Journal of Statistics Applications & Probability Letters
An International Journal

http://dx.doi.org/10.18576/jsapl/120302

Some Properties of Bivariate Modified Weibull Distribution with Competing Risk Application

Sanjay Kumar

Department of Mathematics & Statistics, Indian Institute of Technology Kanpur, Kanpur, Pin 208016, India

Received: 5 May. 2025, Revised: 1 Jun. 2025, Accepted: 30 Jul. 2025
Published online: 1 Sep. 2025

Abstract: El-Bassiouny et al. [1] introduced a flexible distribution, the Marshall-Olkin bivariate modified Weibull distribution (BMWD), and explored several distributional properties. Recently, Kumar et al. [45] studied the estimation of the five unknown parameters of the BMWD. In this paper, we investigate additional interesting statistical properties. The joint distribution of the BMWD is a mixture of an absolutely continuous distribution and a singular distribution. We also discuss other key properties of the BMWD, such as its modal, aging, dependence, ordering, and total positivity properties. The survival copula function associated with the BMWD is derived, and its application to competing risks is explored. We apply the BMWD to analyze a real data set for a dependent competing risks model. Finally, we discuss the multivariate extension of the modified Weibull distribution.

Keywords: Modified Weibull distribution, Singular distribution, Bathtub shaped hazard rate distribution, Mixture distribution, Copula function, Hazard gradient, Competing risks model.

1 Introduction

The bathtub-shaped hazard rate lifetime distributions play an essential role in modeling certain types of real-life data that arise in various fields such as medicine, industry, and engineering etc. These distributions exhibit a combination of monotone and constant hazard rate behavior. Many lifetime distributions and real life data exhibit this property, e.g., Chen distribution, Exponentiated Weibull, see [30]. Lai et al. [2] proposed a three parameters bathtub-shaped hazard rate modified Weibull distribution (MWD). Its hazard rate function (HRF) is increasing, decreasing, constant and bathtub-shaped. The Marshall-Olkin bivariate Weibull (MOBW) distribution, see [21], plays a significant role in life-testing experiments, reliability and survival analysis. Its marginals are Weibull distributions having increasing, decreasing and constant HRFs. The major weakness of the MOBW distribution is that its marginals cannot handle non-monotone hazard rates (in particular, bathtub-shaped unimodal hazard rates). Due to this limitation, El-Bassiouny et al. [1] introduced Marshall-Olkin bivariate modified Weibull distribution (BMWD). The BMWD is a generalization of the MOBW distribution, whose marginals are MWDs that have bathtub-shaped, increasing, decreasing and constant HRFs, see [2]. It offers a flexible framework for modeling dependent lifetimes with a combination of absolutely continuous and singular components. It has equal marginals with a positive probability. For this reason, the BMWD can be used effectively to analyze bivariate data sets with ties. Consequently, the BMWD is more flexible than the MOBW distribution. The BMWD can also be used effectively to analyze dependent competing risks data. Despite its promising potential, much remains to be explored about its statistical properties and practical applicability.

The study of bivariate and multivariate lifetime distributions play a critical role in understanding the joint behavior of two related lifetimes in fields such as reliability engineering, survival analysis, and competing risks models. While classical distributions often rely on independence assumptions, real-world applications frequently involved dependent components, making such assumptions unrealistic. However, existing distributions often lack flexibility to capture the diverse dependency structures and marginal behaviors observed in real-world data. Despite the initial exploration of its distributional properties, key statistical characteristics of the BMWD, such as modal, aging, dependence, and total positivity properties, remain understudied. While Kumar et al. [45] primarily addressed parameter estimation for the BMWD, several important theoretical and practical aspects remain to be explored. A comprehensive investigation of the

* Corresponding author e-mail: kumar.sanjaymk@gmail.com

statistical properties of the BMWD is necessary to enhance understanding of its structural characteristics and practical usefulness. In addition, deriving and studying the corresponding survival copula function would allow for a deeper analysis of the dependence structure in bivariate survival times. Competing risks are a common feature in reliability and survival data, and modeling dependent competing risks with the BMWD offers a natural application. This study aims to demonstrate the practical usefulness of the BMWD for analyzing real-world data arising in such settings. In particular, we illustrate the applicability of the BMWD through a real data analysis, with emphasis on dependent competing risks scenarios. Extending the BMWD to the multivariate setting would substantially enhance its flexibility and applicability to complex systems involving more than two lifetimes.

By addressing these gaps, this study seeks to solidify the theoretical foundation of the BMWD and demonstrate its utility in practical applications, ensuring its relevance in modern reliability and survival analysis. By deriving additional properties and applying the BMWD to real data, this study bridges theoretical advancements with practical insights, emphasizing the distribution's relevance in applied settings.

The rest of the paper is organized as follows. In Section 2, we introduce the BMWD model. In Section 3, we discuss some interesting statistical properties including, distributional properties, modal property, aging property, dependence, ordering and total positivity properties, and study the copula function. Discuss the dependent competing risks application of the BMWD in Section 4. A real data set analysis is presented in Section 5. Finally, the multivariate extension and conclusion appear in Section 6.

2 Preliminary

A random variable T denoting the failure time of a unit is said to follow MWD, see Lai et al.[2], with scale parameter $\alpha > 0$, shape parameter $\beta > 0$ and accelerate parameter $\lambda \geq 0$, denoted as T follows $(\sim)$ $MW(\alpha,\beta,\lambda)$, if for $t > 0$ it has the following respective survival function (SF) and probability density function (PDF);

$$S_{MW}(t;\alpha,\beta,\lambda) = e^{-\alpha t^{\beta} e^{\lambda t}} \quad \text{and} \quad f_{MW}(t;\alpha,\beta,\lambda) = \alpha t^{(\beta-1)}(\beta+\lambda t)e^{\lambda t}e^{-\alpha t^{\beta} e^{\lambda t}}. \tag{1}$$

The hazard rate function (HRF) of the MWD, $h_{MW}(t) = \alpha t^{(\beta-1)}(\beta+\lambda t)e^{\lambda t}$, is bathtub-shaped when $\beta < 1, \lambda > 0$, increasing when $\beta > 1, \lambda \geq 0$, decreasing when $\beta < 1, \lambda = 0$, and constant when $\beta = 1, \lambda = 0$.

The BMWD is defined as follows. Suppose $U_i \sim MW(\alpha_i,\beta,\lambda)$, $i = 1,2,3$, and they are mutually independent random variables. Define $X_1 = \min\{U_1,U_3\}$ and $X_2 = \min\{U_2,U_3\}$, then (X_1,X_2) jointly has the BMWD with parameters $\alpha_1,\alpha_2,\alpha_3,\beta$ and λ, from now it is denoted by $(X_1,X_2) \sim BMW(\alpha_1,\alpha_2,\alpha_3,\beta,\lambda)$. The marginals $X_1 \sim MW(\alpha_1+\alpha_3,\beta,\lambda)$ and $X_2 \sim MW(\alpha_2+\alpha_3,\beta,\lambda)$ with $\min\{X_1,X_2\} \sim MW(\alpha_1+\alpha_2+\alpha_3,\beta,\lambda)$. There is positive correlation between X_1 and X_2 and they are independent if α_3 = 0. The $BMW(\alpha_1,\alpha_2,\alpha_3,\beta,\lambda)$ is a flexible distribution and reduces to the $MOBW(\alpha_1,\alpha_2,\alpha_3,\beta)$ when $\lambda = 0$, the Marshall-Olkin bivariate exponential distribution, denoted by $MOBE(\alpha_1,\alpha_2,\alpha_3)$, when $\beta = 1, \lambda = 0$, and the bivariate Rayleigh distribution when $\beta = 2, \lambda = 0$. The BMWD is not an absolutely continuous distribution. The BMWD is a Marshall–Olkin type bivariate singular distribution that consists of both an absolutely continuous part and a singular part. The joint SF of (X_1,X_2) for $x_1 > 0, x_2 > 0$ is

$$\begin{aligned} S_{(X_1,X_2)}(x_1,x_2) &= P(X_1 > x_1, X_2 > x_2) = P(U_1 > x_1, U_2 > x_2, U_3 > z) \\ &= S_{MW}(x_1;\alpha_1,\beta,\lambda)S_{MW}(x_2;\alpha_2,\beta,\lambda)S_{MW}(z;\alpha_3,\beta,\lambda) \\ &= e^{-\alpha_1 x_1^{\beta} e^{\lambda x_1}} e^{-\alpha_2 x_2^{\beta} e^{\lambda x_2}} e^{-\alpha_3 z^{\beta} e^{\lambda z}}, \end{aligned}$$

where $z = \max\{x_1,x_2\}$, or equivalently the joint SF can be written as

$$S_{(X_1,X_2)}(x_1,x_2) = \begin{cases} e^{-\alpha_1 x_1^{\beta} e^{\lambda x_1}} e^{-(\alpha_2+\alpha_3) x_2^{\beta} e^{\lambda x_2}} & \text{if} \quad x_1 < x_2 \\ e^{-(\alpha_1+\alpha_3) x_1^{\beta} e^{\lambda x_1}} e^{-\alpha_2 x_2^{\beta} e^{\lambda x_2}} & \text{if} \quad x_1 > x_2 \\ e^{-(\alpha_1+\alpha_2+\alpha_3) x^{\beta} e^{\lambda x}} & \text{if} \quad x_1 = x_2 = x. \end{cases} \tag{2}$$

The joint PDF of (X_1,X_2) is given by

$$f_{(X_1,X_2)}(x_1,x_2) = \begin{cases} f_1(x_1,x_2) & \text{if} \quad 0 < x_1 < x_2 < \infty \\ f_2(x_1,x_2) & \text{if} \quad 0 < x_2 < x_1 < \infty \\ f_3(x) & \text{if} \quad 0 < x_1 = x_2 = x < \infty, \end{cases} \tag{3}$$

where

$$\begin{aligned}
f_1(x_1,x_2) &= f_{MW}(x_1;\alpha_1,\beta,\lambda) f_{MW}(x_2;\alpha_2+\alpha_3,\beta,\lambda)\\
&= \alpha_1(\alpha_2+\alpha_3)x_1^{\beta-1}(\beta+\lambda x_1)e^{\lambda x_1-\alpha_1 x_1^{\beta}e^{\lambda x_1}} x_2^{\beta-1}(\beta+\lambda x_2)e^{\lambda x_2-(\alpha_2+\alpha_3)x_2^{\beta}e^{\lambda x_2}}\\
f_2(x_1,x_2) &= f_{MW}(x_1;\alpha_1+\alpha_3,\beta,\lambda) f_{MW}(x_2;\alpha_2,\beta,\lambda)\\
&= (\alpha_1+\alpha_3)\alpha_2 x_1^{\beta-1}(\beta+\lambda x_1)e^{\lambda x_1-(\alpha_1+\alpha_3)x_1^{\beta}e^{\lambda x_1}} x_2^{\beta-1}(\beta+\lambda x_2)e^{\lambda x_2-\alpha_2 x_2^{\beta}e^{\lambda x_2}}\\
f_3(x) &= \frac{\alpha_3}{\alpha_1+\alpha_2+\alpha_3} f_{MW}(x;\alpha_1+\alpha_2+\alpha_3,\beta,\lambda)\\
&= \alpha_3 x^{\beta-1}(\beta+\lambda x)e^{\lambda x}e^{-(\alpha_1+\alpha_2+\alpha_3)x^{\beta}e^{\lambda x}}.
\end{aligned}$$

We have the following the expression

$$P(X_1<X_2)+P(X_1>X_2)+P(X_1=X_2)=1$$

$$\int_0^{\infty}\int_{x_1}^{\infty} f_1(x_1,x_2)dx_2dx_1+\int_0^{\infty}\int_{x_2}^{\infty} f_2(x_1,x_2)dx_1dx_2+\int_0^{\infty} f_3(x)dx=1.$$

Note that the function $f_{(X_1,X_2)}(x_1,x_2)$ may be considered to be a joint PDF of the BMWD, where the frist two components $f_1(x_1,x_2)$ and $f_2(x_1,x_2)$ are the absolutely continuous part which is the PDF with respect to the two-dimensional Lebesgue measure on the first (or positive) quadrant and the third component $f_3(x)$ is the singular part which is a PDF with respect to one-dimensional Lebesgue measure on positive real line, see Bemis et al. [8].

3 Properties of the BMWD

El-Bassiouny et al. [1] discussed the joint SF, joint CDF, joint PDF, joint HRF, and marginal PDFs of the BMWD. Recently, Kumar et al. [45] investigated the classical statistical inferences for the BMWD. In this section, we explore additional important and interesting statistical properties, including distributional properties, modal properties, aging properties, dependence measures, and ordering and positivity properties of the BMWD. Additionally, we derive the associated copula function.

3.1 Distributional Properties

The following results are used to derive the bivariate hazard gradient and the conditional HRF of the BMWD. They are also utilized to develop the dependent competing risks model.

Theorem 1.*If* $(X_1,X_2)\sim BMW(\alpha_1,\alpha_2,\alpha_3,\beta,\lambda)$, *then*

(a).The conditional SF of X_1 *given* $X_2\geq x_2$ *is an absolutely continuous as follows;*

$$S_{X_1|X_2\geq x_2}(x_1)=\begin{cases} e^{-\alpha_1 x_1^{\beta}e^{\lambda x_1}} & \text{if } x_1<x_2\\ e^{-(\alpha_1+\alpha_3)x_1^{\beta}e^{\lambda x_1}}e^{\alpha_3 x_2^{\beta}e^{\lambda x_2}} & \text{if } x_1\geq x_2\end{cases}$$

(b).The conditional PDF of X_1 *given* $X_2\geq x_2$ *is*

$$f_{X_1|X_2\geq x_2}(x_1)=\begin{cases} \alpha_1 x_1^{\beta-1}(\beta+\lambda x_1)e^{\lambda x_1}e^{-\alpha_1 x_1^{\beta}e^{\lambda x_1}} & \text{if } x_1<x_2\\ (\alpha_1+\alpha_3)x_1^{\beta-1}(\beta+\lambda x_1)e^{\lambda x_1}e^{-(\alpha_1+\alpha_3)x_1^{\beta}e^{\lambda x_1}}e^{\alpha_3 x_2^{\beta}e^{\lambda x_2}} & \text{if } x_1\geq x_2\end{cases}$$

*Proof.*The proof is the usual routine so we avoid the proof.

*Remark.*Using the relation $F_{X_1|X_2\geq x_2}(x_1)=1-S_{X_1|X_2\geq x_2}(x_1)$, we can obtain the conditional CDF of X_1 given $X_2\geq x_2$.

*Remark.*If $(X_1,X_2)\sim BMW(\alpha_1,\alpha_2,\alpha_3,\beta,\lambda)$, then similarly we can find the conditional SF and PDF of X_2 given $X_1\geq x_1$ respectively.

Theorem 2. *If $(X_1,X_2) \sim BMW(\alpha_1,\alpha_2,\alpha_3,\beta,\lambda)$, then*
(a). The conditional survival function of X_1 given $X_2 = x_2$ is given as;

$$S_{X_1|X_2=x_2}(x_1) = \begin{cases} e^{-\alpha_1 x_1^{\beta} e^{\lambda x_1}} & \text{if } x_1 < x_2 \\ \frac{\alpha_2}{\alpha_2+\alpha_3} e^{-(\alpha_1+\alpha_3)x_1^{\beta} e^{\lambda x_1}} e^{\alpha_3 x_2^{\beta} e^{\lambda x_2}} & \text{if } x_1 \geq x_2 \end{cases}$$

(b). The conditional PDF of X_1 given $X_2 = x_2$ is

$$f_{X_1|X_2=x_2}(x_1) = \begin{cases} \alpha_1 x_1^{\beta-1}(\beta+\lambda x_1)e^{\lambda x_1} e^{-\alpha_1 x_1^{\beta} e^{\lambda x_1}} & \text{if } x_1 < x_2 \\ \frac{\alpha_2}{\alpha_2+\alpha_3}(\alpha_1+\alpha_3)x_1^{\beta-1}(\beta+\lambda x_1)e^{\lambda x_1} e^{-(\alpha_1+\alpha_3)x_1^{\beta} e^{\lambda x_1}} e^{\alpha_3 x_2^{\beta} e^{\lambda x_2}} & \text{if } x_1 > x_2 \\ \frac{\alpha_3}{\alpha_1+\alpha_3} e^{-\alpha_1 x_2^{\beta} e^{\lambda x_2}} & \text{if } x_1 = x_2 \end{cases}$$

Proof. The proof is the usual routine so we avoid the proof.

Remark. Using the relation $F_{X_1|X_2=x_2}(x_1) = 1 - S_{X_1|X_2=x_2}(x_1)$, we can obtain the conditional CDF of X_1 given $X_2 = x_2$.

Remark. If $(X_1,X_2) \sim BMW(\alpha_1,\alpha_2,\alpha_3,\beta,\lambda)$, then similarly we can find the conditional SF and PDF of X_2 given $X_1 = x_1$ respectively.

Corollary 1. *If $(X_1,X_2) \sim BMW(\alpha_1,\alpha_2,\alpha_3,\beta,\lambda)$, then the conditional SF of X_1 given $X_2 = x_2$ is a convex combination of an absolutely continuous SF and a discrete (degenerate) SF as follows;*

$$S_{X_1|X_2=x_2}(x_1) = pG(x_1) + (1-p)H(x_1),$$

where the absolutely continuous function is

$$G(x_1) = \frac{1}{p}\begin{cases} e^{-\alpha_1 x_1^{\beta} e^{\lambda x_1}} - \frac{\alpha_3}{\alpha_2+\alpha_3} e^{-\alpha_1 x_2^{\beta} e^{\lambda x_2}} & \text{if } x_1 < x_2 \\ \frac{\alpha_2}{\alpha 2+\alpha_3} e^{-(\alpha_1+\alpha_3)x_1^{\beta} e^{\lambda x_1}} e^{\alpha_3 x_2^{\beta} e^{\lambda x_2}} & \text{if } x_1 > x_2 \end{cases}$$

with

$$p = 1 - \frac{\alpha_3}{\alpha_2+\alpha_3} e^{-\alpha_1 x_2^{\beta} e^{\lambda x_2}}$$

and the singular (degenerate at the point $x_1 = x_2$) function is

$$H(x_1) = \begin{cases} 1 & \text{if } x_1 \leq x_2 \\ 0 & \text{if } x_1 > x_2. \end{cases}$$

Proof. The proofs is trivial and therefore it is omitted.

Corollary 2. *If $(X_1,X_2) \sim BMW(\alpha_1,\alpha_2,\alpha_3,\beta,\lambda)$ then conditional PDF of X_1 given $X_2 = x_2$ is a mixture of an absolutely continuous PDF and a pmf (degenerate at $x_1 = x_2$) as follows;*

$$f_{X_1|X_2=x_2}(x_1) = pg(x_1) + (1-p)h(x_1),$$

where the absolutely continuous PDF $g(x_1)$ is

$$g(x_1) = \frac{1}{p}\begin{cases} \alpha_1 x_1^{\beta-1}(\beta+\lambda x_1)e^{\lambda x_1} e^{-\alpha_1 x_1^{\beta} e^{\lambda x_1}} & \text{if } x_1 < x_2 \\ \frac{\alpha_2}{\alpha_2+\alpha_3}(\alpha_1+\alpha_3)x_1^{\beta-1}(\beta+\lambda x_1)e^{\lambda x_1} e^{-(\alpha_1+\alpha_3)x_1^{\beta} e^{\lambda x_1}} e^{\alpha_3 x_2^{\beta} e^{\lambda x_2}} & \text{if } x_1 > x_2 \end{cases}$$

and the degenerate pmf $h(x_1)$ is

$$h(x_1) = P(X_1 = x_1) = \begin{cases} 1 & \text{if } x_1 = x_2 \\ 0 & \text{if } x_1 \neq x_2 \end{cases} \qquad \text{and} \qquad p = 1 - \frac{\alpha_3}{\alpha_2+\alpha_3} e^{-\alpha_1 x_2^{\beta} e^{\lambda x_2}}.$$

*Proof.*The proof is the usual routine and trivial therefore it is omitted.

*Remark.*If $(X_1,X_2) \sim BMW(\alpha_1,\alpha_2,\alpha_3,\beta,\lambda)$, then the joint SF of (X_1,X_2) with the joint PDF $f_{(X_1,X_2)}(x_1,x_2)$ can be written as, for $z = \max\{x_1,x_2\}$

$$\begin{aligned} S_{(X_1,X_2)}(x_1,x_2) =& P(X_1 > x_1, X_2 > x_2) = \int_{x_1}^{\infty}\int_{x_2}^{\infty} f_{(X_1,X_2)}(u,v)dvdu + P(X_1 = X_2 > z) \\ =& \int_{x_1}^{\infty}\int_{x_2}^{\infty} f_{(X_1,X_2)}(u,v)dvdu + \int_{z}^{\infty} f_{(X_1,X_2)}(u,u)du. \end{aligned}$$

We have the following unique decomposition of the joint SF of (X_1,X_2).

Theorem 3.*If $(X_1,X_2) \sim BMW(\alpha_1,\alpha_2,\alpha_3,\beta,\lambda)$, then the joint SF of (X_1,X_2) can be written as mixture of an absolutely continuous part and a singular part as follows;*

$$S_{(X_1,X_2)}(x_1,x_2) = \frac{\alpha_1+\alpha_2}{\alpha_1+\alpha_2+\alpha_3} S_{ac}(x_1,x_2) + \frac{\alpha_3}{\alpha_1+\alpha_2+\alpha_3} S_{si}(x_1,x_2),$$

where for $z = \max\{x_1,x_2\}$, $S_{si}(x_1,x_2) = e^{-(\alpha_1+\alpha_2+\alpha_3)z^{\beta}e^{\lambda z}}$; $z > 0$ and

$$S_{ac}(x_1,x_2) = \frac{\alpha_1+\alpha_2+\alpha_3}{\alpha_1+\alpha_2} e^{-\alpha_1 x_1^{\beta} e^{\lambda x_1}} e^{-\alpha_2 x_2^{\beta} e^{\lambda x_2}} e^{-\alpha_3 z^{\beta} e^{\lambda z}} - \frac{\alpha_3}{\alpha_1+\alpha_2} e^{-(\alpha_1+\alpha_2+\alpha_3)z^{\beta}e^{\lambda z}}. \tag{4}$$

Here $S_{si}(x_1,x_2)$ and $S_{ac}(x_1,x_2)$ are the singular and absolutely continuous parts, respectively.

*Proof.*Let us consider an event $E = \{X_1 = X_2\} = \{(U_3 < U_1) \cap (U_3 < U_2)\}$, then

$P(E) = \frac{\alpha_3}{\alpha_1+\alpha_2+\alpha_3}$ and $P(E^c) = 1 - P(E) = \frac{\alpha_1+\alpha_2}{\alpha_1+\alpha_2+\alpha_3}$. Therefore,

$$\begin{aligned} S_{(X_1,X_2)}(x_1,x_2) &= P(X_1 > x_1, X_2 > x_2) \\ &= P(X_1 > x_1, X_2 > x_2|E)P(E) + P(X_1 > x_1, X_2 > x_2|E^c)P(E^c) \end{aligned}$$

if $z = \max\{x_1,x_2\}$ then

$$S_{si}(x_1,x_2) = P(X_1 > x_1, X_2 > x_2|E) = exp\{-(\alpha_1+\alpha_2+\alpha_3)z^{\beta}e^{\lambda z}\},$$

and the absolutely continuous part we obtain by the subtraction

$$\begin{aligned} S_{ac}(x_1,x_2) &= P(X_1 > x_1, X_2 > x_2|E^c) = \frac{1}{P(E^c)}\left[S_{(X_1,X_2)}(x_1,x_2) - S_{si}(x_1,x_2)\right] \\ &= \frac{\alpha_1+\alpha_2+\alpha_3}{\alpha_1+\alpha_2}\left[S_{(X_1,X_2)}(x_1,x_2) - S_{si}(x_1,x_2)\right] \\ &= \frac{\alpha_1+\alpha_2+\alpha_3}{\alpha_1+\alpha_2} exp\{-\alpha_1 x_1^{\beta} e^{\lambda x_1} - \alpha_2 x_2^{\beta} e^{\lambda x_2} - \alpha_3 z^{\beta} e^{\lambda z}\} \\ &- \frac{\alpha_3}{\alpha_1+\alpha_2} exp\{-(\alpha_1+\alpha_2+\alpha_3)z^{\beta}e^{\lambda z}\} \end{aligned}$$

hence completes the proof.

We have seen that $P(X_1 > x_1, X_2 > x_2|E)$ is a singular part, as its mixed second partial derivative is zero when $x_1 \neq x_2$ and $P(X_1 > x_1, X_2 > x_2|E^c)$ is an absolutely continuous part, as its mixed second partial derivative is a PDF. Hence the joint PDF of (X_1,X_2) can be uniquely decompose as a mixture of the absolutely continuous and singular parts, as given in the following theorem.

Theorem 4.*If $(X_1,X_2) \sim BMW(\alpha_1,\alpha_2,\alpha_3,\beta,\lambda)$, then the joint PDF of (X_1,X_2) can be written as mixture of an absolutely continuous part and a singular part as follows;*

$$f_{(X_1,X_2)}(x_1,x_2) = \frac{\alpha_1+\alpha_2}{\alpha_1+\alpha_2+\alpha_3} f_{ac}(x_1,x_2) + \frac{\alpha_3}{\alpha_1+\alpha_2+\alpha_3} f_{si}(x),$$

where

$$f_{ac}(x_1,x_2)=\frac{\alpha_1+\alpha_2+\alpha_3}{\alpha_1+\alpha_2}\times\begin{cases}f_{MW}(x_1;\alpha_1,\beta,\lambda)f_{MW}(x_2;\alpha_2+\alpha_3,\beta,\lambda) & \text{if } x_1<x_2\\ f_{MW}(x_1;\alpha_1+\alpha_3,\beta,\lambda)f_{MW}(x_2;\alpha_2,\beta,\lambda) & \text{if } x_1>x_2\end{cases} \tag{5}$$

and for $x_1=x_2=x$

$$f_{si}(x)=f_{MW}(x;\alpha_1+\alpha_2+\alpha_3,\beta,\lambda).$$

Here $f_{ac}(x_1,x_2)$ *and* $f_{si}(x)$ *are the absolutely continuous and singular parts, respectively.*

*Proof.*The proof is directly followed by the above Theorem (4) as

$$f_{ac}(x_1,x_2)=\frac{\partial^2}{\partial x_1\partial x_2}S_{ac}(x_1,x_2) \text{ and } f_{si}(x)=\frac{\partial}{\partial x}S_{si}(x_1,x_2).$$

Since the singular part $f_{si}(x)$ is a PDF with respect to one-dimensional Lebesgue measure on positive real line and the absolutely continuous part $f_{ac}(x_1,x_2)$ is a PDF with respect to two-dimensional Lebesgue measure on positive (or first) quadrant.

The singular part occur due to the common shocks and absolutely continuous part due to the individual shock. The function $f_{ac}(x_1,x_2)$ in (5) denotes the joint density of the Block–Basu bivariate modified Weibull (BBBMW) distribution, denoted by $\sim BBBMW(\alpha_1,\alpha_2,\alpha_3,\beta,\lambda)$. This distribution corresponds to the absolutely continuous component of the BMWD, obtained by excluding its singular part. Further details on the BBBMW distribution can be found in Kumar [47]. The function $f_{ac}(x_1,x_2)$ is continuous everywhere on $\{(x_1,x_2): x_1\neq x_2 \text{ and } x_1,x_2>0\}$. It is discontinuous on $x_1=x_2$, unless $\alpha_1=\alpha_2$.

Lemma 1.*Let* $(X_1,X_2)\sim BMW(\alpha_1,\alpha_2,\alpha_3,\beta,\lambda)$. *If* $\alpha_1=\alpha_2$, *then* $f_{ac}(x_1,x_2)$ *is continuous everywhere on* $\{(x_1,x_2): x_1,x_2>0\}$.

*Proof.*It is clear that $f_{ac}(x_1,x_2)$ is continuous everywhere on $\{(x_1,x_2): x_1\neq x_2\}$. Now we will show that $f_{ac}(x_1,x_2)$ is continuous on the line $x_1=x_2$. For $X_1<X_2$, let $c=\frac{\alpha_1+\alpha_2+\alpha_3}{\alpha_1+\alpha_2}$

$$\lim_{x_1\to x_2^-} f_{ac}(x_1,x_2)=c\alpha_1(\alpha_2+\alpha_3)x_2^{2(\beta-1)}(\beta+\lambda x_2)^2e^{2\lambda x_2}e^{-(\alpha_1+\alpha_2+\alpha_3)x_2^{\beta}e^{\lambda x_2}},$$

where $\lim_{x_1\to x_2^-}$ means x_1 approaches x_2 from the left. For $X_1>X_2$

$$\lim_{x_1\to x_2^+} f_{ac}(x_1,x_2)=c\alpha_2(\alpha_1+\alpha_3)x_2^{2(\beta-1)}(\beta+\lambda x_2)^2e^{2\lambda x_2}e^{-(\alpha_1+\alpha_2+\alpha_3)x_2^{\beta}e^{\lambda x_2}},$$

where $\lim_{x_1\to x_2^+}$ means x_1 tends to x_2 from the right. Since $\alpha_1=\alpha_2$, then we have

$$\lim_{x_1\to x_2^-} f_{ac}(x_1,x_2)=\lim_{x_1\to x_2^+} f_{ac}(x_1,x_2)=f_{ac}(x_2,x_2)$$

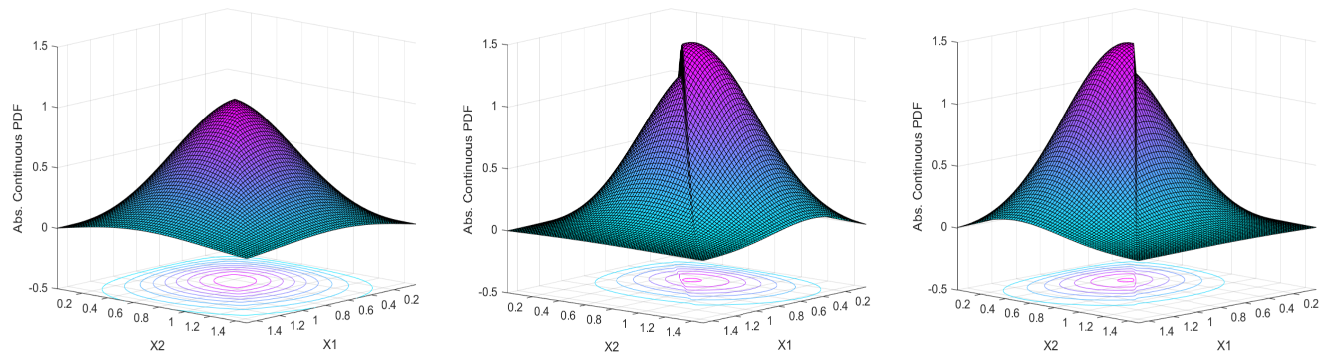
this implies that $f_{ac}(x_1,x_2)$ is also continuous on $x_1=x_2$. Therefore, when $\alpha_1=\alpha_2$, $f_{ac}(x_1,x_2)$ is continuous everywhere on $\{(x_1,x_2): x_1,x_2>0\}$.

3.2 Modal Properties

In this section, we are going to discuss about the mode and shape of the absolutely continuous part of the BMWD. The following result provides the shape of $f_{ac}(x_1,x_2)$, does not depend on constant term.

Theorem 5.*Let* $(X_1,X_2)\sim BMW(\alpha_1,\alpha_2,\alpha_3,\beta,\lambda)$. *For* $\beta>1$, $f(x_1,x_2)$ *is a unimodal.*

(a).If $\alpha_1=\alpha_2$, *then* $f_{ac}(x_1,x_2)$ *is continuous on* $x_1=x_2$ *and the mode lies on* $x_1=x_2$.
(b).If $\alpha_1+\alpha_3<\alpha_2$, *then* $f_{ac}(x_1,x_2)$ *is not continuous on* $x_1=x_2$ *and the mode occurs in the region* $\{(x_1,x_2): x_1>x_2\}$.



(a) BBBMW(1.0, 1.0, 0.5, 2.0, 1.0) **(b)** BBBMW(1.0, 2.0, 0.5, 2.0, 1.0) **(c)** BBBMW(2.0, 1.0, 0.5, 2.0, 1.0)

Fig. 1: Surface and contour plots of PDF of $BBBMW(\alpha_1, \alpha_2, \alpha_3, \beta, \lambda)$.

(c).If $\alpha_2 + \alpha_3 < \alpha_1$, then $f_{ac}(x_1, x_2)$ is not continuous on $x_1 = x_2$ and the mode occurs in the region $\{(x_1, x_2) : x_1 < x_2\}$.

*Proof.*The proof can be obtained in a routine manner by studying the stationary points at different regions namely $\{(x_1, x_2) : x_1 < x_2\}$ and $\{(x_1, x_2) : x_1 > x_2\}$ along the same line as the proof of Theorem 2.3 of Kundu and Gupta [39]. Hence, the details are avoided.

Corollary 3.*In particular when λ = 0, the mode is obtained in the closed form as*

(a)If $\beta > 1$ and $\alpha_1 = \alpha_2$, the mode (x_m, x_m) lies on line $x_1 = x_2$, where $x_m = \left(\frac{2(\beta-1)}{\beta(2\alpha+\alpha_3)}\right)^{1/\beta}$.

(b)If $\beta > 1$, the mode for $\alpha_1 + \alpha_3 < \alpha_2$ is (x_{1m}, x_{2m}), where $x_{1m} > x_{2m}$ and

$$x_{1m} = \left(\frac{\beta - 1}{(\alpha_1 + \alpha_3)\beta}\right)^{1/\beta} \quad and \quad x_{2m} = \left(\frac{\beta - 1}{\alpha_2 \beta}\right)^{1/\beta}.$$

(c)If $\alpha_2 + \alpha_3 < \alpha_1$, the mode is (x_{1m}, x_{2m}), where $x_{1m} < x_{2m}$ and

$$x_{1m} = \left(\frac{\beta - 1}{\alpha_1 \beta}\right)^{1/\beta} \quad and \quad x_{2m} = \left(\frac{\beta - 1}{(\alpha_2 + \alpha_3)\beta}\right)^{1/\beta}.$$

In Figures 1, we provide the surface and the contour plots of $f_{ac}(x_1, x_2)$ of the BMWD for different choice of parameters. It shows the unimodality of the PDF and position of the mode, as stated in Theorem (5).

To determine the mode of the BMWD, we maximize $f_{ac}(x_1, x_2)$ with respect to x_1 and x_2 for fixed parameter values. This maximization is carried out using standard two-dimensional optimization routines, such as *optim* and *nlm*, available in the R software, for various parameter settings. Both optimization methods are applied to identify the mode in each of the three cases described in Theorem (5). The resulting mode estimates, reported in Tables 1, 2, and 3, indicate that $f_{ac}(x_1, x_2)$ is unimodal, with the location of the mode depending on the specific case outlined in Theorem (5).

Table 1: Position of mode and its values when $\alpha_1 = \alpha_2$.

$\alpha_1 = \alpha_2$	$\alpha_3 = 0.025, \beta = 2.0, \lambda = 1.0$ mode(x_{1m}, x_{2m})	$\alpha_3 = 0.05, \beta = 2.5, \lambda = 0.1$ mode(x_{1m}, x_{2m})
0.025	(1.8792, 1.8792)	(2.5509, 2.5509)
0.50	(0.8062, 0.8062)	(1.0328, 1.0328)
1.00	(0.6147, 0.6147)	(0.7944, 0.7944)
1.50	(0.5191, 0.5191)	(0.6793, 0.6793)

From Tables 2 and 3 we also observed the special cases of mode and mention in the following corollary.

Table 2: Position of mode and its values when $\alpha_1 > \alpha_2 + \alpha_3$.

$\alpha_1 > \alpha_2$	$\alpha_3 = 0.025, \beta = 2.0, \lambda = 1.0$ mode(x_{1m}, x_{2m})	$\alpha_3 = 0.05, \beta = 2.5, \lambda = 0.1$ mode(x_{1m}, x_{2m})
(0.04, 0.01)	(1.8468, 1.9141)	(1.5070, 2.3818)
(0.5, 0.02)	(0.8139, 1.7884)	(1.0527, 2.2469)
(1.0, 0.04)	(0.6179, 1.6120)	(0.8022, 2.0424)
(1.5, 1.00)	(0.5209, 0.6116)	(0.6838, 0.7869)

Table 3: Position of mode and its values when $\alpha_2 > \alpha_1 + \alpha_3$.

$\alpha_2 > \alpha_1$	$\alpha_3 = 0.025, \beta = 2.0, \lambda = 1.0$ mode(x_{1m}, x_{2m})	$\alpha_3 = 0.05, \beta = 2.5, \lambda = 0.1$ mode(x_{1m}, x_{2m})
(0.035, 0.005)	(1.9931, 1.19141)	(2.4611, 1.7760)
(0.5, 0.01)	(1.9141, 0.8139)	(2.3818, 1.0527)
(1.0, 0.04)	(1.6120, 0.6179)	(2.0424, 0.8022)
(1.5, 1.00)	(0.6116, 0.5209)	(0.7869, 0.6838)

Corollary 4.*Let $(X_1, X_2) \sim BMW(\alpha_1, \alpha_2, \alpha_3, \beta, \lambda)$. For $\beta > 1$, $f_{ac}(x_1, x_2)$ is a unimodal.*
(a).If $\alpha_1 > \alpha_2$ for all α_3, the mode lies in the region $S_1 = \{(x_1, x_2) : x_1 < x_2\}$.
(b).If $\alpha_1 < \alpha_2$ for all α_3, the mode lies in the region $S_2 = \{(x_1, x_2) : x_1 > x_2\}$.

3.3 Aging Properties

It is worth mentioning that there are several ways to define the bivariate failure rate. Basu [3] was the first to define the bivariate failure rate for an absolutely continuous bivariate distribution. However, Basu's bivariate HRF may not uniquely determine the joint distribution function, which is a less desirable property. Therefore, we have adopted the definition by Johnson and Kotz [4], as it applies when the marginals are absolutely continuous and ensures that the bivariate hazard gradients uniquely define the bivariate joint distribution function.

The Basu's bivariate HRF of the BMWD, as discussed by El-Bassiouny et al. [1], is given as

$$r(x_1, x_2) = \begin{cases} \alpha_1(\alpha_2 + \alpha_3) x_1^{\beta-1} x_2^{\beta-1} (\beta + \lambda x_1)(\beta + \lambda x_2) e^{\lambda(x_1 + x_2)} & \text{if} \quad x_1 < x_2 \\ \alpha_2(\alpha_1 + \alpha_3) x_1^{\beta-1} x_2^{\beta-1} (\beta + \lambda x_1)(\beta + \lambda x_2) e^{\lambda(x_1 + x_2)} & \text{if} \quad x_1 > x_2 \\ \alpha_3 x^{\beta-1} (\beta + \lambda x) e^{\lambda x} & \text{if} \quad x_1 = x_2 = x. \end{cases}$$

The Basu joint HRF plots for different parameters are given in Figure **??** and **??**.

The hazard components of a bivariate distribution are defined as, see Johnson and Kotz [4],

$$h_1(x_1, x_2) = h(x_1 | X_2 > x_2) = -\frac{\partial}{\partial x_1} \ln S(x_1, x_2)$$

$$h_2(x_1, x_2) = h(x_2 | X_1 > x_1) = -\frac{\partial}{\partial x_2} \ln S(x_1, x_2)$$

for all (x_1, x_2) such that $S(x_1, x_2) > 0$. The vector $h(x_1, x_2) = \big(h_1(x_1, x_2), h_2(x_1, x_2)\big)$ is referred to as the hazard gradient of (X_1, X_2). Here, $h_1(x_1, x_2)$ represents the failure rate of X_1 given the additional information $X_2 > x_2$, while $h_2(x_1, x_2)$ represents the failure rate of X_2 given the additional information $X_1 > x_1$. It is well known that the hazard gradient $h(x_1, x_2)$ uniquely determines the joint SF $S(x_1, x_2)$; for further details, see Marshall and Olkin [43].

Theorem 6.*If $(X_1, X_2) \sim BMW(\alpha_1, \alpha_2, \alpha_3, \beta, \lambda)$, then the hazard components of BMWD are*

$$h_1(x_1, x_2) = \begin{cases} \alpha_1 x_1^{\beta-1} (\beta + \lambda x_1) e^{\lambda x_1} & \text{if } x_1 < x_2 \\ (\alpha_1 + \alpha_3) x_1^{\beta-1} (\beta + \lambda x_1) e^{\lambda x_1} & \text{if } x_1 > x_2 \end{cases} \tag{6}$$

$$h_2(x_1, x_2) = \begin{cases} (\alpha_2 + \alpha_3) x_2^{\beta-1} (\beta + \lambda x_2) e^{\lambda x_2} & \text{if } x_1 < x_2 \\ \alpha_2 x_2^{\beta-1} (\beta + \lambda x_2) e^{\lambda x_2} & \text{if } x_1 > x_2 \end{cases} \tag{7}$$

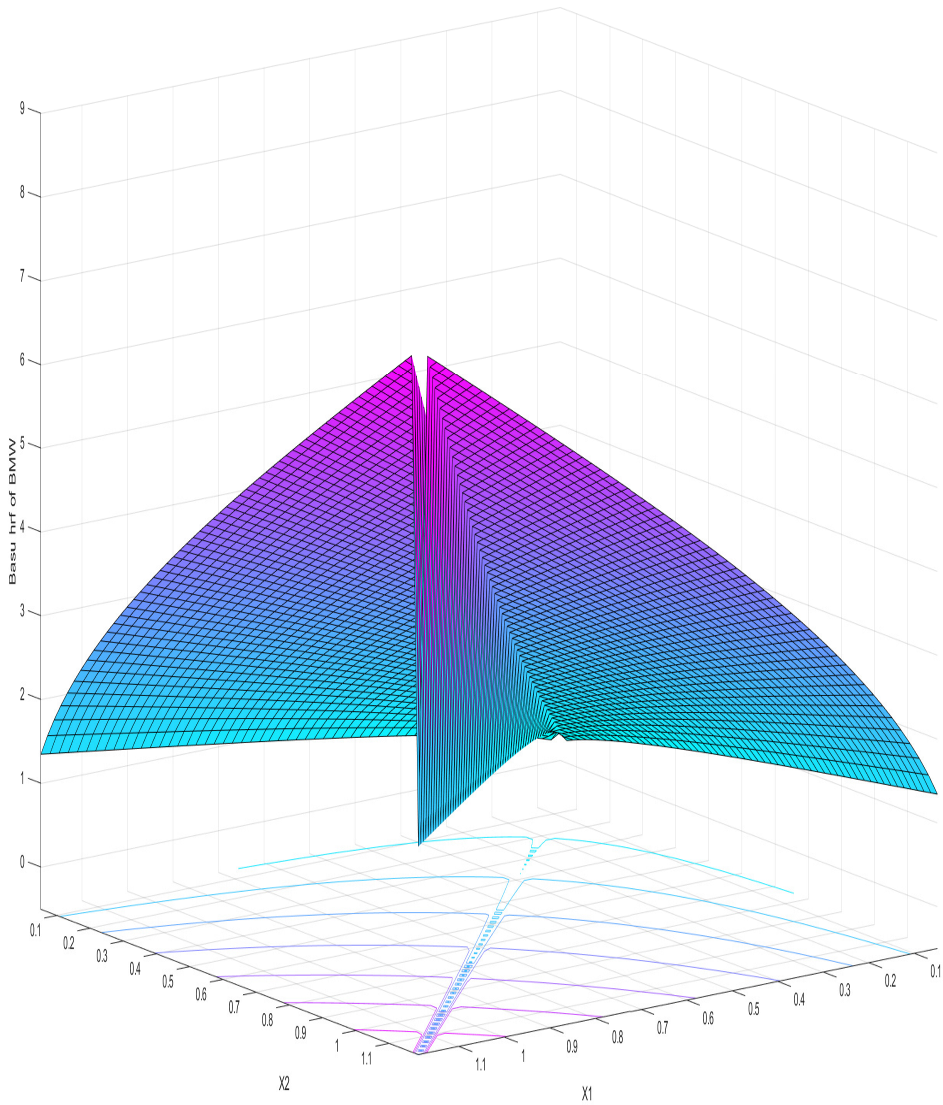


Fig. 2: The joint HRF plots of $BMW(1.0, 1.0, 1.0, 1.5, 0.1)$.

Proof. Using part (a) of the Theorem 1, we get the first component as

$$h_1(x_1, x_2) = h(x_1|X_2 > x_2) = \frac{f_{X_1|X_2>x_2}(x_1)}{S_{X_1|X_2>x_2}(x_1)}$$

similarly the second component is obtained by

$$h_2(x_1, x_2) = h(x_2|X_1 > x_1) = \frac{f_{X_2|X_1>x_1}(x_2)}{S_{X_2|X_1>x_1}(x_2)}$$

hence completes the proof.

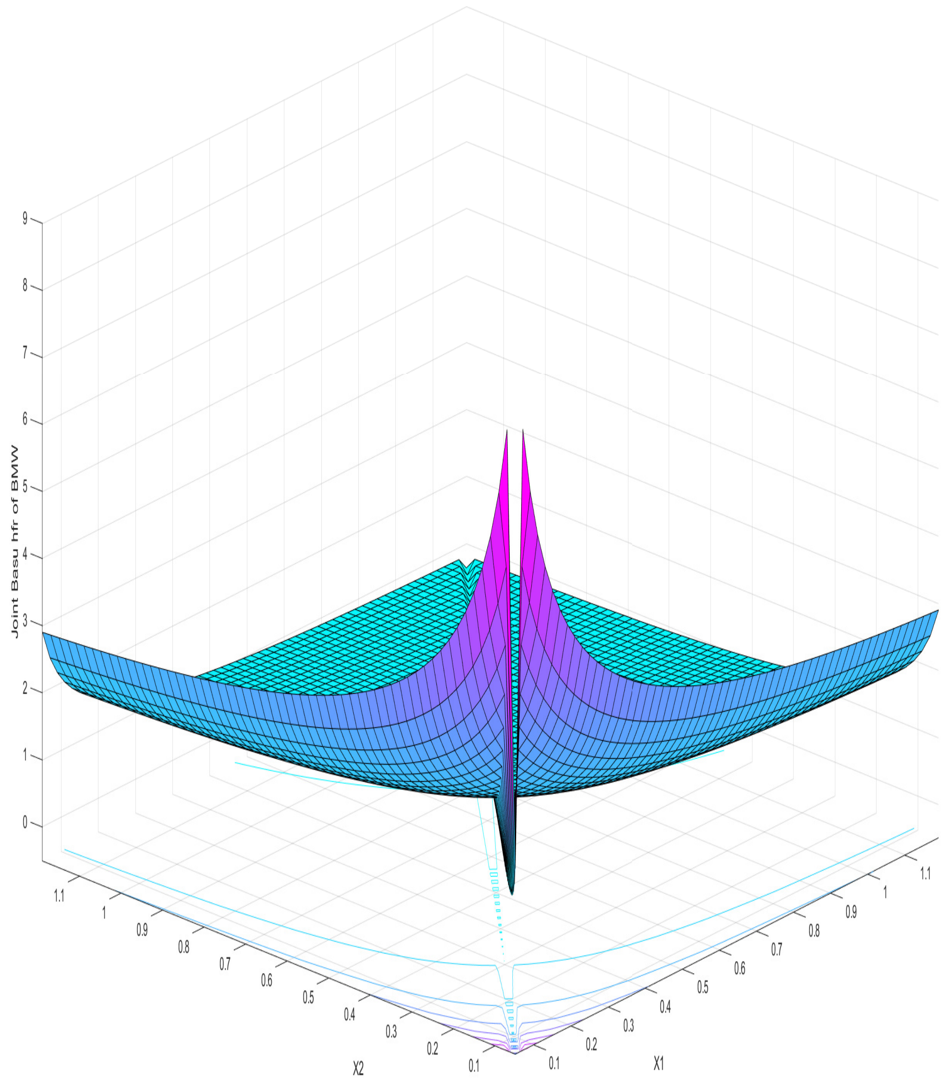


Fig. 3: The joint HRF plots of $BMW(1.0, 1.0, 1.0, 0.5, 0.1)$.

Corollary 5. *If* $(X_1, X_2) \sim BMW(\alpha_1, \alpha_2, \alpha_3, \beta, \lambda)$.

(a).Then the conditional failure (hazard) rate function of X_1 *given* $X_2 = x_2$ *is*

$$r_1(x_1|X_2 = x_2) = \begin{cases} \alpha_1 x_1^{\beta-1}(\beta + \lambda x_1)e^{\lambda x_1} & \text{if } x_1 < x_2 \\ (\alpha_1 + \alpha_3) x_1^{\beta-1}(\beta + \lambda x_1)e^{\lambda x_1} & \text{if } x_1 > x_2 \end{cases} \tag{8}$$

(b).Then the conditional failure (hazard) rate function of X_2 *given* $X_1 = x_1$ *is*

$$r_2(x_2|X_1 = x_1) = \begin{cases} (\alpha_2 + \alpha_3) x_2^{\beta-1}(\beta + \lambda x_2)e^{\lambda x_2} & \text{if } x_1 < x_2 \\ \alpha_2 x_2^{\beta-1}(\beta + \lambda x_2)e^{\lambda x_2} & \text{if } x_1 > x_2 \end{cases} \tag{9}$$

*Proof.*Using Theorem 2, we can easily proof.

3.4 Dependence Properties

Lehmann [6] defined positive quadrant dependence (PQD) between two random variables X_1 and X_2 by

$$R_{F(X_1,X_2)}(x_1,x_2) = \frac{F_{(X_1,X_2)}(x_1,x_2)}{F_{X_1}(x_1)F_{X_2}(x_2)} \geq 1 \quad \forall x_1,x_2. \tag{10}$$

Similarly the negative quadrant dependence is defined as $R_{F(X_1,X_2)}(x_1,x_2) \leq 1$. Using the following relation

$$S_{(X_1,X_2)}(x_1,x_2) = 1 - F_{X_1}(x_1) - F_{X_2}(x_2) + F_{(X_1,X_2)}(x_1,x_2),$$

which implies

$$S_{(X_1,X_2)}(x_1,x_2) - S_{X_1}(x_1)S_{X_2}(x_2) = F_{(X_1,X_2)}(x_1,x_2) - F_{X_1}(x_1)F_{X_2}(x_2),$$

equivalently the equation (10) in terms of survival function can defined by

$$R_{S(X_1,X_2)}(x_1,x_2) = \frac{S_{(X_1,X_2)}(x_1,x_2)}{S_{X_1}(x_1)S_{X_2}(x_2)} \geq (\leq) 1 \quad \forall x_1,x_2. \tag{11}$$

Now we have,

$$\lim_{x_1 \to -\infty} R_{S(X_1,X_2)}(x_1,x_2) = R_{S(X_2)}(x_2) = \frac{S_{X_2}(x_2)}{S_{X_2}(x_2)} = 1.$$

Since

$$\begin{aligned} \frac{\partial}{\partial x_i} \ln R_{S(X_1,X_2)}(x_1,x_2) &= \frac{\partial}{\partial x_i} \ln S_{(X_1,X_2)}(x_1,x_2) - \frac{\partial}{\partial x_i} \ln S_{X_i}(x_i) \\ &= h_{X_i}(x_i) - h_i(x_1,x_2) \qquad i = 1,2 \end{aligned}$$

If $h_{X_i}(x_i) - h_i(x_1,x_2) > (<)\ 0$ for all x_1 and x_2, then the distribution has positive (negative) quadrant dependence.

Theorem 7.*If $(X_1,X_2) \sim BMW(\alpha_1,\alpha_2,\alpha_3,\beta,\lambda)$ and $S_{(X_1,X_2)}(x_1,x_2) \geq (\leq)\ S_{X_1}(x_1)S_{X_2}(x_2)\ \forall x_1,x_2$, the BMWD has positive (negative) quadrant dependence.*

*Proof.*The proof is trivial.

Corollary 6.*If $(X_1,X_2) \sim BMW(\alpha_1,\alpha_2,\alpha_3,\beta,\lambda)$, then there is a positive correlation between X_1 and X_2 which is $\rho = P(X_1 = X_2) = \alpha_3/(\alpha_1+\alpha_2+\alpha_3)$. Thus for $\alpha_3 \to 0$, $\rho \to 0$ this impies that X_1 and X_2 are independent and when $\alpha_3 \to \infty$, $\rho \to 1$.*

An immediate consequence of the PQD property is that for increasing continuous functions $g_1(.)$ and $g_2(.)$, we have $Cov(g_1(X_1), g_2(X_2)) > 0$. Now we discuss some dependence properties of $(X_1, X_2) \sim BMW(\alpha_1,\alpha_2,\alpha_3,\beta,\lambda)$.

(1)From part (a) of Theorem 2, since for every x_1, $P(X_1 > x_1|X_2 = x_2)$ is an increasing function of x_2, thus X_2 is positive regression dependent of X_1. By symmetry it follows that X_1 is positive regression dependent of X_2.

(2)From part (a) of Theorem 1, since for every x_1, $P(X_1 > x_1|X_2 \geq x_2)$ in an increasing function of x_2, thus X_1 is right tail increasing (RTI) in X_2. By symmetry it follows that X_2 is RTI in X_1. It means that the hazard rate of the conditional ditribution of X_1 given X_2 does not exceed the hazard rate of the marginal distribution of X_1, see, e.g., Karia and Deshpande [46].

Note that, RTI $\Rightarrow$ association $\Rightarrow$ PQD.

3.5 Ordering and Total Positivity

If $(X_1, X_2) \sim BMW(\alpha_1, \alpha_2, \alpha_3, \beta, \lambda)$ and $(Y_1, Y_2) \sim BMW(\alpha_1^*, \alpha_2^*, \alpha_3^*, \beta^*, \lambda^*)$ such that $\alpha_1 \geq \alpha_1^*, \alpha_2 \geq \alpha_2^*, \alpha_3 \geq \alpha_3^*, \beta \geq \beta^*, \lambda \geq \lambda^*$, then by using simple algebra for $x_1, x_2 > 0$ we have

$$P(X_1 \geq x_1, X_2 \geq x_2) \leq P(Y_1 \geq x_1, Y_2 \geq x_2).$$

Thus $(X_1, X_2) \leq_{st} (Y_1, Y_2)$, i.e., (X_1, X_2) is less than (Y_1, Y_2) in the usual stochatic ordering. A bivariate random vector (X_1, X_2) and its joint PDF $f(x_1, x_2)$ is said to be totally positive of order two (TP_2) if,

$$f(x_1,x_2)f(y_1,y_2) \geq f(x_1,y_2)f(y_1,x_2) \qquad \forall x_1 < y_1, x_2 < y_2$$

A function $f(\cdot)$ defined on $\mathbb{R}^2$ is saide be supermoduler if, for $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$,

$$f(x_1,x_2) + f(y_1,y_2) \geq f(x_1,y_2) + f(y_1,x_2) \qquad \forall x_1 < y_1, x_2 < y_2$$

Remark. $f(\cdot)$ is supermoduler if and only if $\exp(f)$ is TP_2. Thus $\log(f)$ is superdomuler if and only if $f(\cdot)$ is TP_2.

Theorem 8. *If $(X_1, X_2) \sim BMW(\alpha_1, \alpha_2, \alpha_3, \beta, \lambda)$, then the bivariate random vector (X_1, X_2) and its joint PDF is TP_2.*

The TP_2 property is a stronger concept of dependence. If a copula density $c(u, v)$ possesses TP_2 property, then the associated copula $C(u, v)$ has stochastic increasing (SI), right tail increasing (RTI) and positive quadrant dependence (PQD) properties.

It is well known that TP_2 is a stronger concept of dependence. We have the following implications between TP_2, stochastically increasing (SI), right-tail increasing (RTI), association, PQD, covariance and right corner set increasing (RCSI), see Nelsen [5], and Lai and Balakrishnan [42]

$$TP_2 \Rightarrow SI \Rightarrow RTI \Rightarrow Association \Rightarrow PQD \Rightarrow Cov(X_1, X_2) \geq 0.$$

Also,

$$TP2 \Rightarrow RCSI.$$

Thus, the BMWD family has all these properties. These dependence and ordering properties follow from the Marshall–Olkin construction.

3.6 Copula Associated with BMWD

In this section, we derive the survival copula associated with BMWD. The copula is a function which joint two or more univariate distributions to construct a multivariate distribution, see Nelsen [5]. According to Sklar [7] for a bivariate random vector (X_1, X_2) with the joint SF $S_{(X_1,X_2)}(x_1, x_2)$ and marginal SFs $S_{X_i}(x_i)$ of X_i, i=1,2. For $u, v \in [0, 1]$ the survival copula function is by

$$\bar{C}(u,v) = S_{(X_1,X_2)}\big(S_{X_1}^{-1}(u), S_{X_2}^{-1}(v)\big) \tag{12}$$

To find the survival copula function we use the following lemma.

Lemma 2. *If $X \sim MW(\alpha, \beta, \lambda)$ with SF $S_X(x) = e^{-\alpha x^{\beta} e^{\lambda x}}$; $x > 0$, then for $u \in [0, 1]$ with $u = S_X(x)$, there exist unique increasing continuous function, say $g : \mathbb{R} \to \mathbb{R}^+$ such that*

$$x = S_X^{-1}(u) = g \circ w(u), \quad where \quad w(u) = \log[-(1/\alpha)\log u] \tag{13}$$

and the quantile function is $Q_X(q) = S_X^{-1}(1-q) = g \circ w(1-q)$, where $q = 1 - u = F_X(x)$.
In particular cases,

(a). If $\beta = 0$, then $g \circ w(u) = \frac{w(u)}{\lambda} = \frac{1}{\lambda}\log[-(1/\alpha)\log u]$.
(b). If $\lambda = 0$, then $g \circ w(u) = e^{w(u)/\beta} = \big[-(1/\alpha)\log u\big]^{1/\lambda}$.
(c). If $\lambda = 0$ and $\beta = 1$, then $g \circ w(u) = e^{w(u)} = -(1/\alpha)\log u$.
(d). If $\lambda = 0$ and $\beta = 2$, then $g \circ w(u) = e^{w(u)} = \big[-(1/\alpha)\log u\big]^{1/2}$.

Proof. For $u \in [0,1]$, we have $u = S_X(x)$, this implies that

$$\log[-(1/\alpha)\log u] = \beta \log x + \lambda x. \tag{14}$$

Suppose $h(x) = \beta \log x + \lambda x$, since $h'(x) = \frac{\beta}{x} + \lambda > 0$ for all $\beta, \lambda \geq 0$ this implies that $h : \mathbb{R}^+ \to \mathbb{R}$ is strictly increasing continuous function and $\lim_{x\to 0} h(x) = -\infty$ and $\lim_{x\to\infty} h(x) = \infty$. Thus there exist the unique inverse function of h(x), which is $h^{-1}(w)$, where $w = \log[-1/\alpha \log u]$, so that $x = S^{-1}(u) = h^{-1}(w)$, where $w = \log[-(1/\alpha)\log u]$. Here take $g(.) = h^{-1}(.)$, then completes the proof.

Theorem 9. *If $(X_1, X_2) \sim BMW(\alpha_1, \alpha_2, \alpha_3, \beta, \lambda)$ with the joint SF $S_{(X_1,X_2)}(x_1, x_2)$ given in (2), then there exist a unique function $h^{-1} : \mathbb{R}^+ \to \mathbb{R}$ such that, for $u, v \in [0,1]$ the survival copula function associated with BMWD is*

$$\bar{C}(u,v) = \begin{cases} exp\{-\alpha_1[h^{-1}(w_1)]^\beta e^{\lambda[h^{-1}(w_1)]}\} exp\{-\alpha_{23}[h^{-1}(w_2)]^\beta e^{\lambda[h^{-1}(w_2)]}\} & if\ h^{-1}(w_1) \leq h^{-1}(w_2) \\ exp\{-\alpha_{13}[h^{-1}(w_1)]^\beta e^{\lambda[h^{-1}(w_1)]}\} exp\{-\alpha_2[h^{-1}(w_2)]^\beta e^{\lambda[h^{-1}(w_2)]}\} & if\ h^{-1}(w_1) > h^{-1}(w_2) \end{cases}$$

where, $w_1 = \log[-(1/\alpha_{13})\log u]$ and $w_2 = \log[-(1/\alpha_{23})\log v]$ and $\alpha_{i3} = \alpha_i + \alpha_3$; i = 1, 2.

Proof. Since $(X_1, X_2) \sim BMW(\alpha_1, \alpha_2, \alpha_3, \beta, \lambda)$ with the joint SF in (2), then we have the marginals $X_i \sim MW(\alpha_i + \alpha_3, \beta, \lambda)$ with SF $S_{X_i}(x_i)$, i = 1,2. Let $\alpha_{i3} = \alpha_i + \alpha_3$, by using lemma 2, we get $S^{-1}_{X_i}(u) = h^{-1}(w_i)$; i = 1, 2, where $w_1 = \log[-1/\alpha_{13} \log u]$ and $w_2 = \log[-1/\alpha_{23} \log v]$. The survival copula function associated with BMWD is obtained as in usual manner by using the joint SF, marginal SFs of BMWD and the copula (3.6).

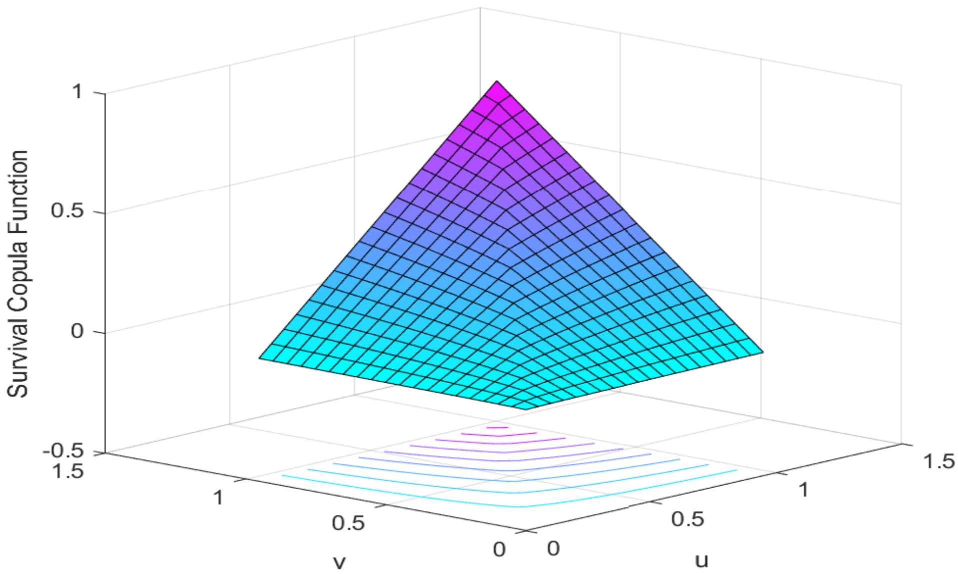


Fig. 4: Plot of Copula function of $BMW(1.5, 1.0, 1.5, 1.0, 0.1)$.

Corollary 7. *If $(X_1, X_2) \sim BMW(\alpha_1, \alpha_2, \alpha_3, \beta, \lambda)$ and the closed form of the function $h^{-1} : \mathbb{R}^+ \to \mathbb{R}$ exists, as given in lemma 2, then the survival copula function is the famous Marshall-Olkin copula, see Nelson [5], which is given by, for $u, v \in [0,1]$*

$$\bar{C}(u,v) = \begin{cases} u^{\frac{\alpha_1}{\alpha_1+\alpha_3}} v & if\ u > v^{\frac{\alpha_1+\alpha_3}{\alpha_2+\alpha_3}} \\ u v^{\frac{\alpha_2}{\alpha_2+\alpha_3}} & if\ u \leq v^{\frac{\alpha_1+\alpha_3}{\alpha_2+\alpha_3}}. \end{cases}$$

Proof.(a). If $\beta = 0$, then the joint SF Of (X_1, X_2) become

$$S_{(X_1,X_2)}(x_1,x_2) = \begin{cases} exp\{-\alpha_1 e^{\lambda x_1}\}exp\{-(\alpha_2+\alpha_3)e^{\lambda x_2}\} & \text{if } x_1 \leq x_2 \\ exp\{-(\alpha_1+\alpha_3)e^{\lambda x_1}\}exp\{-\alpha_2 e^{\lambda x_2}\} & \text{if } x_1 > x_2 \end{cases}$$

and $h^{-1}(w_i) = \frac{w_i}{\lambda}$, i = 1, 2. The survival copula function for $u, v \in [0,1]$ is

$$\bar{C}(u,v) = S_{(X_1,X_2)}\Big(1/\lambda \log[-1/(\alpha_1+\alpha_3)\log u], 1/\lambda \log[-1/(\alpha_2+\alpha_3)\log v]\Big)$$

$$\bar{C}(u,v) = \begin{cases} u^{\frac{\alpha_1}{\alpha_1+\alpha_3}} v & \text{if } u > v^{\frac{\alpha_1+\alpha_3}{\alpha_2+\alpha_3}} \\ uv^{\frac{\alpha_2}{\alpha_2+\alpha_3}} & \text{if } u \leq v^{\frac{\alpha_1+\alpha_3}{\alpha_2+\alpha_3}} \end{cases}$$

Similarly we can proof for all the cases, hence completes the proof.

It should be pointed out that the survival copula is also a copula, that is $\bar{C}(u,v)$ is also a proper distribution function on $[0,1]\times[0,1]$. It may be mentioned that using the copula structures and properties we study several different dependence properties and dependence measures of BMW distribution. It may be useful for developing inference procedures of the unknown parameters. This can have some independent interests also.

4 Application of BMW Model

In survival or reliability analysis, a common focus is the evaluation of one risk in the presence of other risk factors. This is widely recognized in statistical analysis as the competing risks problem. In a competing risks framework, the data typically comprise the failure time and the corresponding cause of failure. Significant research has been conducted in this area under both parametric and non-parametric models, as detailed in the monograph by Crowder [37].

There are two primary approaches to addressing competing risks problems: the latent failure time model, initially introduced by Cox [40], and the cause-specific hazard model proposed by Prentice et al. [44]. For exponential or Weibull lifetime models, Kundu [41] demonstrated that both approaches yield the same likelihood function, despite differing in their interpretations.

In this study, we adopt the latent failure time model, assuming there are only two possible causes of failure. However, this method can be readily extended to accommodate any arbitrary number of causes.

4.1 Competing Risks Model

Let us consider a system with two causes of failure in a lifetime experiment, and the lifetime of the system is $T = \min\{X_1, X_2\}$, where X_i, $i = 1,2$, denotes the lifetime of the i^{th} cause of failure of system. We further suppose that $(X_1,X_2) \sim BMW(\alpha_1,\alpha_2,\alpha_3,\beta,\lambda)$, then the observed random vector is (T,Δ) where

$$\Delta = \begin{cases} 0 & \text{if } X_1 < X_2 \\ 1 & \text{if } X_1 > X_2 \\ 2 & \text{if } X_1 = X_2. \end{cases}$$

Let $\alpha = \alpha_1 + \alpha_2 + \alpha_3$, we can obtain the following useful facts.

Theorem 10. *If $(X_1,X_2) \sim BMW(\alpha_1,\alpha_2,\alpha_3,\beta,\lambda)$, then*
(i).$X_i \sim MW(\alpha_i+\alpha_3,\beta,\lambda)$, i = 1, 2.
(ii).$T = \min\{X_1,X_2\} \sim MW(\alpha,\beta,\lambda)$, where $\alpha = \alpha_1+\alpha_2+\alpha_3$, with SF and HRF

$$S(t;\alpha,\beta,\lambda) = e^{-\alpha t^{\beta} e^{\lambda t}} \qquad and \qquad h(t) = \alpha t^{\beta-1}(\beta+\lambda t)e^{\lambda t}.$$

Based on Theorem 10 and the condtional SF given in Part (b) of the Theorem 2, the joint PDF of (T,Δ) is

$$f(t,\delta) = \begin{cases} f_{X_1}(t)P(X_2 \geq t|X_1 = t) = \alpha_1 t^{\beta-1}(\beta+\lambda t)e^{\lambda t}e^{-\alpha t^{\beta}e^{\lambda t}} & \text{if } \delta = 0 \\ f_{X_2}(t)P(X_1 \geq t|X_2 = t) = \alpha_2 t^{\beta-1}(\beta+\lambda t)e^{\lambda t}e^{-\alpha t^{\beta}e^{\lambda t}} & \text{if } \delta = 1 \\ \alpha_3 t^{\beta-1}(\beta+\lambda t)e^{\lambda t}e^{-\alpha t^{\beta}e^{\lambda t}} & \text{if } \delta = 2 \end{cases} \tag{15}$$

4.1.1 Maximum Likelihood Estimation

It is assumed that the dependent competing risks data $\mathscr{D} = \{(t_1, \delta_1), (t_2, \delta_2), ..., (t_n, \delta_n)\}$ have been observed from BMWD. Let $\theta = (\alpha_1, \alpha_2, \alpha_3, \beta, \lambda)$. Define $I_1 = \{t_i : \delta = 0\}$, $I_2 = \{t_i : \delta = 1\}$ and $I_3 = \{t_i : \delta = 2\}$. Let n_i is the number of observations in the set I_i, $i = 1, 2, 3$. The log-likelihood function is

$$\begin{aligned} l(\theta) =& n_1 \ln \alpha_1 + n_2 \ln \alpha_2 + n_3 \ln \alpha_3 + (\beta - 1)\sum_{i=1}^{n} \ln t_i + \sum_{i=1}^{n} \ln(\beta + \lambda t_i) \\ &+ \lambda \sum_{i=1}^{n} t_i - (\alpha_1 + \alpha_2 + \alpha_3)\sum_{i=1}^{n} t_i^{\beta} e^{\lambda t_i}. \end{aligned} \tag{16}$$

The MLEs of the unknown parameters can be obtained by maximizing (19) with respect to the unknown parameters. For fix $\vartheta = (\beta, \lambda)$, the MLEs of α_1, α_2 and α_3 can be obtained as

$$\hat{\alpha}_1(\vartheta) = \frac{n_1}{\sum_{i=1}^{n} t_i^{\beta} e^{\lambda t_i}} \qquad \hat{\alpha}_2(\vartheta) = \frac{n_2}{\sum_{i=1}^{n} t_i^{\beta} e^{\lambda t_i}} \qquad \hat{\alpha}_3(\vartheta) = \frac{n_3}{\sum_{i=1}^{n} t_i^{\beta} e^{\lambda t_i}}. \tag{17}$$

The MLEs of $\vartheta = (\beta, \lambda)$ are obtained by maximizing the profile log-likelihhod function $l_p(\hat{\alpha}_1(\vartheta), \hat{\alpha}_3(\vartheta), \hat{\alpha}_3(\vartheta), \vartheta)$ using two dimensional optimization technique, like Nelder-Mead method, see Nelder & Mead [22] available in R software under optim, i.e.,

$$\widehat{\vartheta} = \arg\max_{\vartheta}\, l_p\big(\hat{\alpha}_1(\vartheta), \hat{\alpha}_3(\vartheta), \hat{\alpha}_3(\vartheta), \vartheta\big). \tag{18}$$

Moreover, using invariance principle of maximum likelihood estimation, the MLEs of the SF and HRF can be constructed as

$$\hat{S}(t) = exp(-\hat{\alpha} t^{\hat{\beta}} e^{\hat{\lambda} t}) \qquad \text{and} \qquad h(t) = \hat{\alpha} t^{\hat{\beta}-1}(\hat{\beta} + \hat{\lambda} t)e^{\hat{\lambda} t} \quad \text{with} \quad \hat{\alpha} = \hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_3.$$

Remarks: Since any $n_i = 0$, $i = 1, 2, 3$, then the observed data provides no information about α_i and hence the MLE of α_i does not exist, see (17). Thus the condition that all $n_i > 0$, $i = 1, 2, 3$, guarantees the existence of MLE $\hat{\alpha}_i$. The maximum likelihood estimators of $\alpha_1, \alpha_2, \alpha_3$ are mutually independent.
In most cases, we are more interested in the confidence intervals of the parameters instead of point estimation only. Having the Fisher information matrix of $(\alpha_1, \alpha_2, \alpha_3, \beta, \lambda)$ as given following, the confidence intervals can be constructed. The expected Fisher information matrix $\mathscr{I}(\theta)$ is a 5×5 matrix, whose $(i, j)^{th}$ element given as

$$\mathscr{I}_{ij}(\theta) = -E_{\theta}\left\{\frac{\partial^2}{\partial\theta_i \partial\theta_j} l(\theta)\right\}, \quad i, j = 1, 2, 3, 4, 5$$

Since all such expectations exist and are finite, but cannot be obtained analytically, we can approximate them by $-\frac{\partial^2 l(\theta)}{\partial\theta_i \partial\theta_j}$ evaluated at the MLEs $\hat{\theta}$, i.e., the observed Fisher information matrix, denoted as $\mathscr{I}(\hat{\theta})$. Using the asymptotic properties of MLEs we find the asymptotic confidence intervals of the MLEs. The derivation of the obsereved Fisher information matrix is placed in Appendix B. Under mild regularity conditions, the distribution of the MLE $\hat{\theta} = (\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\beta}, \hat{\lambda})$ is asymptotically normal with a mean vector θ and variance-coriance $\mathscr{I}^{-1}(\hat{\theta})$, the inverse matrix of $\mathscr{I}(\hat{\theta})$ i.e.,

$$\hat{\theta} \sim \mathscr{A}\mathscr{N}(\theta, \mathscr{I}^{-1}(\hat{\theta}))$$

For arbitrary $\gamma \in (0, 1)$, the $(1-\gamma)100\%$ asymptotic confidence interval (ACI) of θ_i is given as

$$\left(\hat{\theta}_i - z_{\gamma/2}\sqrt{var(\hat{\theta}_i)},\ \hat{\theta}_i + z_{\gamma/2}\sqrt{var(\hat{\theta}_i)}\right), \quad i = 1, 2, 3, 4, 5,$$

where $z_{\gamma/2}$ is the $\gamma/2 - th$ quantile of the standard normal distribution.

Sometimes, the previous obtained ACIs may have a negative lower bound which further yields worse performance for interval estimation of positive parameter. In order to avoid such drawback, one can use logarithmic transformation method and delta technique to construct the ACI in consequence. The asymptotic normality distribution of $\ln\hat{\theta}_i$ can be obtained as

$$\frac{\ln\hat{\theta}_i - \ln\theta_i}{\mathrm{var}(\ln\hat{\theta}_i)} \to N(0,1), \quad i = 1,2,3,4,5,$$

where $\mathrm{var}(\ln\hat{\theta}_i) = var(\hat{\theta}_i)/\hat{\theta}_i^2$, $i = 1,2,3,4,5$. Therefore, a $(1-\gamma)100\%$ alternative ACI of θ_i can be expressed as

$$\left[\exp\left(\log(\hat{\theta}_i) - \frac{z_{\gamma/2}}{\hat{\theta}_i}\sqrt{var(\hat{\theta}_i)}\right),\ \exp\left(\log(\hat{\theta}_i) + \frac{z_{\gamma/2}}{\hat{\theta}_i}\sqrt{var(\hat{\theta}_i)}\right)\right], \quad i = 1,2,3,4,5.$$

5 Real Data Analysis

In this section, for the illustrative purpose we consider a real data example from medical studies. The diabetic retinopathy is a complication of diabetes that turns out a major cause of blindness and vision loss among diabetic patients. Here we consider a data set obtained from the diabetic retinopathy study (DRS). The study was conducted by the National Eye Institute (NEI) to estimate the effect of laser treatment in delaying the onset of blindness in patients with diabetic retinopathy. There are 71 patients involved in the study. One eye of each patient was randomly selected for laser treatment and the other eye was observed without treatment, for more details, see [33] and [34]. Denote X_1 as the time (in days) to blindness of the laser treated eye and X_2 as the time (in days) to blindness of the other untreated eye. The minimum time to blindness $T_i = \min(X_{1i}, X_{2i})$ and the indicator δ_i ($\delta_i = 1$ when the treated eye has been failed first, $\delta_i = 0$ when the untreated eye has been failed first $\delta_i = 2$ when both the eyes have become blind) have been for the i^{th} patient. For each patient the minimum time to blindness (T_i) and the index for specifying whether treated, untreated or both eyes, firstly is failed (δ_i), is recorded in Table 4. The original data are transformed by $(T-30)/365$ and numerical results are computed in terms of years.

Table 4: Minimum time to blindness in days and its causes for 71 patients with Diabetic retinopathy.

i	T_i	δ	i	T_i	δ	i	T_i	δ	i	T_i	δ	i	T_i	δ
1	266	1	16	125	2	31	717	2	46	663	0	61	503	1
2	91	2	17	777	2	32	642	1	47	567	2	62	423	2
3	154	2	18	306	1	33	141	2	48	966	0	63	285	2
4	285	0	19	415	1	34	407	1	49	203	0	64	315	2
5	583	1	20	307	2	35	356	1	50	84	1	65	727	2
6	547	2	21	637	2	36	1653	0	51	392	1	66	210	2
7	79	1	22	577	2	37	427	2	52	1140	2	67	409	2
8	622	0	23	178	1	38	699	1	53	901	1	68	584	1
9	707	2	24	517	2	39	36	2	54	1247	0	69	355	1
10	469	2	25	272	0	40	667	1	55	448	2	70	1302	1
11	93	1	26	1137	0	41	588	2	56	904	2	71	227	2
12	1313	2	27	1484	1	42	471	0	57	276	1			
13	805	1	28	315	1	43	126	1	58	520	1			
14	344	1	29	287	2	44	350	2	59	485	2			
15	790	2	30	1252	1	45	350	1	60	248	2			

Before going to analyze the dependent competing risks data using the BMW distribution, first we fit the MW distribution to T for model checking and to find the initial guesses. For this purpose, we consider model selection criteria, such as the total time test (TTT) plot and the Weibull probability paper (WPP) plot, to assess whether the MW distribution appropriately fits the given data T. The TTT plot, introduced by Aarset [15], is defined as: $TTT_i = \sum_{j=1}^{i}(n-j+1)(x_{(j)} - x_{(j-1)})$, where $x_{(0)} = 0$ and $x_{(j)}$ are ordered observations. The TTT plot represents TTT_i against i/n. To ensure the values lie between 0 and 1, we use the scaled TTT, given by: $TTT_i^* = \frac{TTT_i}{TTT_n}$. If the scaled TTT graph lies above (or below) the reference line, the data exhibit an increasing (or decreasing) failure rate. If the graph fluctuates above and below the reference line (or aligns with it), the data indicate a bathtub-shaped (or constant) hazard rate. The TTT plots, see Figure 5, for T in the given data set suggest an increasing hazard rate. The WPP plot for the MW distribution, studied by Lai et al. [2]. The WPP plots, see Figure 5, for T in the data set are approximated by convex curve. Thus, based on the TTT and WPP plots, the MW distribution can be considered a suitable fit for T.

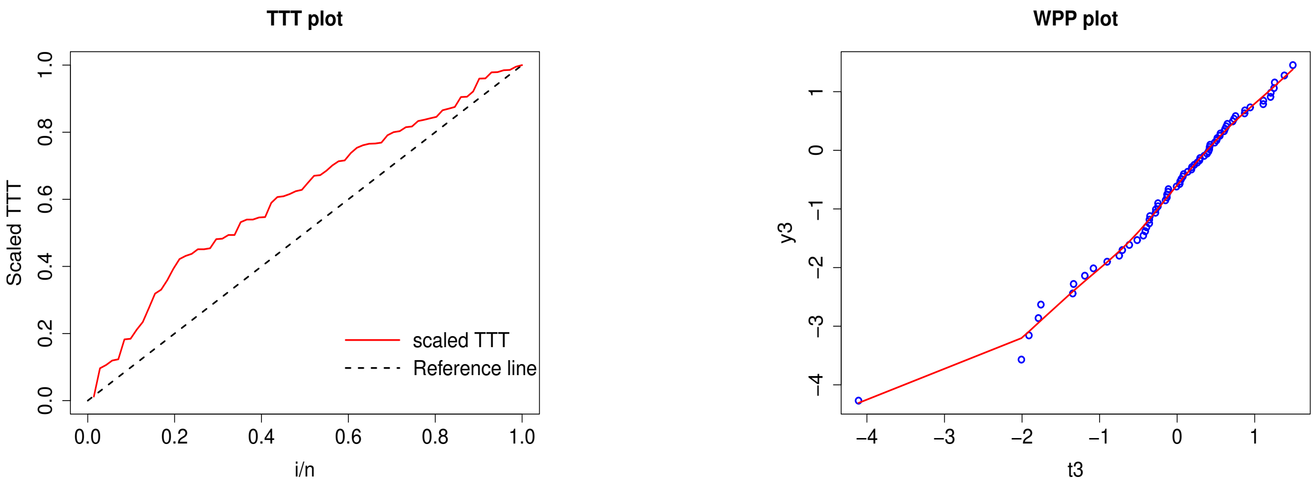


Fig. 5: TTT and WPP plots of DRD

The MLEs of the MW distribution parameters for each case are obtained using three-dimensional standard numerical optimization techniques, such as Nelder-Mead algorithm, see [22], it is available under $optim()$ function in the R software. To initiate the numerical optimization process, appropriate initial guesses for the unknown parameters are required. These initial estimates are derived from the WPP plot for the MW distribution with parameters α, β, and λ, , which can be expressed as the following nonlinear equation: $y = \log(\alpha) + \beta u + \lambda \exp(u)$, where $u = \log(x)$. For given n observations x_i; $i = 1, 2, ..., n$, we have $y_i = \log\left[-\log\left(1 - \frac{i}{(n+1)}\right)\right]$ and $u_i = \log(x_i)$. Note that, u and $\exp(u)$ are not statistically independent. For $T \sim MW(\alpha, \beta, \lambda)$, using the regression procedure, the estimated model is given as $y = -0.8620 + 1.0545u + 0.2207\exp(u)$, where $u = \log(t)$ with coefficient of determination, $R^2 = 0.973$ and hence the initial guess values of (α, β, λ) are (0.4223, 1.0545, 0.2208). Using these initial values, the MLEs of parameters α, β, λ of MW distribution and the corresponding standard errors (SEs), confidence intervals and the KS distances with the associated with p-values are given in the Table 5. Based on the KS distance with the p-value, the MW distribution cannot be rejected for the data T.

Table 5: MLEs, SE, KD distance with p-value

Parameters	MLEs	SE	LL	KS distance	p-value
α	0.054883	0.016772	-446.5452	0.0557	0.8616
β	0.951410	0.130671			
λ	0.007549	0.006314			

Now, we analyze the competing risks data (T, δ) using the BMW distribution. The profile log-likelihood plots for $\alpha_1, \alpha_2, \alpha_3, \beta, \lambda$ are given Figure **??**. These plots ensure that the parameter estimates are unique. To obtain the MLEs of β and λ, we apply a two-dimensional optimization technique using the $optim()$ function with initial guess values (0.951410, 0.007549). The MLEs of $\alpha_1, \alpha_2, \alpha_3, \beta, \lambda$, , along with their corresponding standard errors (SEs) and 95% confidence intervals, are presented in Table 6. For the goodness-of-fit (GoF) test, the KS distance is 0.0556 and the associated p-value is 0.8619. The model demonstrates a good fit to the data, see the KS distance plot and percentile-percentile (P-P) plot as given Figure **??**.

Now we want to test whether the laser treatment has statistical significant effect on the patient or not, i.e.,

$$H_0 : \alpha_1 = \alpha_2 \qquad \text{versus} \qquad H_a : \alpha_1 \neq \alpha_2.$$

Table 6: MLEs, SEs, 95% CIs

Parameter	MLEs	SE	95% CI
α_1	0.013135	0.004574	(0.004170 , 0.022099)
α_2	0.002815	0.001414	(0.000044 , 0.005585)
α_3	0.038935	0.012164	(0.015094 , 0.062775)
β	0.951410	0.131095	(0.694469 , 1.208351)
λ	0.007549	0.006327	(0.003301 , 0.017263)

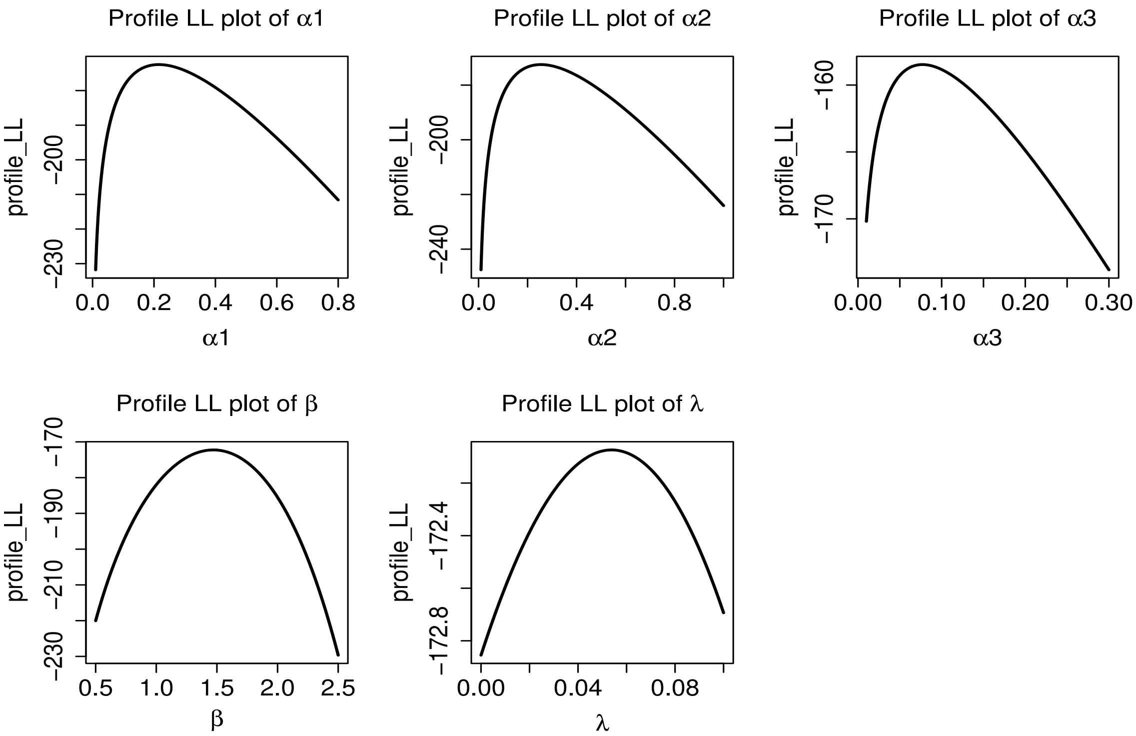


Fig. 6: The profile log-likelihood plots for β and λ.

Under the null hypothesis $\alpha_1 = \alpha_2 = \alpha_{12}$ the log-likelihood function is

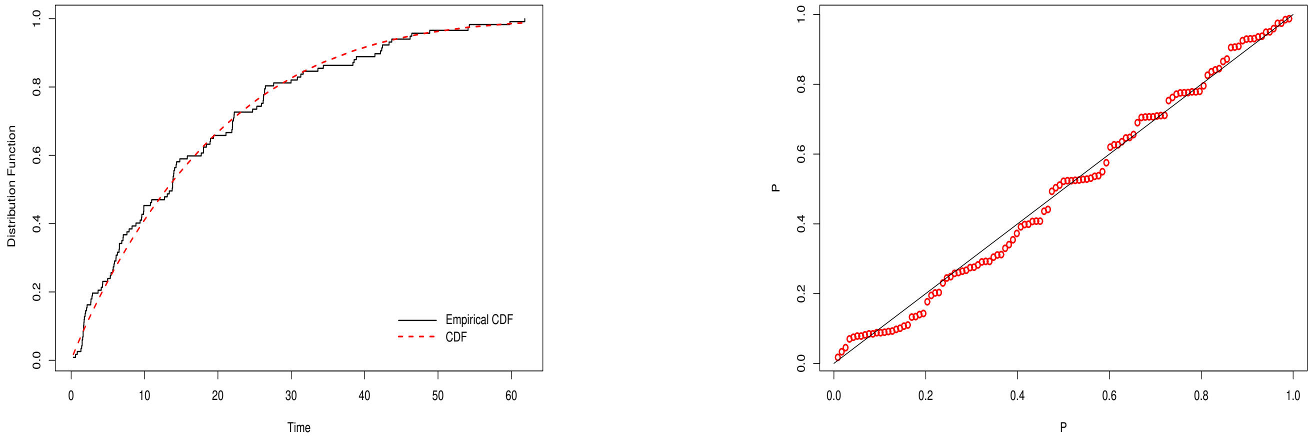
$$l(\theta_0) = (n_1+n_2)\ln\alpha_{12} + n_3\ln\alpha_3 + (\beta-1)\sum_{i=1}^{n}\ln t_i + \sum_{i=1}^{n}\ln(\beta+\lambda t_i) + \lambda\sum_{i=1}^{n} t_i - (2\alpha_{12}+\alpha_3)\sum_{i=1}^{n} t_i^{\beta} e^{\lambda t_i}. \quad (19)$$

where $\theta_0 = (\alpha_{12}, \alpha_3, \beta, \lambda)$. The MLEs of α_{12}, α_3, β and λ are 0.2350, 0.0771, 1.3971 and 0.0147, respectively and corresponding log-likelihood value is -158.6753. The likelihood ratio test statistics value is -2(532.9035-551.0596) = 36.3122 and the associate p-value becomes 0.00001. The null hypothesis is rejected at 5% level of significance, hence the laser treatment is significant.

6 Extension and Conclusion

Multivariate Modified Weibull distribution

The multivariate extension presented in this section follows naturally from the Marshall–Olkin construction by allowing more than two components to be affected by individual as well as common shocks, while retaining modified Weibull



(a) KS distance plot **(b)** P-P plot

Fig. 7: KS distance and P-P plots

marginal distributions. This extension is included mainly for completeness and to demonstrate that the proposed bivariate model can be embedded in a broader multivariate framework. Although the present study focuses on bivariate lifetimes, the multivariate formulation may be useful for modeling more complex systems involving multiple dependent failure times.

In the previous sections, we have been discussed about the univariate and bivariate MW distribution. In this section, we consider its multivariate extension. The extension is quite straight forward and it can done along the same line. First we consider the Marshall-Olkin [20] type multivaite MW distribution. Let $A_p = \{a = (a_1, a_2, ..., a_p) : a_i = 0 \ or \ 1, \text{but } a \neq (0, ..., 0)\}$ be a set of $2^p - 1$ p-dimensional vectors. Let $\{Z_a : a \in A_p\}$, $Z_a \sim MW(\alpha_a, \beta, \lambda)$ are independent random variables. Define, $X_i = \min\{Z_a : a \in A_p, \ a_i = 1\} \ \forall \ i = 1, 2, ..., p$, then $\mathbf{X} = (X_1, X_2, ..., X_p)$ has Marshall-Olkin type multivariate MW distribution with parameters $\alpha^* = \{\alpha_a : a \in A_p\}, \beta$ and λ, and it can be written as $\mathbf{X} = (X_1, X_2, ..., X_p) \sim MOMMW(p, \alpha^*, \beta, \lambda)$, where p denotes the dimension of the distribution, for any $p > 1$, it is not absolutely continuous distribution. The p-variate MOMMW distribution has $2^p + 1$ parameters.

Theorem 11. *If $\mathbf{X} \sim MOMMW(p, \alpha^*, \beta, \lambda)$, then*

(1)The joint SF of $\mathbf{X}$ for $x_i > 0$, $i = 1, 2, ..., p$ is given by,

$$S_{\mathbf{X}}(x_1, x_2, ..., x_p) = exp\left[-\sum_{a \in A_p} \alpha_a z_a^{\beta} e^{\lambda z_a}\right], \tag{20}$$

where $z_a = \max_{1 \le i \le p}(x_i a_i) = \max(x_1 a_1, x_2 a_2, ..., x_p a_p)$.

(2)For $i = 1, 2, ..., p$, $X_i \sim MW(\alpha_i, \beta, \lambda)$, where $\alpha_i = \sum_{(a:a_i=1) \in A_n} \alpha_a$.

(3)$U = \min\{X_1, X_2, ..., X_p\} \sim MW(\alpha, \beta, \lambda)$, where $\alpha = \sum_{a \in A_p} \alpha_a$.

Proof. The proof is the usual rountine so we omitted it.

The MOMMW is more generalization p-dimensional multivariate MW distribution. Using the idea of Proschan and Sullo [10] we introduce the special case of it as follows.

Let us consider a p-component parallel system where there are $(p+1)$ causes of failures. The i^{th} cause affect the i^{th}, $i = 1, 2, ..., p$ component respectively and the $(p+1)^{th}$ cause affect all the components simultaneous. Suppose the failure time due to the j^{th} cause, $U_j \sim MW(\alpha_j, \beta, \lambda)$, $j = 0, 1, 2, 3..., p$ are independent. Let $X_i = \min(U_i, U_0)$, $i = 1, 2, ..., p$, be the failure time of the i^{th} component, then $\mathbf{X} = (X_1, X_2, ..., X_p)$ has multivariate MW distribution with parameters α_i, β, λ, $i = 0, 1, 2, ..., p$, and from now on, we will denote it by $MMW(\alpha_0, \alpha_1, \alpha_2, ..., \alpha_p, \beta, \lambda)$. For $p > 1$ the MMW distribution is singular distribution and it has both absolutely continous and singular parts. The p-variate MMW distribution has p+3 unknown parameters. Let $\alpha = (\alpha_0 + \alpha_1 + \alpha_2 + ... + \alpha_p) = \sum_{i=0}^{p} \alpha_i$.

Theorem 12. *If* $\mathbf{X} = (X_1, X_2, ..., X_p) \sim MMW(\alpha_0, \alpha_1, \alpha_2, ..., \alpha_p, \beta, \lambda)$, *then*

(1) $X_i \sim MW(\alpha_i + \alpha_0, \beta, \lambda)$, $i = 1, 2, ..., p$.
(2) $V = \min\{X_1, X_2, ..., X_p\} \sim MW(\alpha, \beta, \lambda)$.
(3) $P(X_i = \min\{X_1, X_2, ..., X_p\}) = P(X_i < \min\{X_j : 1 \le j (\neq i) \le p\})$
$= P(U_i < \min\{U_j : 0 \le j(\neq i) \le p\}) = \frac{\alpha_i}{\alpha}$, $i = 1, 2, ..., p$.
(4) $P(X_1 = X_2 = ... = X_p) = P(U_0 < \min\{U_j : 1 \le j \le p\}) = \frac{\alpha_0}{\alpha}$.
(5) *If* $\alpha_0 = 0$, *then* $X_1, X_2, ..., X_p$ *are mutually independent.*

Proof. The proof is the usual rountine so we avoid it.

Theorem 13. *If* $\mathbf{X} = (X_1, X_2, ..., X_p) \sim MMW(\alpha_0, \alpha_1, \alpha_2, ..., \alpha_p, \beta, \lambda)$, *then*

(1) *The joint SF of* $\mathbf{X}$ *for* $x_i > 0$, $i = 1, 2, ..., p$ *is given by,*

$$S_{\mathbf{X}}(\mathbf{x}) = exp\left[-\sum_{i=1}^{p} \alpha_i x_i^{\beta} e^{\lambda x_i} - \alpha_0 z^{\beta} e^{\lambda z}\right], \tag{21}$$

where $\mathbf{x} = (x_1, x_2, ..., x_p)$ *and* $z = \max(x_1, x_2, ..., x_p)$, *or*

$$S_{\mathbf{X}}(\mathbf{x}) = \begin{cases} exp\left[-\sum_{k(\neq i)=1}^{p} \alpha_k x_k^{\beta} e^{\lambda x_k} - (\alpha_i + \alpha_0) x_i^{\beta} e^{\lambda x_i}\right] & if \quad x_i = \max(x_1, x_2, ..., x_p) > 0, \\ & \quad i = 1, 2, .., p \\ exp\left[-\alpha x^{\beta} e^{\lambda x}\right] & if \quad x_1 = x_2 = ... = x_p = x > 0. \end{cases}$$

(2) *The joint PDF of* $\mathbf{X}$ *is*

$$f_{\mathbf{X}}(\mathbf{x}) = \begin{cases} f_i(\mathbf{x}) & if \quad x_i = \max(x_1, x_2, ..., x_p) > 0, \quad i = 1, 2, .., p \\ f_0(\mathbf{x}) & if \quad x_1 = x_2 = ... = x_p = x > 0, \end{cases} \tag{22}$$

where

$$f_i(\mathbf{x}) = (1 + \alpha_0/\alpha_i) e^{-\alpha_0 x_i^{\beta} e^{\lambda x_i}} \prod_{k=1}^{p} \alpha_k x_k^{\beta-1} (\beta + \lambda x_k) e^{\lambda x_k} e^{-\alpha_k x_k^{\beta} e^{\lambda x_k}}$$

and

$$\begin{aligned} f_0(\mathbf{x}) &= \frac{\alpha_0}{\alpha} f_{MW}(x; \alpha, \beta, \lambda) \\ &= \alpha_0 x^{\beta-1} (\beta + \lambda x) e^{\lambda x} e^{-\alpha x^{\beta} e^{\lambda x}}. \end{aligned}$$

Proof. The proof is the usual rountine so we avoid it.

Note that the function $f_{\mathbf{X}}(\mathbf{x})$ may be considered to be a joint PDF of the MMW distribution, where the first p components $f_i(\mathbf{x})$, $i = 1, 2, ..., p$ are the absolutely continuous part which is the PDF with respect to the p-D Lebesgue measure and the last $(p+1)^{th}$ component $f_0(\mathbf{x})$ is the singular part which is a PDF with respect to the 1-D Lebesgue measure on the positive real line, see, e.g., Bemis et al. [8]. It will be important to develop both the classical and Bayesian statistical inference of unknown parameters of the model. It has quite effectively application to study the competing risks model when there are p causes of failures. More work needed along that direction.

We have the following unique decomposition of the joint SF of MMW distribution.

Theorem 14. *If* $\mathbf{X} = (X_1, X_2, ..., X_p) \sim MMW(\alpha_0, \alpha_1, \alpha_2, ..., \alpha_p, \beta, \lambda)$, *then the joint SF of* $\mathbf{X}$ *can be written as mixture of an absolutely continuous part and a singular part as follows;*

$$S_{\mathbf{X}}(\mathbf{x}) = \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_p}{\alpha} S_{ac}(\mathbf{x}) + \frac{\alpha_0}{\alpha} S_{si}(x),$$

where for $z = \max\{x_1, x_2, ...x_p\}$

$$S_{si}(x) = \begin{cases} exp[-\alpha z^{\beta} e^{\lambda z}]; & z > 0 \\ 0 & otherwise \end{cases}$$

and

$$S_{ac}(\mathbf{x}) = \frac{\alpha}{\alpha_1+\alpha_2+\cdots+\alpha_p} exp\Big[-\sum_{i=1}^{p}\alpha_i x_i^{\beta} e^{\lambda x_i}\Big] - \frac{\alpha_0}{\alpha} exp\Big[-\alpha z^{\beta} e^{\lambda z}\Big].$$

Here $S_{si}(x)$ is the singular part due to the common shocks and $S_{ac}(\mathbf{x})$ is the absolutely continuous part due to the individual shock.

Proof. Trivial proof.

Theorem 15. *If $\mathbf{X} = (X_1, X_2, ..., X_p) \sim MMW(\alpha_0, \alpha_1, \alpha_2, ..., \alpha_p, \beta, \lambda)$, then the joint PDF of $\mathbf{X}$ can be written as mixture of an absolutely continuous part and a singular part as follows;*

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{\alpha_1+\alpha_2+\cdots+\alpha_p}{\alpha} f_{ac}(\mathbf{x}) + \frac{\alpha_0}{\alpha} f_{si}(x),$$

where,

$$f_{ac}(\mathbf{x}) = \frac{\alpha}{\alpha_1+\alpha_2+\cdots+\alpha_p} f_{MW}(x_i;\alpha_i+\alpha_0,\beta,\lambda) \prod_{k(\neq i)=1}^{p} f_{MW}(x_k;\alpha_k,\beta,\lambda),\ if\ x_i = \max\{X_1, X_2, ..., X_p\}$$

and for $x_1 = x_2 = ... = x_p = x$

$$f_{si}(x) = f_{MW}(x;\alpha,\beta,\lambda).$$

Here $f_{ac}(\mathbf{x})$ and $f_{si}(x)$ are the absolutely continuous and singular parts, respectively.

Proof. Trivial proof.

Acknowledgements:

The author gratefully thanks Prof. Debasis Kundu for his constructive comments, which significantly improved the manuscript. The author also sincerely thanks the anonymous reviewers and the Associate Editor for their valuable suggestions and insightful comments, which greatly enhanced the quality and clarity of this work.

Conflict of Interest and Funding Statements

The author does not have any conflict of interest. This manuscript has not been funded by any funding agent.

Appendix A

To obtain the observed Fisher information matrix under the parameters $(\alpha_1, \alpha_2, \alpha_3, \beta, \lambda)$, we should obtain the second partial derivatives of log likelihhod function, where

$$\frac{\partial l}{\partial \alpha_1} = \frac{n_1}{\alpha_1} - \sum_{i=1}^{n} t_i^{\beta} e^{\lambda t_i} \qquad \frac{\partial l}{\partial \alpha_2} = \frac{n_2}{\alpha_2} - \sum_{i=1}^{n} t_i^{\beta} e^{\lambda t_i} \qquad \frac{\partial l}{\partial \alpha_3} = \frac{n_3}{\alpha_3} - \sum_{i=1}^{n} t_i^{\beta} e^{\lambda t_i}$$

$$\frac{\partial l}{\partial \beta} = \sum_{i=1}^{n} \ln t_i + \sum_{i=1}^{n} (\beta+\lambda t_i)^{-1} - (\alpha_1+\alpha_2+\alpha_3)\sum_{i=1}^{n} t_i^{\beta} \ln t_i e^{\lambda t_i}$$

$$\frac{\partial l}{\partial \lambda} = \sum_{i=1}^{n} t_i(\beta+\lambda t_i)^{-1} + \sum_{i=1}^{n} t_i - (\alpha_1+\alpha_2+\alpha_3)\sum_{i=1}^{n} t_i^{\beta+1} e^{\lambda t_i}.$$

The elements of observed Fisher information matrix $H = ((H_{ij}))$, $(i, j = 1,2,3,4,5)$, which are second partial derivatives log likleihood function evaluated at MLEs, are given as

$$H_{11} = \frac{\partial^2 l}{\partial \alpha_1^2} = -\frac{n_1}{\widehat{\alpha}_1^2} \qquad H_{12} = H_{21} = \frac{\partial^2 l}{\partial \alpha_1 \partial \alpha_2} = 0 \qquad H_{13} = H_{31} = \frac{\partial^2 l}{\partial \alpha_1 \partial \alpha_3} = 0$$

$$H_{14} = H_{41} = \frac{\partial^2 l}{\partial \alpha_1 \partial \beta} = -\sum_{i=1}^{n} t_i^{\hat{\beta}} \ln t_i e^{\hat{\lambda} t_i} \qquad H_{15} = H_{51} = \frac{\partial^2 l}{\partial \alpha_1 \partial \lambda} = -\sum_{i=1}^{n} t_i^{\hat{\beta}+1} e^{\hat{\lambda} t_i}$$

$$H_{22} = \frac{\partial^2 l}{\partial \alpha_2^2} = -\frac{n_2}{\widehat{\alpha}_2^2} \qquad H_{23} = H_{32} = \frac{\partial^2 l}{\partial \alpha_2 \partial \alpha_3} = 0 \qquad H_{24} = H_{42} = H_{14} \qquad H_{25} = H_{52} = H_{15}$$

$$H_{33} = \frac{\partial^2 l}{\partial \alpha_3^2} = -\frac{n_3}{\widehat{\alpha}_3^2} \qquad H_{34} = H_{43} = H_{14} \qquad H_{35} = H_{53} = H_{15}$$

$$H_{44} = \frac{\partial^2 l}{\partial \beta^2} = -\sum_{i=1}^{n} (\hat{\beta} + \hat{\lambda} t_i)^{-2} - \hat{\alpha} \sum_{i=1}^{n} t_i^{\hat{\beta}} (\ln t_i)^2 e^{\hat{\lambda} t_i}$$

$$H_{45} = H_{54} = \frac{\partial^2 l}{\partial \beta \partial \lambda} = -\sum_{i=1}^{n} t_i (\hat{\beta} + \hat{\lambda} t_i)^{-2} - \hat{\alpha} \sum_{i=1}^{n} t_i^{\hat{\beta}+1} \ln t_i e^{\hat{\lambda} t_i}$$

$$H_{55} = \frac{\partial^2 l}{\partial \lambda^2} = -\sum_{i=1}^{n} t_i^2 (\hat{\beta} + \hat{\lambda} t_i)^{-2} - \hat{\alpha} \sum_{i=1}^{n} t_i^{\hat{\beta}+2} e^{\hat{\lambda} t_i}.$$

References

[1] El-Bassiouny, A. H., Shahen, H. S., and Abouhawwash, M. (2018). A new bivariate modified Weibull distribution and its extended distribution. *Journal of Statistics Applications & Probability*, 7(2), 217–231.
[2] Lai, C. D., Xie, M., and Murthy, D. N. P. (2003). A modified Weibull distribution. *IEEE Transactions on Reliability*, 52(1), 33–37.
[3] Basu, A. P. (1971). Bivariate failure rate. *Journal of the American Statistical Association*, 66(333), 103–104.
[4] Johnson, N. L. and Kotz, S. (1975). A vector multivariate hazard rate. *Journal of Multivariate Analysis*, 5(1), 53–66.
[5] Nelsen, R. B. (2007). *An Introduction to Copulas*. Springer.
[6] Lehmann, E. L. (1966). Some concepts of dependence. *The Annals of Mathematical Statistics*, 37(5), 1137–1153.
[7] Sklar, M. (1959). Fonctions de repartition an dimensions et leurs marges. *Publications de l'Institut de Statistique de l'Université de Paris*, 8, 229–231.
[8] Bemis, B. M., Bain, L. J., and Higgins, J. J. (1972). Estimation and hypothesis testing for the parameters of a bivariate exponential distribution. *Journal of the American Statistical Association*, 67(340), 927–929.
[9] Bhattacharyya, G. K. and Johnson, R. A. (1971). Maximum likelihood estimation and hypothesis testing in the bivariate exponential model of Marshall and Olkin. Technical Report.
[10] Proschan, F. and Sullo, P. (1976). Estimating the parameters of a multivariate exponential distribution. *Journal of the American Statistical Association*, 71(354), 465–472.
[11] Arnold, B. C. (1968). Parameter estimation for a multivariate exponential distribution. *Journal of the American Statistical Association*, 63(323), 848–852.
[12] Muliere, P. and Scarsini, M. (1987). Characterization of a Marshall–Olkin type class of distributions. *Annals of the Institute of Statistical Mathematics*, 39, 429–441.
[13] Pena, E. A. and Gupta, A. K. (1990). Bayes estimation for the Marshall–Olkin exponential distribution. *Journal of the Royal Statistical Society: Series B*, 52(2), 379–389.
[14] Ryu, K. (1993). An extension of Marshall and Olkin's bivariate exponential distribution. *Journal of the American Statistical Association*, 88(424), 1458–1465.
[15] Aarset, M. V. (1987). How to identify a bathtub hazard rate. *IEEE Transactions on Reliability*, 36(1), 106–108.

[16] Baxter, L. A. and Rachev, S. T. (1991). The stability of a characterization of the bivariate Marshall–Olkin distribution. *Journal of Mathematical Analysis and Applications*, 160(2), 563–571.

[17] Karlis, D. (2003). ML estimation for multivariate shock models via an EM algorithm. *Annals of the Institute of Statistical Mathematics*, 55, 817–830.

[18] Kundu, D. and Dey, A. K. (2009). Estimating the parameters of the Marshall–Olkin bivariate Weibull distribution by EM algorithm. *Computational Statistics & Data Analysis*, 53(4), 956–965.

[19] Louis, T. A. (1982). Finding the observed information matrix when using the EM algorithm. *Journal of the Royal Statistical Society: Series B*, 44(2), 226–233.

[20] Marshall, A. W. and Olkin, I. (1967). A multivariate exponential distribution. *Journal of the American Statistical Association*, 62(317), 30–44.

[21] Lu, J.-C. (1989). Weibull extensions of the Freund and Marshall–Olkin bivariate exponential models. *IEEE Transactions on Reliability*, 38(5), 615–619.

[22] Nelder, J. A. and Mead, R. (1965). A simplex method for function minimization. *The Computer Journal*, 7(4), 308–313.

[23] Meintanis, S. G. (2007). Test of fit for Marshall–Olkin distributions with applications. *Journal of Statistical Planning and Inference*, 137(12), 3954–3963.

[24] Self, S. G. and Liang, K.-Y. (1987). Asymptotic properties of maximum likelihood estimators and likelihood ratio tests under nonstandard conditions. *Journal of the American Statistical Association*, 82(398), 605–610.

[25] Alam, K., Abernathy, R., and Williams, C. L. (1993). Multivariate goodness-of-fit tests based on statistically equivalent blocks. *Communications in Statistics – Theory and Methods*, 22(6), 1515–1533.

[26] Alam, K. and Williams, C. L. (1995). A multivariate goodness-of-fit test for stochastically ordered distributions. *Biometrical Journal*, 37(8), 945–956.

[27] Rosenblatt, M. (1952). Remarks on a multivariate transformation. *The Annals of Mathematical Statistics*, 23(3), 470–472.

[28] Pyke, R. (1965). Spacings. *Journal of the Royal Statistical Society: Series B*, 27(3), 395–436.

[29] Anderson, T. W. (1996). Some nonparametric multivariate procedure based on statistically equivalent blocks. In *Multivariate Analysis*. Academic Press.

[30] Mudholkar, G. S. and Srivastava, D. K. (1993). Exponentiated Weibull family for analyzing bathtub failure-rate data. *IEEE Transactions on Reliability*, 42(2), 299–302.

[31] Cox, D. R. (1972). Regression models and life-tables. *Journal of the Royal Statistical Society: Series B*, 34(2), 187–202.

[32] Arnold, B. C., Martínez-Flórez, G., and Gómez, H. W. (2022). Bivariate proportional hazard models: Structure and inference. *Symmetry*, 14(10), 2073.

[33] Diabetic Retinopathy Study Research Group. (1976). Preliminary report on effects of photocoagulation therapy. *American Journal of Ophthalmology*, 81(4), 383–396.

[34] Huster, W. J., Brookmeyer, R., and Self, S. G. (1989). Modelling paired survival data with covariates. *Biometrics*, 145–156.

[35] Block, H. W. and Basu, A. P. (1974). A continuous, bivariate exponential extension. *Journal of the American Statistical Association*, 69(348), 1031–1037.

[36] Kundu, D. and Gupta, R. D. (2010). A class of absolutely continuous bivariate distributions. *Statistical Methodology*, 7(4), 464–477.

[37] Crowder, M. J. (2001). *Classical Competing Risks*. CRC Press.

[38] Crowder, M. J. (2012). *Multivariate Survival Analysis and Competing Risks*. CRC Press.

[39] Kundu, D. and Gupta, A. K. (2017). On bivariate inverse Weibull distribution.

[40] Cox, D. R. (1959). The analysis of exponentially distributed life-times with two types of failure. *Journal of the Royal Statistical Society Series B*, 21(2), 411–421.

[41] Kundu, D. (2004). Parameter estimation for partially complete time and type of failure data. *Biometrical Journal*, 46(2), 165–179.

[42] Lai, C. D. and Balakrishnan, N. (2009). *Continuous Bivariate Distributions*. Springer.

[43] Marshall, A. W. (1975). Some comments on the hazard gradient. *Stochastic Processes and their Applications*, 3(3), 293–300.

[44] Prentice, R. L., Kalbfleisch, J. D., Peterson, A. V., Flournoy, N., Farewell, V. T., and Breslow, N. E. (1978). The analysis of failure times in the presence of competing risks. *Biometrics*, 541–554.

[45] Kumar, S., Kundu, D., and Mitra, S. (2024). Estimation of five parameter bivariate modified Weibull singular distribution. *Communications in Statistics – Simulation and Computation*, 1–24.

[46] Karia, R. H. and Deshpande, J. V. (1991). On some dependence concepts. *Communications in Statistics – Theory and Methods*, 20(4), 1157–1167.

[47] Kumar, S. (2026). Absolutely Continuous Bivariate Modified Weibull Distribution: Properties and Estimation. Journal of Statistical Analysis and Applications, 25(12).