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Extended Odd Weibull Inverse Rayleigh Distribution with Application on Carbon Fibres

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Abstract: The aim of this paper is to introduce a new suitable distribution for modeling the carbon fibers data. A new distribution is a combination of the inverse Rayleigh distribution and the extended odd Weibull family to formulate the extended odd Weibull inverse Rayleigh (EOWIR) distribution with three parameters. A simple linear representation, hazard rate function, and moment generating function have been obtained of EOWIR distribution. To estimate the unknown parameters of the distribution of EOWIR, maximum probability, maximum product spacing, and Bayesian estimation methods are applied. For the Bayesian approximation, the MCMC approach is used under the square error loss function. To evaluate the use of estimation methods, a numerical result from the Monte Carlo simulation is obtained.

Keywords: Extended odd Weibull family; inverse Rayleigh distribution; Bayesian; carbon fibers; maximum product spacing

1 Introduction

Modeling real-life events in the form of distributions of probability are one of the key tasks of statistics. Distributions of probability are used to model the phenomena of natural life that are characterized by uncertainty and risk. Many of the distributions of probability are derived since the phenomena of natural life are dynamic and diversified. Established distributions of probability, however, remain unable to accurately describe data for certain natural phenomena. These contribute to the extension and modification of generalized distributions of probability, with the popular existence of having added parameters, generalized probability distributions have advanced. The addition of some parameters to the known distributions of probability improved the consistency of suitability for the data of natural phenomena and improved the accuracy of the distribution tail shape definition.

In inverted distributions, due to their applicability in many fields, the inverse (or inverted) distributions are of great significance, such as biological sciences, life test problems, medical sciences, etc. The density and hazard ratio forms of the inverted distributions illustrate a distinct structure from the non-inverted distributions of conformation. There are discussions with several researchers about the applications of inverted distributions, and the reader may refer to Abd AL-Fattah et al. [1], Barco et al. [2], Hassan and Abd-Allah [3], Muhammed [4], Chesneau et al. [5], Usman and ul Haq [6], Eferhonore et al. [7] among others.

Let X is a random variable with the scale parameter $\vartheta \geq 0$ has inverse Rayleigh distribution (IR) distribution. The cumulative distribution (CDF) and probability density (PDF) functions are as follows:

$$G(x;\vartheta) = e^{-\frac{\vartheta}{x^2}}; \quad x \geq 0, \lambda \geq 0 \tag{1}$$

and,

$$g(x;\vartheta) = \frac{2\vartheta}{x^3} e^{-\frac{\vartheta}{x^2}}; \quad x, \vartheta > 0 \tag{2}$$

In different statistical writings by various authors, a generalization of a different distribution of IR was discussed, mainly applied in reliability estimation. For example, a generalization of the IR distribution known as the exponentiated IR (EIR) distribution was introduced by Rao, and Mbwambo [8]. The transmuted alpha power IR (TAPIR) distribution was implemented by Malik and Ahmad (2019). Al-Omari et al. [9] proposed the distribution of generalized exponential IR (EGIR). By using half-logistic transformation, Almarashi et al. [10] suggested a new extension of IR distribution.

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Kumaraswamy Exponentiated IR (KEIR) distribution was introduced by Ahsan ul Haq [11]. Mahdy et al. [12] introduced the distribution of elicitation IR, called compound IR distribution. We are introducing a new model with three parameters here, called the distribution of extended odd Weibull inverse Rayleigh (EOWIR). Centered on the extended odd Weibull-G (EOW-G) family introduced by Alizadeh et al. [13]. Let $\overline{G}(x;\vartheta) = 1 - G(x;\theta)$ and $g(x;\vartheta) = \frac{dG(x;\theta)}{dx}$ denote the survival function (S) and probability density function (PDF) of a baseline model with parameter vector θ respectively, so the CDF of the EOW-G family is given by:

$$F(x;\Delta) = 1 - \left\{1 + \gamma\left[\frac{G(x;\vartheta)}{\overline{G}(x;\vartheta)}\right]^{\delta}\right\}^{\frac{-1}{\gamma}}, x \in \mathbb{R}. \quad (3)$$

where Δ is a vector of parameters of EOWIR distribution. The corresponding PDF of (3) is defined by

$$f(x;\Delta) = \frac{\delta\, g(x;\vartheta)\, G(x;\vartheta)^{\delta-1}}{\overline{G}(x;\vartheta)^{\delta+1}}\left\{1 + \gamma\left[\frac{G(x;\vartheta)}{\overline{G}(x;\vartheta)}\right]^{\delta}\right\}^{\frac{-1}{\gamma}-1}, \quad (4)$$

where δ and γ are positive shape parameters. The random variable with PDF (4) is denoted by X ~EOW-G$(\delta,\gamma,\vartheta)$. A new versatile three-parameter exponential distribution, called the extended odd Weibull exponential distribution, was introduced by Afify and Mohamed[15]. Under the progressive type-II censoring scheme with random elimination, Alshenawy et al. [14] discussed classical estimation methods of the extended odd Weibull exponential distribution were addressed.

Our aim is to study the point estimation of the unknown EOWIR parameters by using three classical estimation methods. Via simulation, a statistical analysis is carried out between these methods to test the efficiency of these methods and to study how these estimators work for several sample sizes and parameter values. The application of carbon fiber data is discussed.

The remainder of this article is structured as follows. We define the EOWIR distribution under Section 2. In Section 3, along with some of its statistical properties for the EOWIR, is obtained. Section 4 studies three methods of point estimation. To compare the performance of these estimation methods, a simulation study is performed in Section 5. The application of carbon fibers data is discussed in Section 6 to show the efficiency of the distribution of EOWIR with respect to other distributions. Finally, in Section 7, conclusions are provided.

2 EOWIR Distribution

A special model of the EOW-G family with IR distribution as a baseline function is the three-parameter EOWIR distribution. By substituting the IR model CDF and PDF files (1) and (2) of the EOW-G family (3) and (4), the EOWIR distribution CDF and PDF are obtained as;

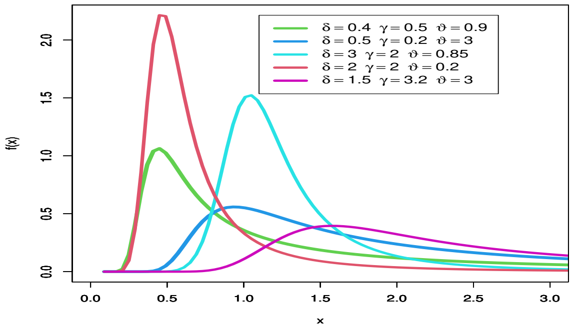

Fig. 1: pdf of EOWIR distribution.

$$F(x;\Delta) = 1 - \left\{1 + \gamma\left[\frac{e^{-\frac{\vartheta}{x^2}}}{1 - e^{-\frac{\vartheta}{x^2}}}\right]^{\delta}\right\}^{\frac{-1}{\gamma}}, \quad (5)$$

where $x > 0, \delta, \gamma, \vartheta > 0$.

$$f(x;\Delta) = \frac{\delta \frac{2\vartheta}{x^3} e^{-\frac{\delta\vartheta}{x^2}}}{\left(1 - e^{-\frac{\vartheta}{x^2}}\right)^{\delta+1}}\left\{1 + \gamma\left[\frac{e^{-\frac{\vartheta}{x^2}}}{1 - e^{-\frac{\vartheta}{x^2}}}\right]^{\delta}\right\}^{\frac{-1}{\gamma}-1}, \quad (6)$$

Therefore, a random variable with PDF (6) is denoted by X ~EOWIR$(\delta,\gamma,\vartheta)$.

The hazard rate function (HR) of the EOWIR distribution are given by

$$h(x;\Delta) = \frac{\delta \frac{2\vartheta}{x^3} e^{-\frac{\delta\vartheta}{x^2}}}{\left(1 - e^{-\frac{\vartheta}{x^2}}\right)^{\delta+1}}\left\{1 + \gamma\left[\frac{e^{-\frac{\vartheta}{x^2}}}{1 - e^{-\frac{\vartheta}{x^2}}}\right]^{\delta}\right\}^{-1}$$

Figures 1 and 2 are different shapes of the PDF and HR of the EOWIR distribution. These figures show that the PDF of the EOWIR distribution can be right-skewed, symmetric, or decreasing curves. The HR of the EOWIR distribution has some important shapes, including constant, decreasing, and upside down curve, which are attractive characteristics for any lifetime model. It can be noticed from the application section that the EOWIR distribution possesses great flexibility and can be used to model skewed data, hence widely applied in different areas such as biomedical studies, biology, reliability, physical engineering, and survival analysis.

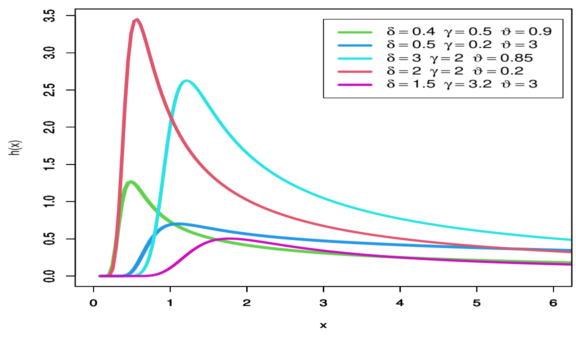

Fig. 2: HR of EOWIR distribution.

3 Statistical Properties of EOWIR Distribution

In this section, we observe some statistical properties of the EOWIR distribution, namely, the linear representation of PDF, which is useful in finding the moments and moment generating function (MGF). Also, we obtain the mean residual life and mean inactivity time.

3.1 Linear Representation

Linear representation for the EOWIR density using series techniques is useful for finding many statistical values and properties of the needed distribution. Now substituting the PDF and CDF of the IR distribution, the above equation can be written as

$$f(x) = \sum_{k,q=0}^{\infty} \varepsilon_{k,q} \left[\delta(q+1)+k\right] \frac{2\vartheta}{x^3} e^{-\frac{\vartheta}{x^2}[\delta(q+1)+k]}, \quad (7)$$

where $\varepsilon_{k,j} = \gamma^q \binom{q+\frac{1}{\gamma}}{q} \binom{\delta(q+1)+k}{k} (-1)^q \frac{\delta}{[\delta(q+1)+k]}$.

Equation (7) denotes the IR density with parameter $\delta\left[\delta(q+1)+k\right]$.

3.2 Quantile for The OWIR Distribution

The quantile function of the EOWIR distribution, say $x = Q(x) = F^{-1}(x,\Theta)$ is derived by inverting (5) as follows:

$$x_u = \sqrt{\frac{\vartheta}{\ln\left\{1+\left[\frac{1}{\gamma}(1-u)^{-\gamma}-\frac{1}{\gamma}\right]^{\frac{-1}{\delta}}\right\}}}; \; 0<u<1 \quad (8)$$

In particular, the first quartile, say Q1, the second quartile, say Q2, and the third quartile, say Q3 are obtained by setting $Q = 0.25, 0.5, 0.75$, respectively, in (8).

3.3 Moments for The EOWIR Distribution

Let X be a random variable having EOWIR distribution. Then the r_{th} moment of X follows simply from Equation (7) as

$$\begin{aligned} \acute{\mu}_r &= E(X^r) \\ &= \sum_{q,k=0}^{\infty} \varepsilon_{q,k} \Gamma\left(1-\frac{r}{2}\right) \vartheta^{\frac{r}{2}} \left[\delta(q+1)+k\right]^{\frac{r}{2}}, \; r<2 \end{aligned} \quad (9)$$

Te moment generating function of EOWIR distribution is given by

$$\begin{aligned} M_X^{\prime}(t) &= E(e^{xt}) \\ &= \sum_{q,j,k=0}^{\infty} \frac{t^j}{j!} \varepsilon_{q,k} \Gamma\left(1-\frac{j}{2}\right) \vartheta^{\frac{j}{2}} \left[\delta(q+1)+k\right]^{\frac{j}{2}}, \; j<2 \end{aligned} \quad (10)$$

4 Estimation Methods

The estimation problem of the OWIR distribution parameters is studied in this section using three different estimation methods called: maximum likelihood estimators (MLE), maximum spacing estimator product (MPSE), and Bayesian estimation based on the function of square error loss.

4.1 Maximum Likelihood Estimators

Let $x_1,\ldots,x_n$ be a random sample with the parameters δ,γ and ϑ from the EOWIR distribution. The log-likelihood feature for the distribution of EOWIR is provided by

$$\begin{aligned} l(\Delta) =& n\left[\ln(\delta)+\ln(2)+\ln(\vartheta)\right] - 3\sum_{i=1}^{n} ln(x_i) - \\ & \delta\vartheta\sum_{i=1}^{n} x_i^{-2} - (\delta+1)\sum_{i=1}^{n}\ln\left[1-e^{-\frac{\vartheta}{x_i^2}}\right] - \\ & \left(\frac{1}{\gamma}+1\right)\sum_{i=1}^{n}\ln\left\{1+\gamma\left[e^{\frac{\vartheta}{x_i^2}}-1\right]^{-\delta}\right\} \end{aligned} \quad (11)$$

The partial derivatives of $l(\Delta)$ with respect to the model parameters δ,γ and ϑ are

$$\begin{aligned} \frac{\partial l(\Delta)}{\partial\delta} =& n\ln(\delta) - \vartheta\sum_{i=1}^{n} x_i^{-2} - \sum_{i=1}^{n}\ln\left[1-e^{-\frac{\vartheta}{x_i^2}}\right] + \\ & \left(\frac{1}{\gamma}+1\right)\sum_{i=1}^{n}\frac{\left[e^{\frac{\vartheta}{x_i^2}}-1\right]^{-\delta}\ln\left[e^{\frac{\vartheta}{x_i^2}}-1\right]}{1+\gamma\left[e^{\frac{\vartheta}{x_i^2}}-1\right]^{-\delta}} \end{aligned} \quad (12)$$

$$\frac{\partial l(\Delta)}{\partial \vartheta} = n\ln(\vartheta) - \delta \sum_{i=1}^{n} x_i^{-2} - (\delta+1) \sum_{i=1}^{n} \frac{e^{-\frac{\vartheta}{x_i^2}} \frac{1}{x_i^2}}{\left[1-e^{-\frac{\vartheta}{x_i^2}}\right]} + \left(\frac{1}{\gamma}+1\right) \sum_{i=1}^{n} \frac{\delta \frac{1}{x_i^2} e^{\frac{\vartheta}{x_i^2}} \left[e^{\frac{\vartheta}{x_i^2}}-1\right]^{-\delta-1}}{1+\gamma\left[e^{\frac{\vartheta}{x_i^2}}-1\right]^{-\delta}} \tag{13}$$

and

$$\frac{\partial l(\Delta)}{\partial \gamma} = -\left(\frac{-4}{\gamma}\right) \sum_{i=1}^{n} \ln\left\{1+\gamma\left[e^{\frac{\vartheta}{x_i^2}}-1\right]^{-\delta}\right\} \left(\frac{1}{\gamma}+1\right) \sum_{i=1}^{n} \ln \frac{\left[e^{\frac{\vartheta}{x_i^2}}-1\right]^{-\delta}}{1+\gamma\left[e^{\frac{\vartheta}{x_i^2}}-1\right]^{-\delta}} \tag{14}$$

It is possible to obtain the MLE of δ, γ and ϑ by maximizing the last equation with respect to δ, γ and ϑ, equal to zero. Using the Newton-Rapshon method, R packages can be used to maximize the log-likelihood function to obtain an MLE.

4.2 Maximum Product of Spacings Method

The MPS method is used as an alternative to the MLE method adopted by Cheng and Amin [17] to estimate the parameters of continuous univariate models. Many authors used MPS to estimate model parameters based on a complete and different censored sample by Almetwally and Almongy [18], Basu et al. [19], Almetwally et al. [18], El-Sherpieny et al. [20] and Alshenawy et al. [14]. Let $x_1 < x_2 < ... < x_n$ then x_i is order of data.

$$D_i(\Delta) = F(x_{(i)}, \Delta) - F(x_{(i-1)}, \Delta);\ i = 1, ..., n+1 \tag{15}$$

where $D_i(\Delta)$ denotes to the uniform spacings, $F(x_{(0)}, \Delta) = 0$, $F(x_{(n+1)}, \Delta) = 1$ and $\sum_{i=1}^{n+1} D_i(\Delta) = 1$. with respect to δ, γ and ϑ. Further, the MPSE of the EOWIR parameters can also be obtained by first derivatives with parameter and equal zero.

4.3 Bayesian Estimation

As random and parameter uncertainties are represented by a previous joint distribution that is established prior to the data collected on the failure, the Bayesian approach deals with the parameters. The ability to incorporate prior knowledge into research makes the Bayesian method very useful in the analysis of reliability, as one of the main problems associated with reliability analysis is the limited availability of data. In the δ, γ and ϑ parameters, as prior gamma distributions, we have to use the insightful before. The δ, γ and ϑ independent joint prior density function can be written as follows:

$$\Pi(\Delta) \propto \delta^{a_1-1} \gamma^{a_2-1} \vartheta^{a_3-1} e^{-(b_1\delta+b_2\gamma+b_3\vartheta)} \tag{16}$$

From the likelihood function and joint prior function, the joint posterior density function of Δ is obtained. The joint posterior of the distribution of EOWIR can then be written as

$$\Pi(\Delta|x) \propto \vartheta^{n+a_3-1} e^{-\vartheta\left(b_3+\sum_{i=1}^{n}\frac{\delta}{x_i^2}\right)} \prod_{i=1}^{n}\left(1-e^{-\frac{\vartheta}{x_i^2}}\right)^{\delta+1} \delta^{n+a_1-1}\gamma^{a_2-1}e^{-(b_1\delta+b_2\gamma)} \prod_{i=1}^{n}\left\{1+\gamma\left[\frac{e^{-\frac{\vartheta}{x_i^2}}}{1-e^{-\frac{\vartheta}{x_i^2}}}\right]^{\delta}\right\}^{\frac{-1}{\gamma}-1}, \tag{17}$$

Using the most common function for symmetric loss, which is a function for squared error loss. Bayes estimators of $\hat{\Delta}$ based on the squared error loss function are defined by the squared error loss function.

$$S(\tilde{\Delta}) = E\left(\tilde{\Delta}-\Delta\right)^2 \int_0^{\infty} ... \int_0^{\infty} \int_{-1}^{1} \left(\tilde{\Delta}-\Delta\right)^2 \Pi(\Delta|x)\, d\Delta_1 d\Delta_2 d\Delta_3 \tag{18}$$

It is noted that the integrals given by (18) can not be directly obtained. As a result, we use the MCMC to find an approximate value of integrals. A significant sub-class of the MCMC techniques is the Gibbs sampling and more general Metropolis within Gibbs samplers. The MH algorithm, together with the Gibbs sampling, are the two most common instances of the MCMC method. The MH algorithm, similar to acceptance-rejection sampling, believes that for each iteration of the algorithm, a candidate value from a proposal distribution can be produced. We use the MH within the Gibbs sampling steps to produce random samples of conditional posterior densities from the OWIR distribution family:

$$\Pi(\delta|\gamma,\vartheta,x) \propto \delta^{n+a_1-1} e^{-\vartheta\sum_{i=1}^{n}\frac{\delta}{x_i^2}} \prod_{i=1}^{n}\left(1-e^{-\frac{\vartheta}{x_i^2}}\right)^{\delta+1} e^{-b_1\delta} \prod_{i=1}^{n}\left\{1+\gamma\left[\frac{e^{-\frac{\vartheta}{x_i^2}}}{1-e^{-\frac{\vartheta}{x_i^2}}}\right]^{\delta}\right\}^{\frac{-1}{\gamma}-1}, \tag{19}$$

$$\Pi(\gamma|\delta,\vartheta,x) \propto \gamma^{a_2-1} e^{-b_2\gamma} \prod_{i=1}^{n}\left\{1+\gamma\left[\frac{e^{-\frac{\vartheta}{x_i^2}}}{1-e^{-\frac{\vartheta}{x_i^2}}}\right]^{\delta}\right\}^{\frac{-1}{\gamma}-1}, \tag{20}$$

Table 1: Bias and MSE of EOWIR distribution for MLE, MPS and Bayesian when $\delta = 0.5$

				MLE		MPS		Bayesian	
γ	ϑ	n		Bias	MSE	Bias	MSE	Bias	MSE
0.5	0.5	50	δ	0.0321	0.1236	-0.0142	0.0117	0.0152	0.0616
			γ	0.0429	0.0728	0.0547	0.0075	0.1233	0.2704
			ϑ	0.0056	0.1302	0.0169	0.0187	0.0343	0.1132
		100	δ	0.0165	0.0807	-0.0076	0.0057	0.0116	0.0573
			γ	0.0453	0.0569	0.0508	0.0039	0.0913	0.2359
			ϑ	0.0020	0.0874	0.0058	0.0079	0.0213	0.0838
		200	δ	0.0057	0.0543	-0.0069	0.0028	0.0067	0.0461
			γ	0.0445	0.0513	0.0487	0.0032	0.0842	0.2531
			ϑ	0.0027	0.0610	0.0039	0.0038	0.0114	0.0583
0.5	2	50	δ	0.0330	0.1353	-0.0134	0.0142	0.0187	0.0616
			γ	0.1981	0.3522	0.2031	0.1735	0.0938	0.2466
			ϑ	0.0413	0.5277	0.0861	0.3056	0.0606	0.2551
		100	δ	0.0145	0.0782	-0.0097	0.0054	0.0116	0.0502
			γ	0.2141	0.2996	0.2376	0.1213	0.0849	0.2350
			ϑ	0.0003	0.3399	0.0161	0.1179	0.0360	0.2201
		200	δ	0.0065	0.0543	-0.0062	0.0028	0.0073	0.0424
			γ	0.2010	0.2506	0.2223	0.1044	0.0859	0.2450
			ϑ	0.0076	0.2488	0.0129	0.0626	0.0303	0.1951
2	0.5	50	δ	0.0327	0.1395	-0.0141	0.0136	0.0132	0.0621
			γ	0.1615	0.1884	0.1795	0.0425	0.0399	0.0782
			ϑ	0.0050	0.1296	0.0160	0.0186	0.0362	0.1160
		100	δ	0.0089	0.0786	-0.0150	0.0057	0.0063	0.0563
			γ	0.1606	0.1708	0.1708	0.0323	0.0325	0.0753
			ϑ	0.0116	0.0888	0.0158	0.0082	0.0289	0.0859
		200	δ	0.0095	0.0567	-0.0032	0.0030	0.0094	0.0474
			γ	0.1639	0.1708	0.1700	0.0310	0.0405	0.0842
			ϑ	0.0009	0.0630	0.0021	0.0040	0.0101	0.0598
2	2	50	δ	0.0307	0.1287	-0.0150	0.0130	0.0191	0.0598
			γ	0.1924	0.3232	0.2089	0.2053	0.0426	0.0781
			ϑ	0.0616	0.5287	0.1034	0.3081	0.0718	0.2695
		100	δ	0.0164	0.0783	-0.0080	0.0054	0.0134	0.0513
			γ	0.2326	0.3248	0.2255	0.1182	0.0408	0.0815
			ϑ	-0.0081	0.3402	0.0083	0.1184	0.0302	0.2273
		200	δ	0.0089	0.0543	-0.0039	0.0027	0.0087	0.0427
			γ	0.2000	0.2623	0.2118	0.0740	0.0404	0.0841
			ϑ	0.0073	0.2466	0.0127	0.0616	0.0350	0.1956

and

$$\Pi(\vartheta|\delta,\gamma,x) \propto \vartheta^{n+a_3-1} e^{-\vartheta\left(b_3+\sum_{i=1}^{n}\frac{\delta}{x_i^2}\right)} \prod_{i=1}^{n}\left(1-e^{-\frac{\vartheta}{x_i^2}}\right)^{\delta+1} \prod_{i=1}^{n}\left\{1+\gamma\left[\frac{e^{-\frac{\vartheta}{x_i^2}}}{1-e^{-\frac{\vartheta}{x_i^2}}}\right]^{\delta}\right\}^{\frac{-1}{\gamma}-1}. \quad (21)$$

5 Simulation

For comparison between the classical estimation methods, the Monte-Carlo simulation procedure is carried out in this section: MLE, MPS, and Bayesian estimation method under square error loss function based on MCMC, for estimation of EOWIR lifetime distribution parameters by R language. Monte-Carlo experiments are performed on the basis of data-generated 10000 random EOWIR distribution samples, where x has EOWIR lifetime for various parameter actual values and different sample sizes n: (50, 100, and 200). We could describe the best methods of estimators as minimizing estimators' Bias and mean squared error (MSE).

The simulation results of the methods presented in this paper for point estimation are summarized in the tables 1, 2. We consider the Bias and MSE values in order to perform the required comparison between various point estimation methods. The following remarks can be noted from these tables:

1. For fixed actual parameters of EOWIR distribution, the Bias, and MSE decrease as n increases.

Table 2: Bias and MSE of EOWIR distribution for MLE, MPS and Bayesian when $\delta = 2$

				MLE		MPS		Bayesian	
γ	ϑ	n		Bias	MSE	Bias	MSE	Bias	MSE
0.5	0.5	50	δ	0.4795	4.8787	-0.0090	0.6859	0.0021	0.0287
			γ	0.0818	4.5221	0.2541	0.3128	0.0875	0.2510
			ϑ	0.0011	0.0866	0.0177	0.0081	0.0112	0.0546
		100	δ	0.1131	0.4952	-0.0359	0.1803	0.0019	0.0367
			γ	0.1549	0.3271	0.2408	0.1793	0.0668	0.2357
			ϑ	0.0023	0.0589	0.0109	0.0036	0.0075	0.0390
		200	δ	0.0414	0.2957	-0.0384	0.0770	0.0018	0.0453
			γ	0.1865	0.2924	0.2294	0.1065	0.0566	0.2282
			ϑ	0.0009	0.0406	0.0057	0.0017	0.0031	0.0276
0.5	2	50	δ	0.4097	3.8492	0.0270	3.0365	0.0030	0.0295
			γ	0.0338	4.1082	0.1124	6.7676	0.0986	0.2635
			ϑ	0.0197	0.3539	0.0840	0.1374	0.0433	0.1948
		100	δ	0.1294	0.4640	-0.0210	0.1572	0.0047	0.0359
			γ	0.1622	0.2534	0.2000	0.0725	0.0725	0.2558
			ϑ	-0.0080	0.2269	0.0265	0.0530	0.0169	0.1335
		200	δ	0.0574	0.3040	-0.0237	0.0791	0.0046	0.0443
			γ	0.1649	0.2291	0.1942	0.0564	0.0674	0.2511
			ϑ	0.0016	0.1619	0.0211	0.0268	0.0146	0.1016
2	0.5	50	δ	0.2626	1.0583	-0.0411	0.5584	0.0016	0.0292
			γ	0.1276	0.6513	0.2432	0.2641	0.0400	0.0842
			ϑ	0.0053	0.0870	0.0216	0.0085	0.0119	0.0542
		100	δ	0.1092	0.4890	-0.0403	0.1774	0.0019	0.0362
			γ	0.1702	0.3361	0.2307	0.1378	0.0394	0.0878
			ϑ	0.0020	0.0590	0.0107	0.0036	0.0063	0.0379
		200	δ	0.0345	0.2955	-0.0450	0.0779	0.0005	0.0461
			γ	0.1939	0.2783	0.2379	0.1121	0.0357	0.0971
			ϑ	0.0019	0.0418	0.0067	0.0018	0.0031	0.0271
2	2	50	δ	0.2947	1.5241	-0.0255	0.9694	0.0032	0.0290
			γ	0.0724	1.8363	0.1928	0.2962	0.0379	0.0848
			ϑ	0.0330	0.3822	0.0983	0.1635	0.0428	0.1937
		100	δ	0.1128	0.5180	-0.0400	0.1919	0.0021	0.0375
			γ	0.1575	0.2684	0.2049	0.0832	0.0376	0.0864
			ϑ	0.0098	0.2379	0.0449	0.0586	0.0226	0.1381
		200	δ	0.0511	0.3031	-0.0297	0.0791	0.0027	0.0446
			γ	0.1686	0.2249	0.1938	0.0571	0.0403	0.0905
			ϑ	-0.0019	0.1658	0.0176	0.0280	0.0091	0.1070

2. For fixed γ, δ and sample size, the Bias, and MSE decreases as ϑ increases.

3. For fixed ϑ, δ and sample size, then the Bias, and MSE decreases as γ increases.

4. For fixed ϑ, γ and sample size, then the Bias, and MSE decreases as δ increases.

5. Bayesian estimation is the best estimation method.
6. MPS estimation is a better alternative method of MLE.

6 Applications to Carbon Fibres data

Two actual carbon fiber breaking data are provided in this Section to evaluate the consistency of the EOWIR distribution. Other related models such as X-Gamma inverse Weibull (XGIW) [Ibrahim and Almetwally [16]], generalized inverse Weibull (GIW) [De Gusmao et al. [21]], Exponentiated generalized inverse Weibull (EGIW) [Elbatal and Muhammed [22]], Topp-Leone inverse Kumaraswamy (TLIK) [Reyad et al. [24]] and Exponential Lomax (EL) [El-Bassiouny et al. [23]] are compared with the EOWIR model. Tables 3 and 5 provide the Akaike Information Criterion (AIC) values, the corrected Akaike Information Criterion (CAIC), the Bayesian Information Criterion (BIC), the Hannan-Quinn Information Criterion (HQIC), and the Kolmogorov-Smirnov (KS) statistics, along with the P-value for all models fitted on the basis of two real data sets.

In their work on the transmuted distribution of Frechet with three parameters, Mahmoud and Mandouh [25] used the first data set to be analyzed. The data set consists of

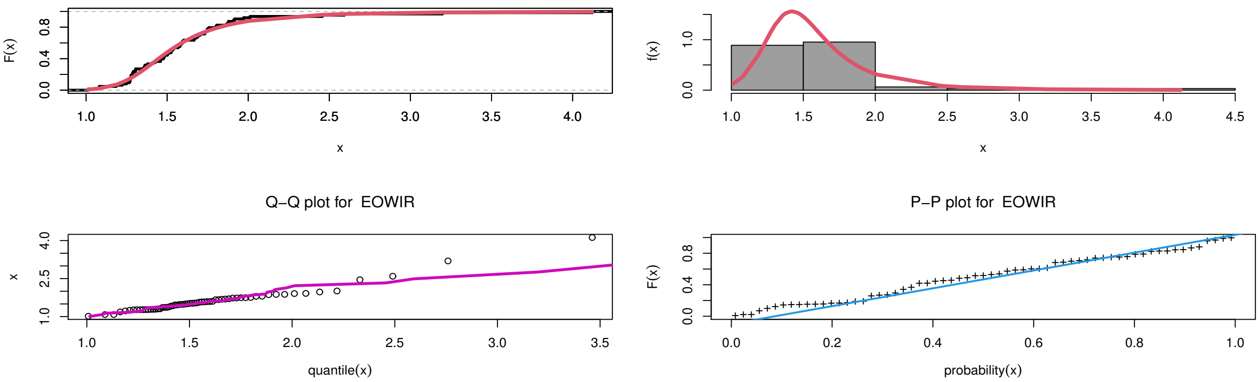

Fig. 3: Estimated PDF, PP-plot and QQ-plot of EOWIR for strengths of glass fibers data.

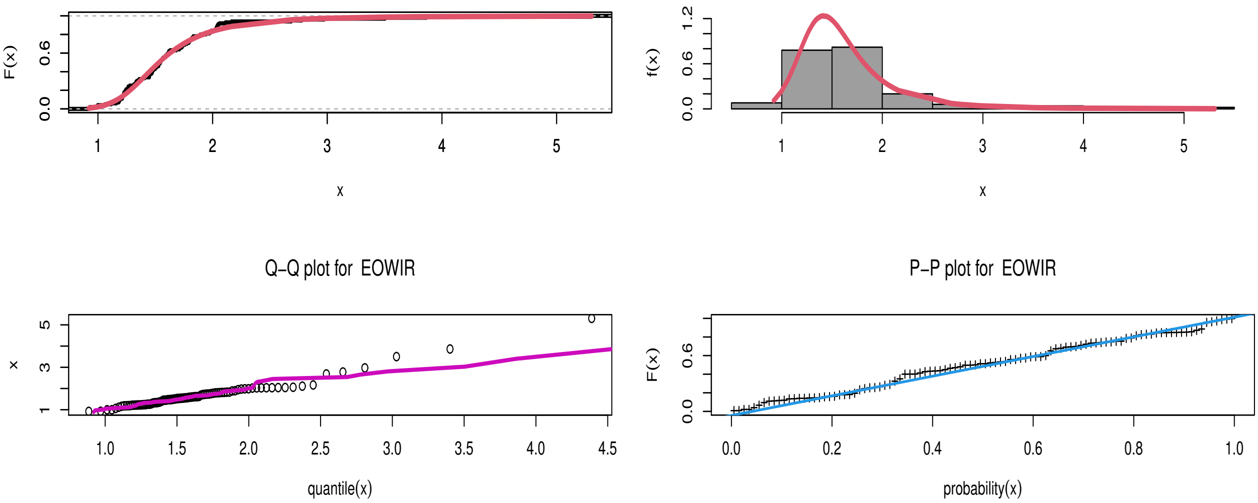

Fig. 4: Estimated PDF, PP-plot and QQ-plot of EOWIR for strengths of carbon fibres data.

100 observations on carbon fiber breaking stress (in Gba) as follows: 0.92, 0.928, 0.997, 0.9971, 1.061, 1.117, 1.162, 1.183, 1.187, 1.192, 1.196, 1.213, 1.215, 1.2199, 1.22, 1.224, 1.225, 1.228, 1.237, 1.24, 1.244, 1.259, 1.261, 1.263, 1.276, 1.31, 1.321, 1.329, 1.331, 1.337, 1.351, 1.359, 1.388, 1.408, 1.449, 1.4497, 1.45, 1.459, 1.471, 1.475, 1.477, 1.48, 1.489, 1.501, 1.507, 1.515, 1.53, 1.5304, 1.533, 1.544, 1.5443, 1.552, 1.556, 1.562, 1.566, 1.585, 1.586, 1.599, 1.602, 1.614, 1.616, 1.617, 1.628, 1.684, 1.711, 1.718, 1.733, 1.738, 1.743, 1.759, 1.777, 1.794, 1.799, 1.806, 1.814, 1.816, 1.828, 1.83, 1.884, 1.892, 1.944, 1.972, 1.984, 1.987, 2.02, 2.0304, 2.029, 2.035, 2.037, 2.043, 2.046, 2.059, 2.111, 2.165, 2.686, 2.778, 2.972, 3.504, 3.863, and 5.306.

In their work on the transmuted distribution of Frechet with three parameters, Mahmoud and Mandouh [25] used the second data set to be analyzed. The data set is of simulated glass fiber strengths and is presented as follows: 1.014, 1.081, 1.082, 1.185, 1.223, 1.248, 1.267, 1.271, 1.272, 1.275, 1.276, 1.278, 1.286, 1.288, 1.292, 1.304, 1.306, 1.355, 1.361, 1.364, 1.379, 1.409, 1.426, 1.459, 1.460, 1.476, 1.481, 1.484, 1.501, 1.506, 1.524, 1.526, 1.535, 1.541, 1.568, 1.579, 1.581, 1.591, 1.593, 1.602, 1.666, 1.670, 1.684, 1.691, 1.704, 1.731, 1.735,

Table 3: MLE estimates, SE, AIC, BIC, ACIC, and HQIC for strengths of glass fibers data

	EOWIR		XGIW		GIW		EGIW		TLIK		EL	
	esimate	SE	esimate	SE	esimate	SE	esimate	SE	esimate	SE	esimate	SE
δ	3.955	0.925	1.964	1.175	2.153	3.513	50.802	43.251	1.042	0.810	243.217	162.346
γ	1.765	0.716	5.014	2.086	0.100	0.890	95.364	37.919	8.635	6.247	26.247	28.681
ϑ	1.501	0.114	6.444	1.164	5.437	0.519	0.272	0.038	5.703	3.453	6.148	8.024
λ							1145.816	145.613				
AIC	44.468		48.143		46.127		48.814		45.434		49.002	
BIC	50.898		54.573		52.557		57.387		51.863		55.431	
CAIC	44.875		48.550		46.534		49.504		45.841		49.408	
HQIC	46.997		50.672		48.656		52.186		47.963		51.530	

Table 4: MLE, MPS, and Bayesian estimates, SE for strengths of carbon fibres data

	MLE		MPS		Bayesian	
	esimate	SE	esimate	SE	esimate	SE
δ	3.9556	0.9253	3.8076	0.9453	4.0693	0.7574
γ	1.7650	0.7159	1.8511	0.6918	1.9270	0.6467
ϑ	1.5006	0.1140	1.4903	0.9043	1.4989	0.1016

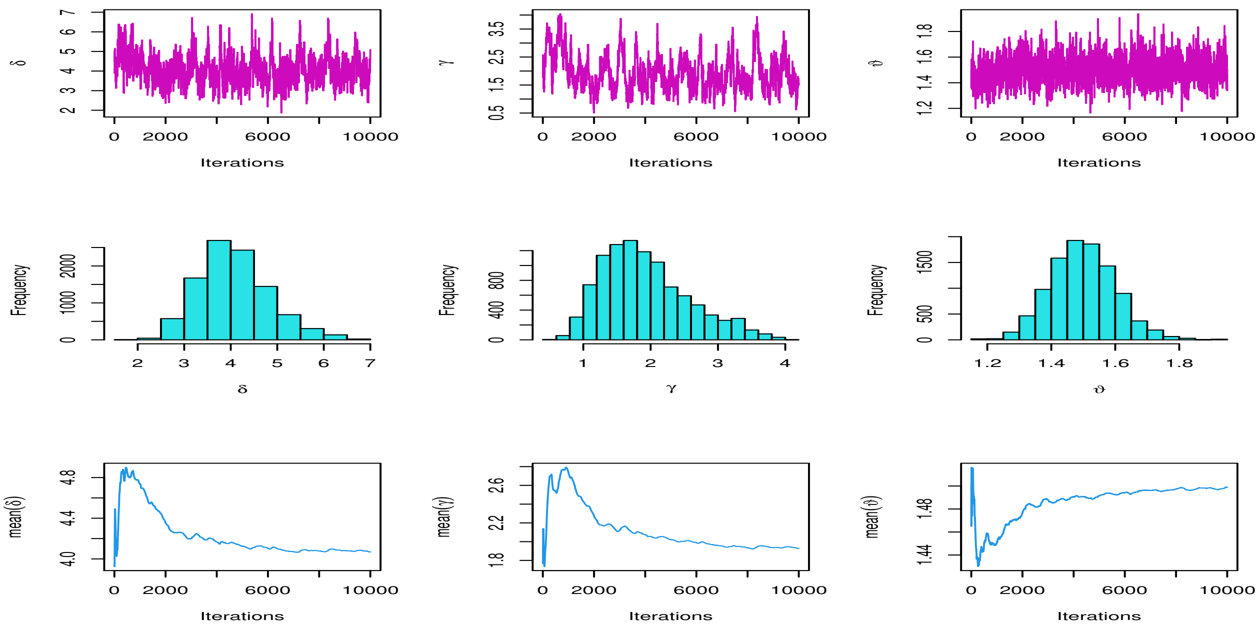

Fig. 5: Convergence of MCMC estimation of EOWIR for strengths of carbon fibres data.

1.747, 1.748, 1.757, 1.800, 1.806, 1.867, 1.876, 1.878, 1.910, 1.916, 1.972, 2.012, 2.456, 2.592, 3.197, and 4.121.

It is evident from Tables 3 and 5 that, in comparison with other distributions, the EOWIR distribution has minimum values for all knowledge parameters. This leads us to believe that EOWIR fits the two real data sets better. The fitted EOWIR PDF, CDF, SF, PP, and QQ-plots of the two data sets are displayed in Figures 3 and 4. Figures 3 and 4 shows that the Q-Q and P-P plots suggest that our distribution is a good choice for modeling the actual data above. By Tables, 4 and 6, the Bayesian estimation method of EOWIR distribution is the best estimation method. History plots, approximate marginal posterior density and MCMC convergence of δ,γ and ϑ are represented in Figures 5 and 6.

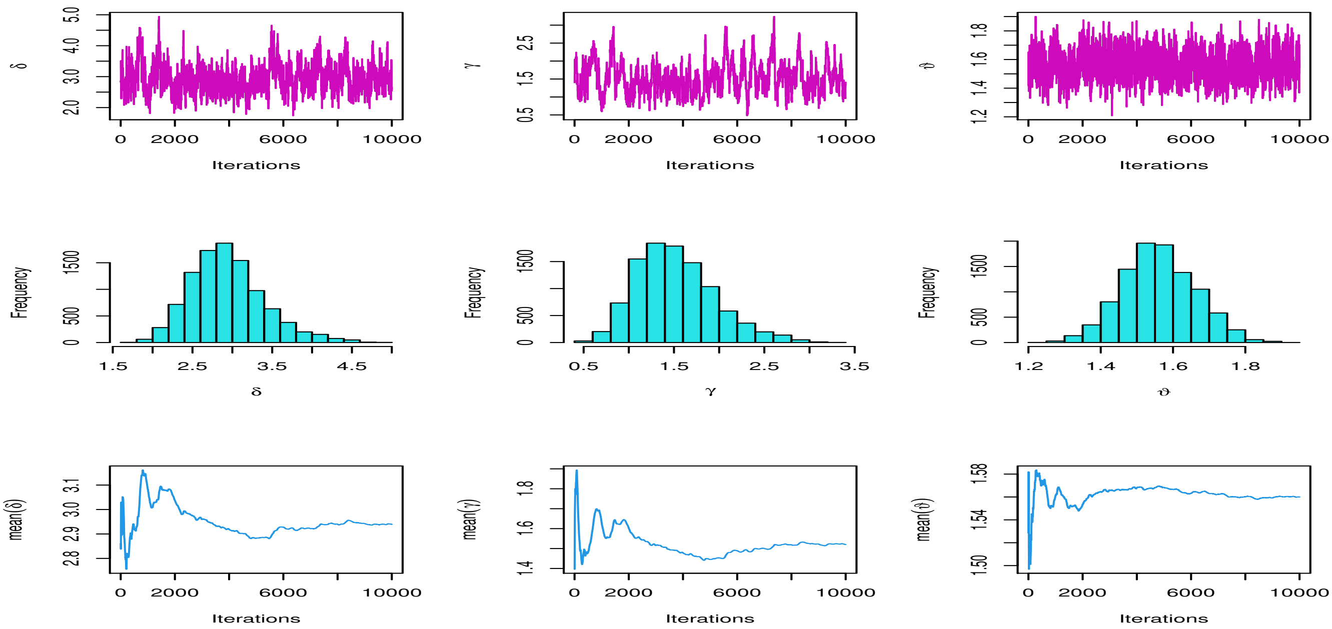

Fig. 6: Convergence of MCMC estimation of EOWIR for stress of carbon fibres data.

Table 5: MLE estimates, SE, KS test, P-value AIC, BIC, ACIC, and HQIC for stress of carbon fibres data

	EOWIR		XGIW		GIW		EGIW		TLIK		EL	
	esimate	SE	esimate	SE	esimate	SE	esimate	SE	esimate	SE	esimate	SE
δ	2.860	0.504	2.679	1.355	2.078	6.587	9.478	21.447	1.178	0.788	116.417	59.669
γ	1.396	0.463	3.338	1.073	0.176	2.438	39.026	64.288	6.912	4.917	14.008	6.908
ϑ	1.566	0.115	4.697	0.615	4.373	0.328	0.500	0.280	4.342	2.180	3.520	2.372
λ							21.370	16.055				
KS	0.064		0.086		0.087		0.073		0.073		0.088	
P-value	0.811		0.456		0.429		0.662		0.667		0.420	
AIC	109.894		114.688		113.383		116.959		111.120		114.774	
BIC	117.709		122.503		121.199		127.380		118.936		122.590	
CAIC	110.144		114.938		113.633		117.381		111.370		115.024	
HQIC	113.057		117.851		116.546		121.177		114.283		117.937	

7 Conclusion

A new generalization of inverse Rayleigh and Weibull distributions called EOWIR distribution is formulated in this paper. We studied its statistical properties and obtained a linear representation for its pdf that was successful in finding the function and quantile function of moments and moment generation. Point estimation of the EOWIR unknown parameters δ, γ, and ϑ was considered by MLE, MPS, and Bayesian estimation methods. To distinguish the performance of different estimation methods, a comparison was carried out through simulation analysis using the R package. For that reason, the MCMC approach was used, real data sets were also considered, and EOWIR was shown to match these data of carbon fibers better compared to other competitive distributions. Bayesian estimation is the best estimation method for estimate parameters of EOWIR distribution.

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Table 6: MLE, MPS, and Bayesian estimates, SE for stress of carbon fibres data

	MLE		MPS		Bayesian	
	esimate	SE	esimate	SE	esimate	SE
δ	2.8604	0.5038	2.8145	0.4919	2.9395	0.4655
γ	1.3958	0.4629	1.4739	0.4671	1.5200	0.4378
ϑ	1.5656	0.1148	0.1455	1.1460	1.5600	0.1008

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