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Estimation in Step-Stress Partially Accelerated Life Tests for the Chen Distribution Using Progressive Type-II Censoring

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Abstract: In this paper, step-stress partially accelerated life tests (SS-PALT) are considered when the lifetime of a product follows a two-parameter bathtub-shaped lifetime distribution (Chen distribution). Based on progressive Type-II censoring, the maximum likelihood estimates (MLEs) are obtained for the distribution parameters and acceleration factor. In addition, asymptotic variance and covariance matrix of the estimators are given. Approximate confidence intervals (CIs) for the parameters based on normal approximation to the asymptotic distribution of the MLEs and the bootstrap (CIs) are derived. An iterative procedure is used to obtain the estimators numerically using (Mathematica Package). A numerical example is presented to illustrate the method of estimation developed here. Finally, a Monte Carlo simulation study is performed to investigate the precision of the MLEs and to compare the CIs considered.

Keywords: Chen distribution; Step-stress partially accelerated life tests; Progressive Type-II censoring; Maximum likelihood estimation; Asymptotic confidence intervals; Bootstrap confidence intervals

1 Introduction

Life tests are usually conducted to assess the reliability of products in many industrial production processes. In industrial experiments, products that are tested are often extremely reliable with large mean times to failure under normal operating conditions. However, for some high reliability products which are designed to operate without failures for an extended period of time, few units would fail at normal condition even censoring schemes are employed. Consequently, with conventional life-testing experiments under Type-II censoring, it is almost impossible to obtain adequate information about the failure time distribution and its associated parameters. To overcome these problems, the experimenter may resort to accelerated life testing (ALT) where in the units are subjected to higher stress levels than normal. The data collected from such an accelerated test may then be extrapolated to estimate the underlying distribution of failure times under design (use) conditions which is non-accelerated. Thus, ALT are widely used to collect information for the assessment of reliability of the tested products. In ALT, test items are run only at accelerated conditions, while in partially accelerated life tests (PALT), they are run at both normal and accelerated conditions. The major assumption in ALT is that the mathematical model relating the lifetime of the unit and the stress are known or can be assumed. In some cases, such life–stress relationships are not known and cannot be assumed, i.e ALT data cannot be extrapolated to use condition. So, in such cases, partially accelerated life test (PALT) is a more suitable test to be performed for which tested units are subjected to both normal and accelerated conditions.

According to Nelson [1] there are mainly three ALT methods. The first method is called the constant-stress ALT, the stress is kept at a constant level throughout the life of test products See [2,3] . The second one is referred to as progressive-stress ALT, the stress applied to a test product is continuously increasing in time [4,5]. The third

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is the step-stress ALT, in which the test condition changes at a given time or upon the occurrence of a specified number of failures, this type has been studied by several authors. In general, the problem of modeling data from ALT and making inference from such data have been studied by many authors. See, for example [6,7].

Models with bathtub shaped or increasing failure rate function (FRF) are useful in reliability analysis and particularly in reliability related decision making and cost analysis. The bathtub shape provides an appropriate conceptual model for the hazard of some electronic and mechanical products. In recent years, some lifetime distributions with bathtub-shaped hazard function have been investigated by several authors. See for example, Xie and Lai [8], Xie et al. [9] and Soliman et al. [10]. In this article, we focus on a two-parameter distribution with the bathtub shape or increasing FRF proposed by Chen [11]. This lifetime distribution has bathtub-shaped FRF if $\alpha < 1$; increasing FRF if $\alpha \geq 1$ and this distribution becomes the exponential power distribution if $\beta = 1$. Wu et al. [12] proposed the optimal estimation of the parameters of this lifetime distribution based on the doubly Type-II censored sample. Let random variable T have a Chen distribution (CD) with parameter β and α, where β is the scale parameter and α is the shape parameter. The probability density function (pdf), cumulative distribution function (cdf), reliability function $S(t)$, and hazard rate function $h(t)$ given by

$$f_1(t) = \alpha\beta t^{\alpha-1}\exp(t^{\alpha}+\beta\left[1-\exp(t^{\alpha})\right]), t>0,\ \alpha,\beta>0, \quad (1)$$

$$F_1(t)=1-\exp(\beta\left[1-\exp(t^{\alpha})\right]), \quad (2)$$

$$S_1(t) = \exp(\beta\left[1-\exp(t^{\alpha})\right]), \quad (3)$$

and

$$h_1(t) = \alpha\beta t^{\alpha-1}\exp\left(t^{\alpha}\right). \quad (4)$$

The CD with known shape parameter has been considered in literature and applied in practice. Based on the progressive Type II censoring scheme, Ahmadi et al [13] used the max p-value method to select the optimum value of the shape parameter of the Weibull distribution and hence supposed that shape parameter is known. They constructed the ML estimator for and developed a testing procedure for the lifetime performance index of the products with CD on the basis of the progressive Type II censored sample with max p-value method

Censoring is very common in life tests. There are several types of censored tests. The most common censoring schemes are Type-I (time) censoring and Type-II (failure) censoring. However, the conventional Type-I and Type-II censoring schemes do not have the flexibility of allowing removal of units at points other than the terminal point of the experiment. Because of this lack of flexibility, a more general censoring scheme called progressive Type-II right censoring, for extensive reviews of the literature on progressive censoring see Balakrishnan and Aggarwala [14]. Suppose n units are placed on a life testing experiment and let T_1, T_2, ..., T_n be their corresponding lifetimes. We assume that T_i, $i = 1, 2$, ..., n are independent and identically distributed with pdf $f(t)$ and cdf $F(t)$. With progressively Type II censoring, n units are placed on test. Consider that $T_{1;m,n} < T_{2;m,n} <$ ... $< T_{m;m,n}$ is the corresponding progressively Type II censored sample, with censoring scheme $\mathbf{R} = (R_1, R_2, ..., R_m)$. Since the joint pdf of $T_{1;m,n} < T_{2;m,n}$ $<$ ... $< T_{m;m,n}$ is given by

$$f_{1,2,\ldots,m}(t_{1;m,n},t_{2;m,n},\ldots,t_{m;m,n}) = A\prod_{i=1}^{m} f(t_{i;m,n}) \left[S(t_{i;m,n})\right]^{Ri}, \quad (5)$$

$$0 < t_{1;m,n} < t_{2;m,n} < \ldots < t_{m;m,n} < \infty,$$

where

$$A = \prod_{i=1}^{m}\left(n-\sum_{j=1}^{i-1}R_j - i + 1\right). \quad (6)$$

ALTs are preferred to be used in manufacturing industries to obtain enough failure data, in a short period of time, necessary to make inferences regarding its relationship with external stress variables. In ALTs, the test items are tested only at accelerated conditions. According to Nelson [1] there are mainly three ALT methods. The first method is called the constant-stress ALT, the stress is kept at a constant level throughout the life of test products See [9,10]. The second one is referred to as progressive-stress ALT, the stress applied to a test product is continuously increasing in time [4,5]. The third is the step-stress ALT, in which the test condition changes at a given time or upon the occurrence of a specified number of failures, this type has been studied by several authors. Several authors have dealt with this type of ALT, including [6,7].

2 Model Description and Basic Assumptions

In SS-PALT, all of the n units are tested first under normal condition, if the unit does not fail for a prespecified time , then it runs at accelerated condition until failure. This means that if the item has not failed by some prespecified time , the test is switched to the higher level of stress and it is continued until items fails. The effect of this switch is to multiply the remaining lifetime of the item by the inverse of the acceleration factor . In this case the switching to the higher stress level will shorten the life of test item. Thus the total lifetime of a test item, denoted by Y, passes through two stages, which are the normal and accelerated conditions. Then the lifetime of the unit in SS-PALT is given as follows

$$Y = \begin{cases} T, & T < \tau^* \\ \tau^* + \lambda^{-1}(T-\tau^*), & T > \tau^*, \end{cases} \quad (7)$$

where T, is the lifetime of an item at use condition, τ^* is the stress change time and λ is the acceleration factor which is the ratio of mean life at use condition to that at accelerated condition, usually $\lambda > 1$. Assume that the lifetime of the test item follows CD with parameters α and β. Therefore, the probability density function of total lifetime Y of an item is given by

$$f(y) = \begin{cases} 0, & y < 0 \\ f_1(y), & 0 < y \le \tau^* \\ f_2(y), & y > \tau^*, \end{cases} \tag{8}$$

where $f_1(y)$, is given by (1) and,

$$f_2(y) = \alpha\beta\lambda(\tau^* + \lambda(y-\tau^*))^{\alpha-1} \\ \exp\left\{(\tau^* + \lambda(y-\tau^*))^{\alpha} + \beta 1 - \exp(\tau^* + \lambda(y-\tau^*))^{\alpha}\right\} \tag{9}$$

is obtained by the transformation variable technique using equations (1) and (7). cdf , $S_2(t)$, and $h_2(t)$ of is given by

$$F_2(x)=1-\exp(\beta\,[1-\exp((\tau^*+\lambda(y-\tau^*))^{\alpha})]), \tag{10}$$

$$S_2(t) = \exp(\beta\,[1-\exp((\tau^*+\lambda(y-\tau^*))^{\alpha})]), \tag{11}$$

and

$$h_2(t) = \alpha\beta t^{\alpha-1}\exp\left((\tau^*+\lambda(y-\tau^*))^{\alpha}\right). \tag{12}$$

In progreessive Type II censoring the test terminates when the number of observations is reached to $m < n$. The observed values of the total lifetime Y are : $y_1 < y_2 < ... < y_J < \tau^* < y_{J+1} < ... < y_m$ where J are the number of items failed at normal conditions and $m-J$ at accelerated conditions. Let us define the two indicator functionsalign

$$\delta_{1i} = \begin{cases} 1, \, y_i \le \tau^* \\ 0, \text{ o.w.} \end{cases}, \delta_{2i} = \begin{cases} 1, \, y_i > \tau^* \\ 0, \text{ o.w.} \end{cases}, \quad i = 1, 2, ..., m \tag{13}$$

For the lifetimes $y_1 < y_2 < ... < y_m$ of m items are independent and identically distributed random variables, then from (5) and (13) the likelihood function is given by

$$L(\alpha,\beta,\lambda|\underline{y}) = A\prod_{i=1}^{m}\left[f_1(y_i)[S_1(y_i)]^{Ri}\right]^{\delta_{1i}}\left[f_2(y_i)[S_2(y_i)]^{Ri}\right]^{\delta_{2i}} \\ 0 < y_1 < y_2 < ... < y_J < \tau^* < y_{J+1} < ... < y_m < \infty, \tag{14}$$

where A given in (6)

3 Maximum Likelihood Estimation

3.1 MLEs

From two populations whose cdfs and pdfs given in (1), (2) (9) and (10), with $\mathbf{R} = (R_1, R_2, ..., R_m)$. the likelihood function $L(\alpha,\beta,\lambda|\underline{y})$ in (14) without normalized constant is then given by

$$L(\alpha,\beta,\lambda|\underline{y}) \propto (\alpha\beta)^m\lambda^{m-J}\exp\left\{(\alpha-1)\sum_{i=1}^{J}\log y_i \right. \\ +(\alpha-1)\sum_{i=1}^{J}\log y_i + \beta\sum_{i=1}^{J}(R_i+1)(1-\exp(y_i^{\alpha})) + (\alpha-1) \\ \times\sum_{i=J+1}^{m}\log[\tau^* + \lambda(y_i-\tau^*)] + \sum_{i=J+1}^{m}(\tau^*+\lambda(y_i-\tau^*))^{\alpha} \\ +\beta\sum_{i=J+1}^{m}(R_i+1)\left(1-\exp(\tau^*+\lambda(y_i-\tau^*))^{\alpha}\right). \tag{15}$$

Then the log-likelihood function of $L(\alpha,\beta,\lambda|\underline{y})$ given by

$$\ell(\alpha,\beta,\lambda|\underline{y}) = m\log\alpha\beta + (m-J)\log\lambda + \sum_{i=1}^{J}y_i^{\alpha} \\ +(\alpha-1)\sum_{i=1}^{J}\log y_i + \beta\sum_{i=1}^{J}(R_i+1)(1-\exp(y_i^{\alpha})) + (\alpha-1) \\ \times\sum_{i=J+1}^{m}\log[\tau^* + \lambda(y_i-\tau^*)] + \sum_{i=J+1}^{m}(\tau^*+\lambda(y_i-\tau^*))^{\alpha} \\ +\beta\sum_{i=J+1}^{m}(R_i+1)\left(1-\exp(\tau^*+\lambda(y_i-\tau^*))^{\alpha}\right). \tag{16}$$

Calculating the first partial derivatives of (16) with respect to α, β and λ and equating each to zero, we get the likelihood equations as

$$\frac{\partial\ell(\alpha,\beta,\lambda|\underline{y})}{\partial\alpha} = \frac{m}{\alpha} + \sum_{i=1}^{J}\log y_i + \sum_{i=1}^{J}y_i^{\alpha}\log y_i - \beta\sum_{i=1}^{J}(R_i+1)y_i^{\alpha} \\ \times\log y_i\exp(y_i^{\alpha}) + \sum_{i=J+1}^{m}\log[\tau^*+\lambda(y_i-\tau^*)] \\ + \sum_{i=J+1}^{m}[\tau^*+\lambda(y_i-\tau^*)]^{\alpha}\log[\tau^*+\lambda(y_i-\tau^*)] \\ -\beta\sum_{i=J+1}^{m}(R_i+1)[\tau^*+\lambda(y_i-\tau^*)]^{\alpha} \\ \times\log[\tau^*+\lambda(y_i-\tau^*)]\exp([\tau^*+\lambda(y_i-\tau^*)]^{\alpha}, \tag{17}$$

$$\frac{\partial\ell(\alpha,\beta,\lambda|\underline{y})}{\partial\beta} = \frac{m}{\beta} + \sum_{i=1}^{J}(R_i+1)(1-\exp(y_i^{\alpha})) \\ + \sum_{i=J+1}^{m}(R_i+1)\left(1-\exp(\tau^*+\lambda(y_i-\tau^*))^{\alpha}\right) = 0, \tag{18}$$

and

$$\frac{\partial\ell(\alpha,\beta,\lambda|\underline{t})}{\partial\lambda} = \frac{m-J}{\lambda} + (\alpha-1)\sum_{i=1}^{J}\frac{(y_i-\tau^*)}{\tau^*+\lambda(y_i-\tau^*)} \\ +\alpha\sum_{i=1}^{J}(y_i-\tau^*)(\tau^*+\lambda(y_i-\tau^*))^{\alpha-1} - \alpha\beta\sum_{i=J+1}^{m}(R_i+1) \\ \times(y_i-\tau^*)(\tau^*+\lambda(y_i-\tau^*))^{\alpha-1}\exp\left((\tau^*+\lambda(y_i-\tau^*))^{\alpha}\right) \\ = 0. \tag{19}$$

From (18), we can write

$$\beta(\alpha,\lambda)=\frac{m}{D} \tag{20}$$

where

$$D=\sum_{i=1}^{J}(R_i+1)\left(\exp(y_i^{\alpha})-1\right)+\sum_{i=J+1}^{m}(R_i+1)\exp\left(\tau^*+\lambda(y_i-\tau^*)^{\alpha}-1\right). \tag{21}$$

By substituting (20) in (17) and (19) we have

$$\frac{m}{\alpha}+\sum_{i=1}^{J}\log y_i+\sum_{i=1}^{J}y_i^{\alpha}\log y_i-\frac{m}{D}\sum_{i=1}^{J}(R_i+1)y_i^{\alpha}\log y_i\exp(y_i^{\alpha})+\sum_{i=J+1}^{m}\log\left[\tau^*+\lambda(y_i-\tau^*)\right]+\sum_{i=J+1}^{m}\left(\tau^*+\lambda(y_i-\tau^*)\right)^{\alpha}\times\log\left[\tau^*+\lambda(y_i-\tau^*)\right]-\frac{m}{D}\sum_{i=1}^{m}(R_i+1)(\tau^*+\lambda(y_i-\tau^*))^{\alpha}\times\log\left[\tau^*+\lambda(y_i-\tau^*)\right]\exp\left(\tau^*+\lambda(y_i-\tau^*)\right)^{\alpha}=0, \tag{22}$$

and

$$\frac{m-j}{\lambda}+(\alpha-1)\sum_{i=J+1}^{m}\frac{(y_i-\tau^*)}{\tau^*+\lambda(y_i-\tau^*)}+\alpha\sum_{i=J+1}^{m}(y_i-\tau^*)\times(\tau^*+\lambda(y_i-\tau^*))^{\alpha-1}-\frac{m\alpha}{D}\sum_{i=J+1}^{m}(R_i+1)(y_i-\tau^*)\times(\tau^*+\lambda(y_i-\tau^*))^{\alpha-1}\exp\left((\tau^*+\lambda(y_i-\tau^*))^{\alpha}\right)=0. \tag{23}$$

Thus, likelihoods equations are reduced to a two nonlinear equation (22) and (23) which could be solved numerically with respect to α and λ using any iteration procedure such as quasi-Newton Raphson, to get the MLE, $\hat{\alpha}$ and $\hat{\lambda}$, and hence $\hat{\beta}$ by using (20).

3.2 Approximate interval estimation

From the log-likelihood function in (16), we have

$$\frac{\partial^2\ell(\alpha,\beta,\lambda|\underline{y})}{\partial\alpha^2}=\frac{-m}{\alpha^2}+\sum_{i=1}^{J}y_i^{\alpha}(\log y_i)^2+\sum_{i=J+1}^{m}\left[\tau^*+\lambda(y_i-\tau^*)\right]^{\alpha}\left(\log\left[\tau^*+\lambda(y_i-\tau^*)\right]\right)^2-\beta\sum_{i=1}^{J}(R_i+1)y_i^{\alpha}(1+y_i^{\alpha})(\log y_i)^2\exp(y_i^{\alpha})-\beta\sum_{i=J+1}^{m}(R_i+1)\left(\log\left[\tau^*+\lambda(y_i-\tau^*)\right]\right)^2\times\left[\tau^*+\lambda(y_i-\tau^*)\right]^{\alpha}\left(1+\left[\tau^*+\lambda(y_i-\tau^*)\right]^{\alpha}\right)\times\exp(\left[\tau^*+\lambda(y_i-\tau^*)\right]^{\alpha}) \tag{24}$$

$$\frac{\partial^2\ell(\alpha,\beta,\lambda|\underline{y})}{\partial\beta^2}=-\frac{m}{\beta^2} \tag{25}$$

$$\frac{\partial^2\ell(\alpha,\beta,\lambda|\underline{y})}{\partial\lambda^2}=-\alpha(\alpha-1)\beta\sum_{i=J+1}^{m}(R_i+1)(y_i-\tau^*)^2+\alpha(\alpha-1)\sum_{i=J+1}^{m}(y_i-\tau^*)^2\left[\tau^*+\lambda(y_i-\tau^*)\right]^{\alpha-2}-\frac{(m-J)}{\lambda^2}-(\alpha-1)\sum_{i=J+1}^{m}\frac{(y_i-\tau^*)^2}{\left[\tau^*+\lambda(y_i-\tau^*)\right]^2}\times\left[\tau^*+\lambda(y_i-\tau^*)\right]^{\alpha-2}\exp\left(\left[\tau^*+\lambda(y_i-\tau^*)\right]^{\alpha}\right)-\alpha^2\beta\sum_{i=J+1}^{m}(R_i+1)(y_i-\tau^*)^2\left[\tau^*+\lambda(y_i-\tau^*)\right]^{2(\alpha-1)}\times\exp\left(\left[\tau^*+\lambda(y_i-\tau^*)\right]^{\alpha}\right) \tag{26}$$

$$\frac{\partial^2\ell(\alpha,\beta,\lambda|\underline{y})}{\partial\alpha\partial\beta}=\frac{\partial^2\ell(\alpha,\beta,\lambda|\underline{y})}{\partial\beta\partial\alpha}=-\sum_{i=1}^{J}(R_i+1)y_i^{\alpha}\times\log y_i\exp(y_i^{\alpha})-\sum_{i=J+1}^{m}(R_i+1)(\tau^*+\lambda(y_i-\tau^*))^{\alpha}\times\log\left[\tau^*+\lambda(y_i-\tau^*)\right]\exp\left(\tau^*+\lambda(y_i-\tau^*)\right)^{\alpha}, \tag{27}$$

$$\frac{\partial^2\ell(\alpha,\beta,\lambda|\underline{y})}{\partial\alpha\partial\lambda}=\frac{\partial^2\ell(\alpha,\beta,\lambda|\underline{y})}{\partial\lambda\partial\alpha}=\sum_{i=J+1}^{m}\frac{(y_i-\tau^*)}{(\tau^*+\lambda(y_i-\tau^*))^2}+\alpha\sum_{i=J+1}^{m}(y_i-\tau^*)(\tau^*+\lambda(y_i-\tau^*))^{\alpha-1}\log\left[\tau^*+\lambda(y_i-\tau^*)\right]+\sum_{i=J+1}^{m}(y_i-\tau^*)(\tau^*+\lambda(y_i-\tau^*))^{\alpha-1}-\alpha\beta\sum_{i=J+1}^{m}(R_i+1)\times(y_i-\tau^*)(\tau^*+\lambda(y_i-\tau^*))^{\alpha-1}\log\left[\tau^*+\lambda(y_i-\tau^*)\right]\times\exp\left(\tau^*+\lambda(y_i-\tau^*)\right)^{\alpha}-\beta\sum_{i=J+1}^{m}(R_i+1)(y_i-\tau^*)\times(\tau^*+\lambda(y_i-\tau^*))^{\alpha-1}\exp\left(\tau^*+\lambda(y_i-\tau^*)\right)^{\alpha}-\alpha\beta\sum_{i=J+1}^{m}(R_i+1)(y_i-\tau^*)(\tau^*+\lambda(y_i-\tau^*))^{2\alpha-1}\times\log\left[\tau^*+\lambda(y_i-\tau^*)\right]\times\exp\left(\tau^*+\lambda(y_i-\tau^*)\right)^{\alpha}, \tag{28}$$

and

$$\frac{\partial^2\ell(\alpha,\beta,\lambda|\underline{y})}{\partial\beta\partial\lambda}=\frac{\partial^2\ell(\alpha,\beta,\lambda|\underline{y})}{\partial\lambda\partial\beta}=-\alpha\sum_{i=J+1}^{m}(R_i+1)\times(y_i-\tau^*)\left[\tau^*+\lambda(y_i-\tau^*)\right]^{\alpha-1}\times\exp(\left[\tau^*+\lambda(y_i-\tau^*)\right]^{\alpha}. \tag{29}$$

The observed Fisher information matrix $I(\alpha,\beta,\lambda)$, for the MLEs ($\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$), see [1], is the 3×3 symmetric matrix of negative second partial derivatives of the log-likelihood function with respect to (α, β and λ). In practice , we usually estimate $I^{-1}(\alpha,\beta,\lambda)$ by $I_0^{-1}(\hat{\alpha},\hat{\beta},\hat{\lambda})$, where

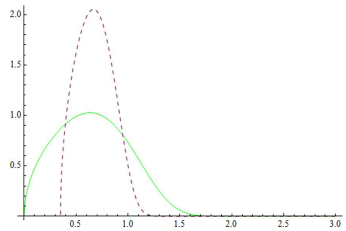

Fig. 1: The probability density function under normal and accelerate condition.

$$I_0(\alpha,\beta,\lambda) = -\begin{bmatrix} \frac{\partial^2\ell(\alpha,\beta,\lambda|\underline{t})}{\partial\alpha 2} & \frac{\partial^2\ell(\alpha,\beta,\lambda|\underline{t})}{\partial\alpha\partial\beta} & \frac{\partial^2\ell(\alpha,\beta,\lambda|\underline{t})}{\partial\alpha\partial\lambda} \\ \frac{\partial^2\ell(\alpha,\beta,\lambda|\underline{t})}{\partial\beta\partial\alpha} & \frac{\partial^2\ell(\alpha,\beta,\lambda|\underline{t})}{\partial\beta 2} & \frac{\partial^2\ell(\alpha,\beta,\lambda|\underline{t})}{\partial\beta\partial\lambda} \\ \frac{\partial^2\ell(\alpha,\beta,\lambda|\underline{t})}{\partial\lambda\partial\alpha} & \frac{\partial^2\ell(\alpha,\beta,\lambda|\underline{t})}{\partial\lambda\partial\beta} & \frac{\partial^2\ell(\alpha,\beta,\lambda|\underline{t})}{\partial\lambda 2} \end{bmatrix}$$

$$\text{at } (\alpha,\beta,\lambda) = (\hat{\alpha},\hat{\beta},\hat{\lambda}). \quad (30)$$

The observed Fisher information matrix enables us to construct confidence intervals for the parameters based on limiting normal distribution. Thus, the 100(1-γ)% approximate confidence intervals for α,β and λ are

$$\hat{\alpha} \mp z_{\frac{\gamma}{2}}\sqrt{v_{11}},\ \hat{\beta} \mp z_{\frac{\gamma}{2}}\sqrt{v_{22}} \text{ and } \hat{\lambda} \mp z_{\frac{\gamma}{2}}\sqrt{v_{33}} \text{ and} \quad (31)$$

respectively, where v_{11}, v_{22} and v_{33} are the elements on the main diagonal of the covariance matrix $I^{-1}(\hat{\alpha},\hat{\beta},\hat{\lambda})$ and $z_{\frac{\alpha}{2}}$ is the percentile of the standard normal distribution with right-tail probability $\frac{\gamma}{2}$.

4 Bootstrap Confidence Intervals

The bootstrap is a resampling method for statistical inference. It is commonly used to estimate confidence intervals, but it can also be used to estimate bias and variance of an estimator or calibrate hypothesis tests. Mor survey of the nonparametric and parametric bootstrap methods Davison and Hinkley [15], Efron and Tibshirani [16]. In this section, the two confidence intervals based on the parametric bootstrap methods are proposed: (i) percentile bootstrap method Efron [17], (ii) bootstrap-*t* method Hall, [18]. The algorithms for estimating the confidence intervals of parameters using both methods are illustrated below.

1. Based on the original progressively Type-II sample, $\underline{y} = (y_1 < y_2 < ... < y_J < \tau^* < y_{J+1} < ... < y_m)$, obtain $\hat{\alpha}$ and $\hat{\lambda}$ from (21) and (22) and hence $\hat{\beta}$ from (20).
2. Based on $\hat{\alpha}$ and $\hat{\beta}$ and the values of n and m and τ^* with the same values of $\mathbf{R}, (i = 1,2,...,m_j)$, generate $\underline{t}^* = (t_1^* < t_2^* < ... < t_m^*)$ using the algorithm described in Balakrishnan and Sandhu [18], and hence $\hat{\lambda}$ in (7) the sample obtained $\underline{y}^* = (y_1^* < y_2^* < ... < y_J^* < \tau^* < y_{J+1}^* < ... < y_m^*)$.
3. As in step 1 based on $\underline{y}^*$ compute the bootstrap sample estimates of $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\lambda}$ say $\hat{\alpha}^*$, $\hat{\beta}^*$ and $\hat{\lambda}^*$.
4. Repeat the above steps 2 and 3 N times representing N different bootstrap samples. The value of N has been taken to be 1000.
5. Arrange all $\hat{\alpha}^*$, $\hat{\beta}^*$ and $\hat{\lambda}^*$ in an ascending order to obtain the bootstrap sample $(\hat{\varphi}_k^{*[1]}, \hat{\varphi}_k^{*[2]}, ..., \hat{\varphi}_k^{*[N]}), k = 1,2,3$ where $(\varphi_1^* = \alpha^*, \varphi_2^* = \beta^*, \varphi_3^* = \lambda^*)$.

Percentile bootstrap confidence interval:

Let $G(z) = P(\hat{\varphi}_k^* \leqslant z)$ be cumulative distribution function of $\hat{\varphi}_k^*$. Define $\hat{\varphi}_{kboot}^* = G^{-1}(z)$ for given z. The approximate bootstrap $100(1-\gamma)\%$ confidence interval of $\hat{\varphi}_k^*$ given by

$$\left[\hat{\varphi}_{kboot}^*(\frac{\gamma}{2}), \hat{\varphi}_{kboot}^*(1-\frac{\gamma}{2})\right]. \quad (32)$$

Bootstrap-t confidence interval

First, find the order statistics $\delta_k^{*[1]} < \delta_k^{*[2]} < ... < \delta_k^{*[N]}$, where

$$\delta_k^{*[j]} = \frac{\hat{\varphi}_k^{*[j]} - \hat{\varphi}_k}{\sqrt{\operatorname{var}\left(\hat{\varphi}_k^{*[j]}\right)}},\ j = 1,2,...,N,\ k = 1,2,3, \quad (33)$$

where $\hat{\varphi}_1 = \hat{\alpha}$, $\hat{\varphi}_2 = \hat{\beta}$, $\hat{\varphi}_3 = \hat{\lambda}$.

Let $H(z) = P(\delta_k^* < z)$ be the cumulative distribution function of δ_k^*. For a given z, define

$$\hat{\varphi}_{\text{kboot}-t} = \hat{\varphi}_k + \sqrt{\operatorname{Var}(\hat{\varphi}_k)}H^{-1}(z). \quad (34)$$

The approximate $100(1-\gamma)\%$ confidence interval of $\hat{\varphi}_k$ is given by $\left(\hat{\varphi}_{\text{kboot}-t}(\frac{\gamma}{2}), \hat{\varphi}_{\text{kboot}-t}(1-\frac{\gamma}{2})\right)$.

5 Illustrative Example

In this section, we present an example to illustrate the estimation procedure of MLE and the two considered bootstrap CIs methods for the parameters α, β and λ. In this example, we simulate a samples of size ($m = 30$ from $n = 50$) are generated from Chen distribution with (α, β, λ) = (1.5, 1.0, 2) and censoring scheme (CS) R ={2, 0, 0, 3, 0, 0, 1, 0, 0, 3, 2, 0, 0, 2, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 2, 0, 0, 2, 0} by using the algorithm described in Balakrishnan and Sandhu [18] and using tranformation (8) with $\tau^* = 0.7$. The data are presented in Table 1 below. Fig 1 show the probability density functions under normal conditions and accelarate coinditions. We can use any iteration procedure such as quasi-Newton Raphson to solve the two non-linear equation in (22) and (23). The point estimates of the parameters as well as (95%) approximate confidence intervales are presented in Table 2. Also the point estimates and relate of the parameters as well as (95%) Percentile bootstrap (BPCIs) and bootstrap-t (BTCIs) confidence intervales are presented also in Table 2.We, observed that the BTCIs and ACIs intervals are narrower than the PBCIs and always include the population parameters values.

Table 1: Simulated progressively censored samples with SS-PALTs.

0.253584	0.260912	0.302932	0.30793	0.35838	0.360785	0.416776	0.418423	0.43369	0.441461
0.461691	0.484631	0.510462	0.552298	0.565323	0.600714	0.607132	0.643923	0.651039	0.719479
0.736579	0.785256	0.794652	0.795427	0.803143	0.833382	0.840279	0.845124	0.877254	0.978564

Table 2: MLEs, bootstrap and (95%) approximate confidence intervales and length

Pa.s	$(.)_{ML}$	$(.)_{Boot}$	95%(ACI)	length	95%(BPCI)	length	95%(BPCI)	length
$\alpha=1.5$	2.3171	2.7178	(1.4423, 3.1919)	1.7496	(1.2420, 4.3371)	3.0951	(1.0429, 3.7772)	2.7313
$\beta=1.0$	1.0667	1.2234	(0.4396, 1.6938)	1.2542	(0.4541, 2.9882)	2.5341	(0.3691, 1.9241)	1.5550
$\lambda=2$	1.5534	1.7021	(0.2484, 2.8584)	2.6099	(0.1289, 4.9321)	4.8032	(0.3325, 2.9998)	2.6673

Table 3: MLEs and MSEs for the parameters (α,β,λ) at (0.8, 1.2, 2.5).

τ^*			MLE						Boot					
0.4	(n,m)	CS	AVG			MSE			AVG			MSE		
			$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
	(30,20)	*I*	0.871	1.437	2.611	0.411	0.810	1.274	1.095	1.830	2.211	0.931	1.213	1.472
		II	0.863	1.442	2.561	0.312	0.691	0.908	0.899	1.502	2.411	0.495	0.856	1.008
		III	0.891	1.346	2.633	0.384	0.731	0.999	0.993	1.496	2.213	0.722	0.999	1.325
		IV	0.896	1.221	2.617	0.378	0.776	1.002	0.990	1.524	2.011	0.569	0.987	0.996
	(50,30)	*I*	0.842	1.452	2.665	0.310	0.621	0.872	0.954	1.732	2.311	0.701	0.800	0.954
		II	0.823	1.243	2.513	0.100	0.321	0.601	0.865	1.425	2.200	0.401	0.550	0.751
		III	0.834	1.314	2.421	0.230	0.444	0.741	0.964	1.477	2.223	0.522	0.562	0.902
		V	0.845	1.325	2.600	0.330	0.452	0.720	0.983	1.472	2.621	0.622	0.524	0.801
0.8	(30,20)	*I*	0.864	1.399	2.321	0.358	0.785	0.999	0.987	1.547	2.313	0.839	1.000	1.112
		II	0.834	1.295	2.430	0.300	0.599	0.801	0.812	1.533	2.477	0.408	0.754	0.908
		III	0.855	1.347	2.333	0.399	0.621	0.947	0.914	1.591	2.313	0.777	0.823	0.999
		IV	0.854	1.330	2.601	0.377	0.699	0.955	0.845	1.533	2.213	0.533	0.725	0.965
	(50,30)	*I*	0.833	1.312	2.607	0.299	0.555	0.772	0.974	1.533	2.414	0.623	0.654	0.752
		II	0.813	1.202	2.483	0.108	0.300	0.524	0.815	1.412	2.445	0.371	0.452	0.654
		III	0.834	1.311	2.431	0.201	0.480	0.642	0.903	1.408	2.277	0.466	0.535	0.802
		V	0.842	1.337	2.613	0.300	0.423	0.666	0.922	1.431	2.423	0.472	0.501	0.833

6 Simulation Studies

Simulation studies have been performed to illustrating the theoretical results of estimation problem. The performance of the resulting estimators of the acceleration, shape and scale parameters has been considered in terms of their average (AVG) and mean square error (MSE), where

$$\overline{\hat{\varphi}_k}=\frac{1}{M}\sum_{i=1}^{M}\hat{\varphi}_k^{(i)},(\varphi_1=\alpha,\varphi_2=\beta,\varphi_3=\lambda) \qquad (35)$$

,and

$$\text{MSE}=\frac{1}{M}\sum_{i=1}^{M}\left(\hat{\varphi}_k^{(i)}-\varphi_k\right)^2. \qquad (36)$$

We also compare different confidence intervals, namely the confidence intervals obtained by using asymptotic distributions of the MLEs and the two different bootstrap confidence intervals in terms of the average confidence lengths (AC) and coverage percentages (CP). For each simulated sample under a particular setting, we computed 95% confidence intervals and checked whether the true value lay within the interval and recorded the length of the confidence interval. This procedure was repeated 1000 times. The estimated coverage probability was computed as the number of confidence intervals that covered the true values divided by 1000 while the estimated expected width of the confidence interval was computed as the sum of the lengths for all intervals divided by 1000. In our study we have used three different censoring schemes (C.S), namely:

scheme I : $R_m=n-m,\quad R_i=0$ for $i\neq m$.

scheme II : $R_1=n-m,\quad R_i=0$ for $i\neq 1$.

scheme III : $R_{\frac{m+1}{2}}=n-m,\quad R_i=0$ for $i\neq\frac{m+1}{2}$; if m odd, and

$R_{\frac{m}{2}}=n-m,\ R_i=0$ for $i\neq\frac{m}{2}$; if m even.

scheme IV : $R_{\frac{2m-n}{2}+1}=\ldots=R_{\frac{n}{2}}=1$, other $R_i=0$.

scheme V : $R_{2m-n+1}=\ldots=R_{\frac{m}{2}+5}=1$, other $R_i=0$.

In simulation studies, we consider the population parameter values $(\alpha=0.8,\beta=1.2,\lambda=2.5)$ and two case separately. **(i)** $\tau^*=0.4$. and $(ii)\tau^*=0.8$.

7 Concluding Remarks

A simulation study was conducted to examine and compare the performance of the proposed methods for different sample sizes, different censoring schemes, different acceleration factor and different change time. From the results, we observe the following.

Table 4: Comparisons of (AC) and (CP) of 95% CIs (α,β,λ) at (0.8, 1.2, 2.5).

τ^*	(n,m)	CS	MLE			Boot-P			Boot-*t*		
			α	β	λ	α	β	λ	α	β	λ
0.4	(30,20)	*I*	2.595	3.324	4.119	2.999	4.801	6.033	2.449	3.309	4.088
			(0.90)	(0.90)	(0.88)	(0.91)	(0.87)	(0.89)	(0.91)	(0.91)	(0.90)
		II	2.006	3.021	3.091	2.057	4.021	4.013	2.009	3.006	4.000
			(0.93)	(0.90)	(0.91)	(0.89)	(0.94	(0.89)	(0.92)	(0.90)	(0.91)
		III	2.443	3.411	4.521	2.822	4.615	5.233	2.321	3.417	4.108
			(0.92)	(0.90)	(0.89)	(0.91)	(0.93)	(0.92)	(0.93)	(0.92)	(0.90)
		IV	2.405	3.402	4.499	2.811	4.555	5.245	2.381	3.408	4.111
			(0.93)	(0.90)	(0.91)	(0.93)	(0.90)	(0.91)	(0.93)	(0.94)	(0.91)
	(50,30)	*I*	2.205	2.329	3.218	2.325	3.899	5.093	2.330	2.359	3.188
			(0.91)	(0.91)	(0.90)	(0.93)	(0.90)	(0.90)	(0.93)	(0.93)	(0.93)
		II	2.006	2.081	3.000	2.401	3.890	3.853	2.011	2.116	3.011
			(0.92)	(0.92)	(0.93)	(0.90)	(0.92)	(0.92)	(0.91)	(0.91)	(0.93)
		III	2.213	2.911	3.920	2.459	3.775	5.003	2.329	2.499	3.899
			(0.92)	(0.92)	(0.92)	(0.93)	(0.91)	(0.91)	(0.93)	(0.94)	(0.94)
		V	2.115	2.801	3.488	2.411	3.957	4.233	2.221	2.709	3.551
			(0.92)	(0.91)	(0.92)	(0.93)	(0.91)	(0.92)	(0.92)	(0.93)	(0.92)
0.8	(30,20)	*I*	2.465	3.258	4.108	3.000	4.724	6.122	2.420	3.296	4.72
			(0.91)	(0.90)	(0.89)	(0.91)	(0.90)	(0.90)	(0.92)	(0.92)	(0.91)
		II	2.102	3.011	3.080	2.040	4.001	3.999	2.011	2.987	3.900
			(0.92)	(0.91)	(0.90)	(0.90)	(0.93)	(0.90)	(0.91)	(0.93)	(0.93)
		III	2.422	3.409	4.503	2.811	4.599	5.198	2.299	3.311	4.100
			(0.91)	(0.92)	(0.89)	(0.92)	(0.93)	(0.93)	(0.92)	(0.92)	(0.91)
		IV	2.396	3.203	4.385	2.717	4.556	5.233	2.381	3.411	4.101
			(0.92)	(0.91)	(0.92)	(0.93)	(0.91)	(0.94)	(0.92)	(0.96)	(0.93)
	(50,30)	*I*	2.300	2.274	3.199	2.204	3.785	5.091	2.289	2.299	3.200
			(0.97)	(0.92)	(0.92)	(0.92)	(0.93)	(0.91)	(0.94)	(0.94)	(0.96)
		II	2.001	2.066	3.011	2.333	3.785	3.759	2.012	2.122	3.006
			(0.92)	(0.93)	(0.94)	(0.92)	(0.91)	(0.91)	(0.92)	(0.92)	(0.94)
		III	2.205	2.881	3.895	2.369	3.655	5.000	2.400	2.485	3.823
			(0.91)	(0.96)	(0.94)	(0.97)	(0.92)	(0.94)	(0.94)	(0.94)	(0.96)
		V	2.201	2.745	3.501	2.401	3.882	4.185	2.201	2.711	3.523
			(0.95)	(0.93)	(0.94)	(0.93)	(0.91)	(0.91)	(0.93)	(0.94)	(0.95)

1. For fixed values of the sample size, by increasing the observed failure times the MSEs decrease.
2. For fixed values of the sample size, the scheme II in which the censoring occurs after the first observed failure gives more accurate results through the MSEs than the other schemes.
3. Results in the censoring schemes III and IV are closed to other.
4. The approximate CIs and bootstrap-*t* CIs give more accurate results than the bootstrap-p CIs since the lengths of the former are less than the lengths of latter, for different sample sizes, and different schemes.
5. For fixed sample sizes and observed failures, the second scheme II , in which censoring occurs after the first observed failure, gives smallest lengths of the CIs for all methods.

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