $\{\Upsilon_2(n)\}_{n=0}^{\infty}$,$\{\Upsilon_3(n)\}_{n=0}^{\infty}$ are the bounded multiple sequences of arbitrary real or complex numbers, $\nu, \zeta, \sigma, \frac{\nu+\zeta-1}{2}, \frac{\nu+\zeta}{2} \neq 0, -1, -2, \ldots$ and $(1-\rho) \neq 0, \pm 1, \pm 2, \pm 3, \ldots$.
Then

$$\sum_{m,n,p,q=0}^{\infty}\Upsilon_1(m+n+p+q)\Upsilon_2(m+n)\Upsilon_3(p+q)\times$$

$$\times \frac{(\rho)_p(\rho)_q(-1)^{n+q} x^{m+n} y^{p+q}}{(\nu)_m(\zeta)_m(\nu)_n(\zeta)_n(\sigma)_p(\sigma)_q m!n!p!q!}$$

$$= \sum_{m,p=0}^{\infty}\Upsilon_1(2m+2p)\Upsilon_2(2m)\Upsilon_3(2p)\times$$

$$\times \frac{(\rho)_p(\sigma-\rho)_p(\frac{\nu+\zeta-1}{3})_m(\frac{\nu+\zeta}{3})_m(\frac{\nu+\zeta+1}{3})_m}{(\nu)_m(\frac{\nu}{2})_m(\frac{\nu+1}{2})_m(\zeta)_m(\frac{\zeta}{2})_m(\frac{\zeta+1}{2})_m}\times$$

$$\times \frac{(\frac{-27x^2}{64})^m(\frac{y^2}{4})^p}{(\sigma)_p(\frac{\sigma}{2})_p(\frac{\sigma+1}{2})_p(\frac{\nu+\zeta-1}{2})_m(\frac{\nu+\zeta}{2})_m m!p!},, \quad (33)$$

provided that each of the series involved is absolutely convergent.

We conclude our present investigation by observing that various special cases associated with double, triple, quadruple hypergeometric functions scattered in the literature of special functions, can also be deduced in an analogous manner.

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