A Controlled Contraction Principle in Partial S-Metric Spaces

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Abstract: In this paper, we introduce the notion of a partially $\alpha$--contractive self mapping and prove the existence and uniqueness of a fixed point for such mapping. Our results improve and generalize many results in S-metric spaces.

Keywords: Fixed point theory, Partial S-metric space, S-metric space

1 Introduction

The existence and uniqueness of fixed point for a self mapping was first introduced by Banach on a metric space. That was the starting point for many research work on this topic. Under different contraction principle and different types of metric space, such as partial metric space, and b-metric space, see [[3]-[19]]. In this article, we work in partial S-metric space.

The existence and uniqueness of a fixed point for a self mapping on different types of metric spaces were the main topic for many research papers [[2]-[18]]. The notion of S-metric space was introduced by Sedghi [4]. A generalization of S-metric space was given by Nabil in [1], where he introduced partial S-metric spaces. Moreover, he proved the existence of a fixed point for a self mapping in partial S-metric space. In this paper, we generalize the results in [1] by adding a control function to the contraction principle, which makes the results in [1] a direct consequences of our theorems.

Before proceeding to the main results, we set forth some definitions that will be used in the sequel.

Definition 1. [5] Let $X$ be a nonempty set and $p : X \times X \longrightarrow [0, +\infty)$. We say that $(X, p)$ is a partial metric space if for all $x, y, z \in X$ we have:

1. $x = y$ if and only if $p(x, y) = p(x, x) = p(y, y);$
2. $p(x, x) \leq p(x, y);$
3. $p(x, y) = p(y, x);$
4. $p(x, z) \leq p(x, y) + p(y, z) - p(y, y).$

Definition 2. [4] Let $X$ be a nonempty set. An S-metric space on $X$ is a function $S : X^3 \longrightarrow [0, \infty)$ that satisfies the following conditions, for all $x, y, z, a \in X$:

$- S(x; y; z) \geq 0,$
$- S(x; y; z) = 0$ if and only if $x = y = z,$
$- S(x; y; z) \leq S(x; x; a) + S(y; y; a) + S(z; z; a).$

The pair $(X; S)$ is called an S-metric space.

Next, we give the definition of partial S-metric space.

Definition 3. [1] Let $X$ be a nonempty set. A partial S-metric space on $X$ is a function $S_p : X^3 \longrightarrow [0, \infty)$ that satisfies the following conditions, for all $x, y, z, t \in X$:

(i) $x = y$ if and only if $S_p(x, x, x) = S_p(y, y, y) = S_p(x, y, y)$
(ii) $S_p(x, y, z) \leq S_p(x, x, t) + S_p(y, y, t) + S_p(z, z, t) - S_p(t, t, t)$
(iii) $S_p(x, x, x) \leq S_p(x, y, z)$
(iv) $S_p(x, x, y) = S_p(y, y, x).$

The pair $(X, S_p)$ is called a partial S-metric space.

Definition 4. A sequence $\{x_n\}^\infty_{n=0}$ of elements in $(X, S_p)$ is called $p$-Cauchy if the limit $\lim_{m,n \longrightarrow \infty} S_p(x_n, x_m)$ exists and finite. The partial S-metric space $(X, S_p)$ is called complete if for each $p$-Cauchy sequence $\{x_n\}^\infty_{n=0}$ there exists $z \in X$ such that $S_p(z, z, z) = \lim_{n} S_p(z, x_n, x_n) = \lim_{m,n} S_p(x_n, x_m, X).$

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Moreover, \((X, S_p)\) is a complete partial S-metric space if and only if \((X, S_p^n)\) is a complete S-metric space. A sequence \(\{x_n\}_n\) in a partial S-metric space \((X, S_p)\) is called 0-Cauchy if \(\lim_{n,m \to \infty} S_p(x_n, x_m) = 0\). We say that \((X, S_p)\) is 0-complete if every 0-Cauchy in \(X\) converges to a point \(x \in X\) such that \(S_p(x, x, x) = 0\).

One can easily construct an example of a partial S-metric space by using the ordinary partial metric space.

Example 1. \([1]\) Let \(X = [0, \infty)\) and \(p\) be the ordinary partial metric space on \(X\). Define the mapping on \(X^3\) to be \(S_p(x, y, z) = p(x, y) + p(y, z)\). Then \(S_p\) defines a partial S-metric space.

Definition 5. Let \((X, S_p)\) be a partial S-metric space and \(T : X \to X\) be a given mapping. We say that \(T\) is partially \(\alpha\)-contractive if there exists a constant \(k \in (0, 1)\) and a function \(\alpha : X \times X \to (0, +\infty)\) such that for all \(x, y \in X\) we have
\[
\alpha(x, y)S_p(Tx, Ty) \leq \max\{kS_p(x, x, y), S_p(x, x, x), S_p(y, y, y)\},
\]
for all \(n \geq 0\). Also, since \(T\) is \(\alpha\)-admissible; \(\alpha(x_0, x_0) \geq 1\) implies \(\alpha(x_0, x_1) = \alpha(x_0, T(x_0), x_0) \geq 1\) and hence \(\alpha(x_1, x_0) = \alpha(T(x_0), x_0) \geq 1\). By induction on \(n\) we get
\[
\alpha(x_n, x_{n+1}) \geq 1,
\]
for all \(n \geq 0\).

Example 2. Let \(X = [0, +\infty)\). Define \(T : X \to X\) by \(T(x) = \sqrt{x}\) and \(\alpha : X \times X \to (0, +\infty)\) by
\[
\alpha(x, y) = \begin{cases} 
  e^{x-y} & \text{if } x \geq y \\
  0 & \text{if } x < y.
\end{cases}
\]

Therefore, \(\{S_p(x_n, x_n, x_n)\}_{n \geq 0}\) is a nonincreasing sequence. Define
\[
r_0 := \lim_{n \to \infty} S_p(x_n, x_n, x_n) = \inf_{n \geq 0} S_p(x_n, x_n, x_n) \geq 0,
\]
and
\[
M_0 := \frac{2}{1-k} S_p(x_0, x_0, x_1) + S_p(x_0, x_0, x_0).
\]

2 Main Result

In this section, we prove the existence of a fixed point in partial S-metric space. We prove relevant corollary. This next theorem is considered to be our main result.

Theorem 1. Let \((X, S_p)\) be a complete partial S-metric space, \(T\) be a self mapping on \(X\) and assume that \(T\) is partially \(\alpha\)-contractive. If \(T\) is \(\alpha\)-admissible and \(R_\alpha\)-admissible and if \(X_p(\alpha)\) is nonempty, then \(Z_p(\alpha)\) is nonempty. Also, assume that there exists \(x_0 \in X\) such that \(\alpha(x_0, x_0) \geq 1\) then:

1. The set \(X_p(\alpha)\) is nonempty;
2. There exists \(x \in X_p(\alpha)\) such that \(T(a) = a\).

Moreover, if for all \(u, v \in X_p(\alpha)\) with the property \(Tu = u\) and \(Tv = v\) we have \(\alpha(u, v) \geq 1\), then \(T\) has a unique fixed point in \(X_p(\alpha)\).

Proof. Let \(x_0 \in X\) such that \(\alpha(x_0, x_0) \geq 1\). Define a sequence \(\{x_n\}\) for all \(n \geq 0\) in \(X\) such that \(x_1 = T(x_0), x_2 = T(x_1), \ldots, x_{n+1} = T(x_n)\). Since \(T\) is \(R_\alpha\)-admissible and \(\alpha\)-admissible, we have \(\alpha(x_0, x_1) = \alpha(x_0, T(x_0)) \geq 1\) and hence \(\alpha(x_1, x_2) = \alpha(T(x_0), T(x_1)) \geq 1\). So, by induction on \(n\) we get
\[
\alpha(x_n, x_{n+1}) \geq 1,
\]
for all \(n \geq 0\). Also, since \(T\) is \(R_\alpha\)-admissible; \(\alpha(x_0, x_0) \geq 1\) implies \(\alpha(x_0, x_1) = \alpha(x_0, T(x_0)) \geq 1\). By induction on \(n\), we also conclude that
\[
\alpha(x_n, x_n) \geq 1.
\]

Therefore, \(\{S_p(x_n, x_n, x_n)\}_{n \geq 0}\) is a nonincreasing sequence. Define
\[
r_0 := \lim_{n \to \infty} S_p(x_n, x_n, x_n) = \inf_{n \geq 0} S_p(x_n, x_n, x_n) \geq 0,
\]
and
\[
M_0 := \frac{2}{1-k} S_p(x_0, x_0, x_1) + S_p(x_0, x_0, x_0).
\]

Next, we need to show that \(S_p(x_0, x_0, x_n) \leq M_0\), for any \(n \geq 0\). If \(n = 0\); the case is trivial. For \(n = 1\) and using the fact that \(k \in (0, 1)\) we deduce that
\[
S_p(x_0, x_0, x_1) \leq \frac{2}{1-k} S_p(x_0, x_0, x_1) \leq M_0.
\]
So, we may assume that is true for all \(n \leq n_0 - 1\) and prove it for \(n = n_0 \geq 2\).
Also, by induction assumption, we have
\[ S_p(x_0, x_{n_0}, x_{n_0-1}) \leq \frac{2\varepsilon}{1-k} S_p(x_0, x_1, x_1) + S_p(x_0, x_0, x_0). \]
So, we have
\[
S_p(x_0, x_0, x_{n_0}) \leq 2S_p(x_0, x_0, x_1) + \max \left\{ \frac{2\varepsilon}{1-k} S_p(x_0, x_1, x_1) + kS_p(x_2, x_2, x_0), S_p(x_0, x_0, x_0) \right\} \\
\leq 2S_p(x_0, x_0, x_1) + \frac{2\varepsilon}{1-k} S_p(x_0, x_0, x_1) + S_p(x_0, x_0, x_0) = 2 \frac{\varepsilon}{1-k} S_p(x_0, x_0, x_1) + S_p(x_0, x_0, x_0) = M_0.
\]

Hence, by induction we conclude that \( S_p(x_0, x_0, x_m) \leq M_0 \). Next, we need to show that
\[
\lim_{n,m} S_p(x_n, x_n, x_m) = r_0.
\]
For all \( n,m \) we have \( S_p(x_n, x_n, x_m) \geq S_p(x_n, x_n, x_n) \geq r_0 \). Let \( \varepsilon > 0 \) find a natural number \( n_0 \) such that \( S_p(x_0, x_0, x_0) < r_0 + \varepsilon \) and \( 2M_k \varepsilon < r_0 + \varepsilon \). Now for any \( n,m \geq n_0 \), since \( T \) is \( R_k \)-admissible and using the fact that \( \alpha(x_n, x_{n+1}) \geq 1 \) we deduce that \( \alpha(x_n, x_m) \geq 1 \). Hence,
\[
S_p(x_n, x_n, x_n) \leq \alpha(x_n, x_n) S_p(x_n, x_n, x_n) \\
\leq \max \left\{ \alpha(x_n, x_{n-1}, x_{n-1}) S_p(x_{n-1}, x_{n-1}, x_{n-1}), S_p(x_{n-1}, x_{n-1}, x_{n-1}) \right\} \\
\leq \max \left\{ k \alpha(x_{n-1}, x_{n-2}, x_{n-2}) S_p(x_{n-2}, x_{n-2}, x_{n-2}), S_p(x_{n-2}, x_{n-2}, x_{n-2}) \right\} \\
\leq \cdots \leq \max \left\{ k^n \alpha(x_{n-n_0}, x_{n-n_0}, x_{n-n_0}) S_p(x_{n-n_0}, x_{n-n_0}, x_{n-n_0}) \right\} \\
\leq r_0 + \varepsilon.
\]

Hence,
\[
\lim_{n,m} S_p(x_n, x_n, x_m) = r_0.
\]
Since \( (X, p) \) is a complete partial \( S \)-metric space; there exists \( x \in X \) such that
\[
r_0 = S_p(x, x, x) = \lim_{n} S_p(x, x, x_n) = \lim_{n} S_p(x_n, x_n, x, x) \quad (n, m) \to \infty.
\]
Next, we show that \( S_p(x, x, x) = S_p(x, x, T x) \). For each natural number \( n \) we have
\[
S_p(x, x, x) = \sum_{i=0}^{n} S_p(x, x, x_i) \leq 2S_p(x, x, x_0) + S_p(x, x, x_n) + S_p(T x, T x, x_0).
\]
Using the property that \( T \) is \( \alpha \)-contractive we deduce that there exists a subsequence of natural numbers \( \{n_i\} \) such that
\[
S_p(T x, T x, x_{n_i}) \leq \alpha(x_{n_i-1}, x_{n_i}) S_p(T x, T x, x_{n_i}) \\
\leq \max \left\{ k S_p(x, x, x_{n_i-1}), S_p(x, x, x_{n_i}), S_p(x_{n_i-1}, x_{n_i-1}, x_{n_i-1}) \right\}.
\]
So, for \( l \geq 1 \), we have either \( S_p(T x, T x, x_{n_i}) \leq k S_p(x, x, x_{n_i-1}) \) or less than or equal \( S_p(x, x, x) \) or less than or equal \( S_p(x_{n_i-1}, x_{n_i-1}, x_{n_i-1}) \).
In all of these cases, if we take the limit as \( l \) goes toward \( \infty \) we get \( S_p(x, x, T x) \leq S_p(x, x, x) \). But, we know by the property \( (ii) \) of the partial \( S \)-metric space definition that \( S_p(x, x, x) \leq S_p(x, x, T x) \). Therefore,
\[
S_p(x, x, x) = S_p(x, x, T x).
\]

Now, we show that \( X_S^\alpha(\alpha) \) is nonempty. For each natural number \( l \) pick \( x_l \in X \) with \( \alpha(x_l, x_l) \geq 1 \) and \( S_p(x_l, x_l, x_l) < \rho_{S_p}^\alpha + \frac{1}{n_0} \) and show that
\[
\lim_{n,m} S_p(x_n, x_n, x_m) = \rho_{S_p}^\alpha.
\]
Let \( \varepsilon > 0 \) put \( n_0 := \left( \frac{1}{\rho_{S_p}^\alpha} \right) + 1 \) if \( l \geq n_0 \) then we have:
\[
\rho_{S_p}^\alpha \leq S_p(x_l, x_l, T x_l) \leq S_p(x_l, x_l, T x_l) \leq r_l \leq S_p(x_l, x_l, T x_l) \leq \rho_{S_p}^\alpha + \frac{1}{n_0} < \rho_{S_p}^\alpha(\alpha) + \frac{1}{n_0}. \]
Hence, we deduce that:
\[
U_l := S_p(x_l, x_l, T x_l) - S_p(T x_l, T x_l, T x_l) < \frac{\varepsilon}{3},
\]
for \( l \geq n_0 \). Also, if \( l \geq n_0 \), then \( S_p(x_l, x_l, T x_l) = r_l \leq S_p(x_l, x_l, x_l) < \rho_{S_p}^\alpha(\alpha) + \frac{1}{n_0} \).

We know that \( S_p(x_l, x_l, x_l) = S_p(x_l, x_l, T x_l) \) which implies:
\[
U_l := S_p(x_l, x_l, T x_l) - S_p(T x_l, T x_l, T x_l) < \frac{\varepsilon}{3},
\]
for \( l \geq n_0 \). Also, if \( l \geq n_0 \), then \( S_p(x_l, x_l, x_l) = r_l \leq S_p(x_l, x_l, x_l) < \rho_{S_p}^\alpha(\alpha) + \frac{1}{n_0} \).

Hence,
\[
\rho_{S_p}^\alpha(\alpha) \leq \rho_{S_p}^\alpha(\alpha) + \frac{\varepsilon}{3} < \rho_{S_p}^\alpha(\alpha) + \varepsilon.
\]
Thus,
\[
\lim_{n,m} S_p(x_n, x_n, x_m) = \rho_{S_p}^\alpha(\alpha).
\]
Since \( (X, S_p) \) is complete there exists \( a \in X \) such that
\[
S_p(a, a, a) = \lim_{n} S_p(a, a, x_n) = \lim_{n} S_p(a, a, x_n) = \rho_{S_p}^\alpha(\alpha).
\]
Therefore, we conclude that \( a \in X_S^\alpha(\alpha) \) and thus \( X_S^\alpha(\alpha) \) is nonempty. Therefore, \( Z_S^\alpha(\alpha) \) is nonempty.

Now, let \( x_0 \in Z_S^\alpha(\alpha) \) be arbitrary. Then by the above argument we have
\[
\rho_{S_p}^\alpha(\alpha) \leq S_p(T x, T x, T x) \leq S_p(x, x, T x) = r_0 = \rho_{S_p}^\alpha(\alpha).
\]
Thus, \( T x = x \). Now, assume that \( T \) has two fixed points in \( Z_S^\alpha(\alpha) \) say \( u \) and \( v \). By our hypothesis, we know that \( \alpha(u, v) \geq 1 \). Thus,
\[
S_p(u, u, v) = S_p(T u, T u, T v) \leq \alpha(u, v) S_p(T u, T u, T v) \leq \max \left\{ k S_p(u, u, v), S_p(u, u, u), S_p(v, v, v) \right\}.
\]
Now, if \( S_p(u, u, v) \leq kS_p(u, u, v) \) we deduce that \( S_p(u, u, v) = 0 \) and in this case \( u = v \), or condition (ii) of the definition of the partial S-metric space we obtain \( S_p(u, u, u) = S_p(v, v, v) \) and in this case by condition (i) of the same definition we conclude that \( u = v \). Therefore, we obtain the uniqueness as desired.

As a consequence of the above result, the following corollary follows easily.

**Corollary 1.** Let \((X, S_p)\) be a 0-complete partial S-metric space, \( k \in [0, 1) \) and consider the map \( T : X \rightarrow X \) to be \( \alpha\)-admissible and \( R_{\alpha}\)-admissible, and there exists \( x_0 \in X \) such that \( \alpha(x_0, x_0) \geq 1 \), also for every \( x, y \in X \) we have \( \alpha(x, y)S_p(Tx, Tx, Ty) \leq kS_p(x, y, y) \). Then there exists \( x \in X \) such that \( T^\infty x = x \).

**Proof.** Using the same technique and notation in the proof of Theorem 1, we deduce that \( S_p(x_0, x_0, x_0) \leq \alpha(x_0, x_0)S_p(x_0, x_0, x_0) \leq k^2S_p(x_0, x_0, x_0) \).

This implies that \( S_p(x, x, x) = \lim_{n \to \infty} S_p(x, x, x) = \lim_{n \to \infty} S_p(x, x, x) = 0 \).

In closing, we change the contraction principle in Theorem 1, to show that there exist a unique fixed point in the whole space \( X \).

**Theorem 2.** Let \((X, S_p)\) be a complete partial S-metric space, \( k \in [0, 1) \) and assume that there exists \( x_0 \in X \) such that \( \alpha(x_0, x_0) \geq 1 \). Consider the map \( T : X \rightarrow X \) to be \( \alpha\)-admissible and \( R_{\alpha}\)-admissible. Assume that for every \( x, y \in X \) we have

\[
\alpha(x, y)S_p(Tx, Tx, Ty) \leq \max \{kS_p(x, y, y), \frac{S_p(x, x, x) + S_p(y, y, y)}{2}\},
\]

then there exists a unique \( u \in X \) such that \( Tu = u \).

**Proof.** Note that, for every \( x, y \in X \) we have:

\[
\alpha(x, y)S_p(Tx, Tx, Ty) \leq \max \{kS_p(x, x, x), \frac{S_p(x, x, x) + S_p(y, y, y)}{2}\},
\]

Thus, all the conditions of Theorem 1 are satisfied. Hence, there exists two fixed points \( u \), \( v \in X \) for \( T \) such that \( \alpha(u, v) \geq 1 \). Hence,

\[
S_p(u, u, v) = S_p(Tu, Tu, Tv) \leq \alpha(u, v)S_p(Tu, Tu, Tv) \leq \max \{kS_p(u, u, v), \frac{S_p(u, u, u) + S_p(v, v, v)}{2}\}.
\]

Thus, we either have \( S_p(u, u, v) \leq kS_p(u, u, v) \) which implies that \( S_p(u, u, u) = 0 \) and hence \( u = v \), or \( 0 = 2S_p(u, u, v) - S_p(u, u, u) - S_p(v, v, v) \) which also implies that \( u = v \) as desired.

**Example 3.** Let \((X, S_p)\) be a partial S-metric space, where \( X = [0, 1] \cup [2, 3] \) and the partial S-metric space \( S_p : X^3 \rightarrow [0, +\infty) \) is defined by

\[
S_p(x, y, z) = \begin{cases} 
\max \{x, y, z\} - z & \text{if } \{x, y, z\} \cap [2, 3] \neq \emptyset, \\
|x - y - z| & \text{if } \{x, y, z\} \subseteq [0, 1].
\end{cases}
\]

Define the functions \( T : X \rightarrow X \) and \( \alpha : X \times X \rightarrow [0, +\infty) \) as follows \( Tx = \frac{x + 1}{2} \) if \( 0 \leq x \leq 1, T2 = 1, \) and \( Tx = \frac{x + 2}{2} \) if \( 2 < x \leq 3 \).

It is easy to see that \( T \) is \( \alpha\)-admissible and \( R_{\alpha}\)-admissible. Note that, we can always pick our \( x, y, z \) such that \( \max \{x, y\} > z \). Also \( T \) is an increasing function. So, for every \( x \geq y \in X \) we have:

\[
S_p(Tx, Tx, Ty) \leq \alpha(x, y)S_p(Tx, Tx, Ty) \leq \frac{1}{2}S_p(x, y, y),
\]

and

\[
S_p(Tx, Tx, Ty) \leq \frac{1}{2}S_p(x, x, x) + \frac{1}{2}S_p(y, y, y),
\]

\[
\{x, y\} \cap [2, 3] \neq \emptyset.
\]

One can verify that the function \( T \) in this example satisfies the conditions of Theorem 2 and the unique fixed point will be 1.

**3 Conclusion**

In closing, the author would like to bring to the reader’s attention the possibility of obtaining the same result of Theorem 2.1 by changing the hypothesis where \( T \) is partially \( \alpha\)-contractive with the following contraction principle \( \alpha(x, y)S_p(Tx, Tx, Ty) \leq \psi(S_p(x, x, y)) \), where \( \psi \) is a self-function on \( (0, +\infty) \).

**References**


