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Generalized Implicit Strong Vector Equilibrium Problem with Vector Relaxed Monotonicity

Mijanur Rahaman and Rais Ahmad*

Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India

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Abstract: In this paper, we introduce a new concept of vector relaxed α-C-monotonicity for bi-mappings. Using the KKM technique and other appropriate assumptions, we establish existence results for generalized implicit strong vector equilibrium problem with vector relaxed α-C-monotonicity and weaker coercivity condition in topological vector spaces. Finally, we give some applications of generalized implicit strong vector equilibrium problem to convex differentiable optimization problem and convex minimization problem. An example is also given.

Keywords: Vector equilibrium problem, vector relaxed α-C-monotone, KKM theorem, generalized vector 0-diagonally convexity

1 Introduction

Equilibrium problems introduced and investigated by *Blum* and *Oettli* [4] are being used to study a wide class of unrelated problems appearing in pure and applied sciences in a natural, novel and unified framework. The equilibrium problems include many mathematical problems as particular cases for examples, mathematical programming problems, complementarity problems, variational inequality problems, minimax inequality problems, fixed point problems, Nash equilibrium problems etc., see e.g. [2,3,4].

Several application oriented problems occurring in optimization, economics, engineering and general sciences involve equilibria in their description. In optimization theory, equilibrium problems have got into the centre of theoretical approach. From a mathematical outlook, it invites us to ponder that many advances in optimization theory in recent years ought to have major involvements for equilibrium problems. The technique involved in the study of equilibrium problems are applicable to a variety of diverse area and proved to be fruitful and unconventional.

During the recent years many generalizations of monotonicity such as pseudomonotonicity, relaxed monotonicity, relaxed α-monotonicity, quasimonotonicity etc., (see [5,6,12]) have been introduced to study equilibrium problems and variational inequalities and based on these different relaxed monotonicity notions, existence results for equilibrium problems and variational inequalities are obtained by many authors, see [1,10,9] and references therein.

Inspired and motivated by the recent development discussed above for equilibrium problems and generalized monotonicities, in this report, we introduced the notion of vector relaxed α-C-monotonicity to solve generalized implicit strong vector equilibrium problem in topological vector spaces by using KKM theorem for bounded and unbounded sets. As an application, we applied our problem to convex differentiable optimization problem and convex minimization problem. Our results of this paper extend and improve many known results and the concept of α-C-monotonicity is much more general than the existing monotonicities.

2 Preliminaries

Let X,Y and Z be the topological vector spaces and $K \subset X, D \subset Z$ nonempty subsets of X and Z, respectively. Let $C \subset Y$ be a closed convex and pointed cone in Y. Let $f : D \times K \times K \longrightarrow Y$ be a tri-mapping, $g : K \longrightarrow Y$ a single-valued mapping and $T : K \longrightarrow 2^D$ a multi-valued mapping. We consider the following generalized implicit

* Corresponding author e-mail: mrahman96@yahoo.com

strong vector equilibrium problem of finding $\bar{x} \in K$ and $\bar{z} \in T(\bar{x})$ such that

$$f(\bar{z}, \bar{x}, y) + g(y) - g(\bar{x}) \in C, \ \forall y \in K. \quad (1)$$

We call this $\bar{x}$ a strong solution for the generalized implicit strong vector equilibrium problem (2).

Some special cases of problem (2) are listed below.

(i)If $g = 0$, then problem (2) becomes generalized strong vector quasi equilibrium problem with multi-valued mapping considered by *Fu et al.* [11] which is to find $\bar{x} \in K$ and $\bar{z} \in T(\bar{x})$ such that

$$f(\bar{z}, \bar{x}, y) \in C, \ \forall y \in K. \quad (2)$$

(ii)If T is a single-valued mapping such that $f(z,x,y) = \langle T(x), y-x \rangle$ and $Y = \mathbb{R}$, then problem (2) reduces to the following mixed variational inequality problem of finding $\bar{x} \in K$ such that

$$\langle T(x), y-x \rangle + g(y) - g(\bar{x}) \geq 0, \ \forall y \in K.$$

This problem was introduced and studied by *Noor* [13].

Now, we present some definitions and concepts which are essential to prove our main results.

Definition 1. *A mapping $g: K \longrightarrow Y$ is said to be hemicontinuous if, for any fixed $x, y \in K$, the mapping $\lambda \mapsto g(x + \lambda(y-x))$ is continuous at 0^+.*

Definition 2. *A multi-valued mapping $F: K \longrightarrow 2^Y$ is said to be KKM-mapping if, for each finite subset $\{x_1, \cdots, x_n\}$ of K, $Co\{x_1, \cdots, x_n\} \subseteq \bigcup_{i=1}^{n} F(x_i)$, where $Co\{x_1, \cdots, x_n\}$ denotes the convex hull of $\{x_1, \cdots, x_n\}$.*

Theorem 1([8]). *Let K be a subset of a topological vector space X and let $F: K \longrightarrow 2^Y$ be a KKM-mapping. If for each $x \in K, F(x)$ is closed and if for at least one point $x \in K, F(x)$ is compact, then $\bigcap_{x \in K} F(x) \neq \emptyset$.*

Definition 3. *A mapping $g: K \longrightarrow Y$ is said to be completely continuous if, for any sequence $\{x_n\} \in K$, $x_n \to x_0 \in K$ weakly, then $g(x_n) \to g(x_0)$.*

Definition 4. *Let $h: K \times K \longrightarrow Y$ be a mapping. Then h is said to be C-convex in the first argument if, for each pair $x, y \in K$ and $\lambda \in [0,1]$,*

$$h(\lambda x + (1-\lambda)y, z) \in \lambda h(x,z) + (1-\lambda)h(y,z) - C, \ \forall z \in K.$$

Definition 5. *If K is an affine set, then $g: K \longrightarrow Y$ is said to be affine if,*

$$g(tu_1 + (1-t)u_2) = tg(u_1) + (1-t)g(u_2), \ \forall u_1, u_2 \in K, t \in \mathbb{R};$$

with $u = tu_1 + (1-t)u_2 \in K$.

Now, we extend the definition of relaxed α-monotonicity [12] to vector relaxed α-C-monotonicity for bi-mappings.

Definition 6. *A bi-mapping $\phi: K \times K \longrightarrow Y$ is said to be vector relaxed α-C-monotone if, there exists a mapping $\alpha: X \longrightarrow Y$ with $\alpha(tx) = t^p \alpha(x)$, for all $t > 0$ such that*

$$\phi(x,y) + \phi(y,x) \in \alpha(y-x) - C, \ \forall x, y \in K,$$

where $\lim_{t \to 0} \frac{t^p \alpha(y-x)}{t} = 0$ and $p > 1$ is a constant.

Example 2.1. Let $X = K = \mathbb{R}$, $Y = \mathbb{R}^2$ and $C = \{(x,y) \in \mathbb{R}^2 : x, y \leq 0\}$. Let the function ϕ and α be defined by

$$\phi(x,y) = \left(\frac{x}{y}, x^2 + y^2\right),$$
$$\alpha(x) = (-x^2, x^2).$$

Now,

$$\begin{aligned}
&\phi(x,y) + \phi(y,x) - \alpha(y-x) \\
&= \left(\frac{x}{y}, x^2 + y^2\right) + \left(\frac{y}{x}, y^2 + x^2\right) - \left(-(y-x)^2, (y-x)^2\right) \\
&= \left(\frac{x^2 + y^2}{xy}, 2(x^2 + y^2)\right) - \left(-(y-x)^2, (y-x)^2\right) \\
&= \left(\frac{x^2 + y^2}{xy} + (y-x)^2, (x+y)^2\right) \\
&\in -C.
\end{aligned}$$

Therefore, ϕ is vector relaxed α-C-monotone.

Definition 7([7]). *Let $T: K \longrightarrow 2^D$ be a multi-valued mapping. A tri-mapping $f: D \times K \times K \longrightarrow Y$ is said to be generalized vector 0-diagonally convex with respect to T if, for any finite set $\{x_1, \cdots, x_n\} \in K$ and any $\bar{x} = \sum_{i=1}^{n} t_i x_i$ with $t_i \geq 0$ for $i = 1, \cdots, n$ and $\sum_{i=1}^{n} t_i = 1$, there exists $z \in T(\bar{x})$ such that*

$$\sum_{i=1}^{n} t_i f(z, \bar{x}, x_i) \notin -C.$$

3 Existence Results

In this section, we discuss the existence results for generalized implicit strong vector equilibrium problem (2).

Theorem 2. *Let $K \subset X$ and $D \subset Z$ be the nonempty bounded closed convex subsets of the topological vector spaces X and Z, respectively and Y an another topological vector space. Suppose that $C \subset Y$ is a closed convex pointed cone in Y. Let $f: D \times K \times K \longrightarrow Y$ be a tri-mapping, $g: K \longrightarrow Y$ a single-valued mapping and $T: K \longrightarrow 2^D$ a closed compact continuous multi-valued mapping. Assume that*

(i) for each $x \in K$, there exists $z \in T(x)$ such that $f(z,x,x) = 0$;

(ii) for fixed $z \in D$, the mapping $f(z, \cdot, \cdot): K \times K \longrightarrow Y$ is vector relaxed α-C-monotone;

(iii) *f is generalized vector 0-diagonally convex with respect to T;*
(iv) *f is hemicontinuous in the second argument and C-convex in the third argument;*
(v) *for fixed $y \in K$, the mapping $x \mapsto f(z,x,y)$ is completely continuous;*
(vi) *g is affine, hemicontinuous and completely continuous mapping;*
(vii) *the mapping $\alpha : X \longrightarrow Y$ is completely continuous;*

Then, there exists $\bar{x} \in K$ and $\bar{z} \in T(\bar{x})$ such that

$$f(\bar{z},\bar{x},y) + g(y) - g(\bar{x}) \in C, \ \forall y \in K.$$

For the proof of Theorem 2, we need the following lemma, for which all the assumptions of Theorem 2 are remain same.

Lemma 1. *The following two problems are equivalent:*

(I) Find $x \in K$ and $z \in T(x)$ such that $f(z,x,y) + g(y) - g(x) \in C; \ \forall y \in K;$
(II) Find $x \in K$ and $z \in T(x)$ such that $f(z,y,x) + g(x) - g(y) \in \alpha(y-x) - C; \ \forall y \in K.$

Proof. Suppose that (I) has a solution, i.e., there exist $x \in K$ and $z \in T(x)$ such that

$$f(z,x,y) + g(y) - g(x) \in C; \ \forall y \in K.$$

Since $f(z,\cdot,\cdot)$ is vector relaxed α-C-monotone, we have

$$f(z,x,y) + f(z,y,x) \in \alpha(y-x) - C.$$

Now,

$$\begin{aligned} & f(z,y,x) + g(x) - g(y) \\ \in\ & \alpha(y-x) - f(z,x,y) + g(x) - g(y) - C \\ \in\ & \alpha(y-x) - C - C \\ =\ & \alpha(y-x) - C. \end{aligned}$$

Hence (II) follows.

Conversely, suppose $x \in K$ and $z \in T(x)$ is the solution of (II) and $y \in K$ is any point. Letting $x_t = ty + (1-t)x$, $t \in (0,1]$ and due to convexity of K, $x_t \in K$. Therefore,

$$f(z,x_t,x) + g(x) - g(x_t) \in \alpha(x_t - x) - C. \quad (3)$$

Since f is C-convex in the third argument, $0 \in f(z,x,x)$, g is affine and using (3), we have

$$\begin{aligned} 0 =\ & f(z,x_t,x_t) + g(x_t) - g(x_t) \\ \in\ & t f(z,x_t,y) + (1-t) f(z,x_t,x) + t g(y) + \\ & (1-t)g(x) - g(x_t) - C \\ =\ & t\{f(z,x_t,y) + g(y) - g(x_t)\} + (1-t)\{f(z,x_t,x) + \\ & g(x) - g(x_t)\} - C \\ \in\ & t\{f(z,x_t,y) + g(y) - g(x_t)\} + (1-t)\alpha(x_t - x) - \\ & (1-t)C - C \\ \in\ & t\{f(z,x_t,y) + g(y) - g(x_t)\} + (1-t)\alpha(x_t - x) - C. \end{aligned}$$

This implies that

$$t\{f(z,x_t,y) + g(y) - g(x_t)\} + (1-t)\alpha(x_t - x) \in C.$$

Since C is a convex cone, we have for $p > 1$

$$f(z,x_t,y) + g(y) - g(x_t) + (1-t)\frac{t^p \alpha(y-x)}{t} \in C.$$

Since g, f are hemicontinuous in the second argument, letting $t \to 0$, we get

$$f(z,x,y) + g(y) - g(x) \in C, \ \ \forall y \in K;$$

and therefore (I) follows.

Proof of Theorem 2. For any $y \in K$, define two multi-valued mappings $F, G : K \longrightarrow 2^X$ as follows:

$$\begin{aligned} F(y) &= \{x \in K : f(z,x,y) + g(y) - g(x) \in C, \text{ for some } z \in T(x)\}; \\ G(y) &= \{x \in K : f(z,y,x) + g(x) - g(y) \in \alpha(y-x) - C, \text{ for some } z \in T(x)\}. \end{aligned}$$

We claim that F is a KKM mapping. In fact, if F is not a KKM mapping, then there exists $\{y_1, \cdots, y_n\} \subset K$ such that $Co\{y_1, \cdots, y_n\} \nsubseteq \cup_{i=1}^n F(y_i)$, that means there exists $\bar{x} \in Co\{y_1, \cdots, y_n\}$, $\bar{x} = \sum_{i=1}^n t_i y_i$, where $t_i \geq 0$, $i = 1, \cdots, n$ with $\sum_{i=1}^n t_i = 1$, but $\bar{x} \notin \cup_{i=1}^n F(y_i)$.

Thus, we have for $\bar{z} \in T(\bar{x})$

$$f(\bar{z},\bar{x},y_i) + g(y_i) - g(\bar{x}) \notin C;$$

and hence using the affinity of g, we have

$$\sum_{i=1}^n t_i\{f(\bar{z},\bar{x},y_i) + g(y_i) - g(\bar{x})\} = \sum_{i=1}^n t_i\{f(\bar{z},\bar{x},y_i)\} + g(\bar{x}) - g(\bar{x}) \notin C;$$

which contradicts generalized vector 0-diagonally convexity of f with respect to T and hence F is a KKM mapping.

Now, we will prove that $F(y) \subseteq G(y)$, for all $y \in K$. For any given $y \in K$, let $x \in F(y)$. Then, there exists $z \in T(x)$ such that

$$f(z,x,y) + g(y) - g(x) \in C.$$

Since $f(z,\cdot,\cdot)$ is vector relaxed α-C-monotone, using similar arguments as in the proof of Lemma 1, we have

$$f(z,y,x) + g(x) - g(y) \in \alpha(y-x) - C.$$

Therefore $x \in G(y)$ and hence $F(y) \subseteq G(y)$, for all $y \in K$. This implies that G is also a KKM mapping.

Since K is bounded, closed and convex, therefore we deduce that K is weakly compact. From the assumptions, we know that $G(y)$ is weakly closed for all $y \in K$. In fact, since $x \mapsto f(z,x,y)$, g and α are completely continuous, we know that $G(y)$ is weakly closed for all $y \in K$ and so $G(y)$ is weakly compact in K for all $y \in K$. It follows from Theorem 1 and Lemma 1 that

$$\bigcap_{y \in K} F(y) = \bigcap_{y \in K} G(y) \neq \emptyset.$$

This implies that there exist $x \in K$ and $z \in T(x)$ such that

$$f(z,x,y) + g(y) - g(x) \in C, \quad \forall y \in K.$$

This completes the proof. □

We prove the following result for generalized implicit strong vector equilibrium problem (2) in unbounded setting.

Theorem 3. *Let K be an unbounded closed convex subset of a topological vector space X; Y,Z be two other topological vector spaces and $D \subset Z$ is a nonempty closed subset. Suppose that $f : D \times K \times K \longrightarrow Y$, $g : K \longrightarrow Y$ are the single-valued mapping and $T : K \longrightarrow 2^D$ is a closed continuous multi-valued mapping such that all the assumptions (i)-(vii) of Theorem 2 are fulfilled. Additionally, if f satisfied the weakly coercivity condition, i.e., there exists $\tilde{x} \in K$ such that*

$$f(z,x,\tilde{x}) + g(\tilde{x}) - g(x) \in -intC, \tag{4}$$

whenever $x \in K$, $z \in T(x)$ and $\|x\|$ is large enough, then generalized implicit strong vector equilibrium problem (2) has a solution.

Proof. For $\mu > 0$, assume that $K_\mu = \{y \in K : \|y\| \le \mu\}$. Now, consider the problem to find $x_\mu \in K \cap K_\mu$ and $z_\mu \in T(x_\mu)$ such that

$$f(z_\mu, x_\mu, y) + g(y) - g(x_\mu) \in C, \quad \forall y \in K \cap K_\mu. \tag{5}$$

Since K_μ is bounded, therefore by Theorem 2, we can see that problem (5) has at least one solution $x_\mu \in K \cap K_\mu$.

For $\tilde{x}$ in the weakly coercivity condition (4), we take $\|\tilde{x}\| \le \mu'$. From (5), we have

$$f(z_{\mu'}, x_{\mu'}, \tilde{x}) + g(\tilde{x}) - g(x_{\mu'}) \in C. \tag{6}$$

Since $x_{\mu'} \in K_{\mu'}$, we conclude that $\|x_{\mu'}\| \le \mu'$. If $\|x_{\mu'}\| = \mu'$, we may choose μ' large enough so that by the weakly coercivity condition, we have

$$f(z_{\mu'}, x_{\mu'}, \tilde{x}) + g(\tilde{x}) - g(x_{\mu'}) \in -intC,$$

which contradicts (6). Therefore, we must have μ' such that $\|x_{\mu'}\| < \mu'$.

Now, for any $y \in K$, we can choose $0 < \lambda < 1$ small enough such that $\lambda y + (1-\lambda)x_{\mu'} \in K \cap K_{\mu'}$. From (5), we conclude that

$$\begin{aligned} & f(z_{\mu'}, x_{\mu'}, \lambda y + (1-\lambda)x_{\mu'}) + g(\lambda y + (1-\lambda)x_{\mu'}) - \\ & \quad g(x_{\mu'}) \in C \\ \Rightarrow\ & \lambda f(z_{\mu'}, x_{\mu'}, y) + (1-\lambda) f(z_{\mu'}, x_{\mu'}, x_{\mu'}) + \lambda g(y) + \\ & \quad (1-\lambda) g(x_{\mu'}) - g(x_{\mu'}) - C \in C \\ \Rightarrow\ & \lambda f(z_{\mu'}, x_{\mu'}, y) + \lambda g(y) - \lambda g(x_{\mu'}) \in C, \end{aligned}$$

which implies that

$$f(z_{\mu'}, x_{\mu'}, y) + g(y) - g(x_{\mu'}) \in C, \quad \forall y \in K.$$

Therefore, generalized implicit strong vector equilibrium problem (2) has a solution. This completes the proof.

As it is well known, a vector equilibrium result in our context can be reformulated as a generalized vector variational inequality. Let $L(X,Y)$ be the space of all continuous linear operators from X to Y. Assume that $\phi : K \longrightarrow 2^{L(X,Y)}$ is a continuous closed multi-valued mapping with compact values. For $\phi \in L(X,Y)$, we write $\langle \phi, x \rangle := \phi(x)$. In fact, setting $f(z,x,y) = \langle z, y-x \rangle$, for all $z \in \phi(x)$ and $x,y \in K$. Then all the conditions of Theorem 2 are satisfied and we have the following.

Corollary 1. *Let X, Y, K and C be the same as in Theorem 2. Let $\phi : K \longrightarrow 2^{L(X,Y)}$ be a mapping such that*

(i) ϕ is vector relaxed α-C-monotone and 0-diagonally convex;

(ii) ϕ is hemicontinuous, C-convex and for any given $y,z \in K$, the mapping $x \mapsto \langle z, y-x \rangle$ is completely continuous;

(iii) g is affine, hemicontinuous and completely continuous mapping;

(iv) the mapping $\alpha : K \longrightarrow Y$ is completely continuous;

Then there exists $\bar{x} \in K$ and $\bar{z} \in \phi(\bar{x})$ such that

$$\langle \bar{z}, y - \bar{x} \rangle + g(y) - g(\bar{x}) \in C, \ \forall y \in K.$$

On taking $Y = \mathbb{R}$, $C = \mathbb{R}^+$ and the mapping g as zero mapping, vector relaxed α-C-monotone reduced to relaxed α-monotone [12] and consequently, we have the following corollary.

Corollary 2. *Let $\phi : K \longrightarrow 2^{X^*}$ be a mapping such that ϕ is a relaxed α-monotone, convex, hemicontinuous and completely continuous mapping, and $\alpha : X \longrightarrow Y$ is completely continuous. Then, there exists $\bar{x} \in K$ and $\bar{z} \in \phi(\bar{x})$ such that*

$$\langle \bar{z}, y - \bar{x} \rangle \ge 0, \ \forall y \in K.$$

4 Applications

4.1 Application to the vector convex differentiable optimization problem

The vector convex differentiable optimization problem

$$\begin{cases} \min\limits_{C} \psi(x) \\ \text{subject to } x \in K, \end{cases} \tag{7}$$

where $\psi : D \longrightarrow K$ is convex and differentiable and has an optimality criterion. It is well known that a vector $\bar{x}$ solves (7) if and only if it solves the following vector variational inequality problem: Find $\bar{x} \in C$ such that

$$\langle \nabla \psi(\bar{x}), y - \bar{x} \rangle \in C, \quad \forall y \in K. \tag{8}$$

By putting $f(\bar{z}, \bar{x}, y) = \langle \nabla \psi(\bar{z}), y - \bar{x} \rangle$ in (2), we see that the generalized implicit strong vector equilibrium problem (2) reduces to the following mixed vector variational inequality problem

$$\langle \nabla \psi(\bar{z}), y - \bar{x} \rangle + g(y) - \phi(\bar{x}) \in C, \ \forall y \in K, \qquad (9)$$

which can be viewed as analogous to the vector convex differentiable optimization problem.

We can derive the following existence result vector convex differentiable optimization problem from Theorem 2.

Theorem 4. *Let $\psi : D \longrightarrow K$ be a C-convex and differentiable mapping, $g : K \longrightarrow Y$ a single-valued mapping and $T : K \longrightarrow 2^D$ a closed compact continuous multi-valued mapping. Assume that*

(ii) *ψ is vector relaxed α-C-monotone and 0-diagonally convex;*
(iii) *ψ is hemicontinuous and completely continuous;*
(vi) *g is affine, hemicontinuous and completely continuous mapping;*
(vii) *the mapping $\alpha : X \longrightarrow Y$ is completely continuous;*

Then, there exists $\bar{x} \in K$ and $\bar{z} \in T(\bar{x})$ such that

$$\langle \nabla \psi(\bar{z}), y - \bar{x} \rangle + g(y) - \phi(\bar{x}) \in C, \ \forall y \in K.$$

4.2 Application to the convex minimization problem

If f is a single-valued mapping, $g \equiv 0$, and for all $x, y \in K$, the mapping $z \mapsto f(z, x, y)$ is constant in (2), then we have the following strong vector equilibrium problem of finding $\bar{x} \in K$ such that

$$f(\bar{x}, y) \in C, \ \forall y \in K. \qquad (10)$$

If we take $f(x, y) = \phi(y) - \phi(x)$ and $Y = \mathbb{R}$ in (10), we have the following convex minimization problem which is to find $\bar{x} \in K$ such that

$$\phi(y) \geq \phi(\bar{x}), \ \forall y \in K. \qquad (11)$$

Now, we have the following existence result for convex minimization problem (11), which is a special case of Theorem 2.

Theorem 5. *Let $\phi : K \longrightarrow \mathbb{R}$ be a convex, relaxed α-monotone, hemicontinuous and completely continuous mapping, and $\alpha : X \longrightarrow Y$ is completely continuous. Then, there exists $\bar{x} \in K$ such that*

$$\phi(y) \geq \phi(\bar{x}), \ \forall y \in K.$$

5 Conclusion

The motivation of this paper is to introduce the concept of vector relaxed α-C-monotonicity supported by an example. This new notion of vector relaxed α-C-monotonicity is more general than the concept of various generalized monotonicities considered by many authors, see e.g., [5,6,12]. Using this concept, generalized vector 0-diagonally convexity for tri-mappings and some other concepts, we solve a generalized implicit strong vector equilibrium problem. Two existence results are proved for this problem, one in bounded setting and other in unbounded setting. Some applications of our problem is also discussed. It is also ensured that our results are more general by two corollaries. The improvement and development of our results need further research endeavour and we hope our considered problem may stimulate future research and applications in this ongoing field.

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Mijanur Rahaman received his Ph.D degree from Aligarh Muslim University in 2016. Currently, he is guest faculty lecturer in the Department of Mathematics at Aligarh Muslim University, Aligarh, India. His research interests are in the areas of nonlinear analysis, variational inequalities and optimization.

Rais Ahmad received his Ph.D degree in Mathematics at Aligarh Muslim University, Aligarh, India. He is a full Professor in the Department of Mathematics of Aligarh Muslim University, India. His research interests are nonlinear functional analysis and optimization, equilibrium problems, complementarity and fixed point problems. He has visited a number of countries for research purposes. He has published more than 100 research articles in various journals of international repute.