Third-Order Differential Subordination and Differential Superordination Results for Analytic Functions Involving the Srivastava-Attiya Operator

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Abstract: In this article, by making use of the linear operator introduced and studied by Srivastava and Attiya [16], suitable classes of admissible functions are investigated and the dual properties of the third-order differential subordinations are presented. As a consequence, various sandwich-type theorems are established for a class of univalent analytic functions involving the celebrated Srivastava-Attiya transform. Relevant connections of the new results presented here with those that were considered in earlier works are pointed out.

Keywords: Analytic functions, univalent functions, differential subordination, differential superordination, srivastava-Attiya operator, sandwich-type theorems, admissible functions

1 Introduction

Let be the class of functions which are analytic in the open unit disk

\[ U := \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \} . \]

Also let

\[ H[a, n] \quad (n \in \mathbb{N} := \{1, 2, 3, \ldots \}; a \in \mathbb{C}) \]

be the subclass of the analytic function class \( H \) consisting of functions of the following form:

\[ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \quad (z \in U) . \]

Let \( \mathcal{A} (\subset H) \) be the class of functions which are analytic in \( U \) and have the normalized Taylor-Maclaurin series of the form:

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in U) . \]

Suppose that \( f \) and \( g \) are in \( \mathcal{H} \). We say that \( f \) is subordinate to \( g \) (or \( g \) is superordinate to \( f \)), written as follows:

\[ f \prec g \text{ in } U \quad \text{or} \quad f(z) \prec g(z) \quad (z \in U), \]

if there exists a function \( \omega \in \mathcal{H} \), satisfying the conditions of the Schwarz lemma, namely

\[ \omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in U) \]

such that

\[ f(z) = g(\omega(z)) \quad (z \in U). \]

It follows that

\[ f(z) \prec g(z) \quad (z \in U) \implies f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U). \]

In particular, if \( g \) is univalent in \( U \), then the reverse implication also holds true (see, for details, [11]).

The concept of differential subordination is a generalization of various inequalities involving complex
variables. We recall here some more definitions and terminologies from the theory of differential subordination and differential superordination.

**Definition 1.** (see [1]) Let \( \psi : \mathbb{C}^4 \times U \rightarrow \mathbb{C} \) and suppose that the function \( h(z) \) is univalent in \( U \). If the function \( p(z) \) is analytic in \( U \) and satisfies the following third-order differential subordination:

\[
\psi(p(z),zp'(z),z^2p''(z),z^3p'''(z);z) \prec h(z),
\]

then \( p(z) \) is called a solution of the differential subordination (2). Furthermore, a given univalent function \( q(z) \) is called a dominant of the solutions of the differential subordination (2) or, more simply, a dominant if \( p(z) \prec q(z) \) for all \( p(z) \) satisfying (2). A dominant \( \bar{q}(z) \) that satisfies \( \bar{q}(z) \prec q(z) \) for all dominants \( q(z) \) of (2) is said to be the best dominant.

**Definition 2.** (see [23]) Let \( \psi : \mathbb{C}^4 \times U \rightarrow \mathbb{C} \) and let the function \( h(z) \) be univalent in \( U \). If the function \( p(z) \) and

\[
\psi(p(z),zp'(z),z^2p''(z),z^3p'''(z);z)
\]

are univalent in \( U \) and satisfy the following third-order differential superordination:

\[
h(z) \prec \psi(p(z),zp'(z),z^2p''(z),z^3p'''(z);z),
\]

then \( p(z) \) is called a solution of the differential superordination given by (3). An analytic function \( q(z) \) is called a subordinant of the solutions of the differential superordination given by (3) (or, more simply, a subordinant) if \( q(z) \prec p(z) \) for all \( p(z) \) satisfying (3).

A univalent subordinant \( \bar{q}(z) \) that satisfies \( q(z) \prec \bar{q}(z) \) for all subordinants \( q(z) \) of (3) is said to be the best subordinant of the differential superordination given by (3). We note that both the best dominant and the best subordinant are unique up to rotation of \( U \). The monograph by Miller and Mocanu [11] and the more recent book of Bulboaca [2] provide detailed expositions on the theory of differential subordination and differential superordination.

With a view to defining the Srivastava-Attiya operator, we recall here the general Hurwitz-Lerch Zeta function, which is defined by the following series (see, for example, [17]):

\[
\Phi(z,\mu,b) = \sum_{n=0}^{\infty} \frac{z^n}{(b+n)^\mu} \quad (b \in \mathbb{C} \setminus \mathbb{Z}_0^+; \mu \in \mathbb{C} \text{ when } z \in U; \Re(\mu) > 0 \text{ when } z \in \partial U),
\]

where \( \mathbb{Z}_0^+ \) denotes the set of non-positive integers.

Special cases of the function \( \Phi(z,\mu,b) \) include, for example, the Riemann Zeta function given by

\[
\xi(\mu) = \Phi(1,\mu,1),
\]

the Hurwitz Zeta function given by

\[
\zeta(\mu,b) = \Phi(1,\mu,b),
\]

the Lerch Zeta function given by

\[
\ell_\mu \vartheta = \Phi(\frac{2\pi i \vartheta}{\mu},1) \quad (\vartheta \in \mathbb{R}; \Re(\mu) > 1),
\]

the Polylogarithm function given by

\[
\text{Li}_\mu = z \Phi(z,\mu,1),
\]

and so on (see, for further details, [19]).

Srivastava and Attiya [16] considered the following normalized function:

\[
R_{\mu,b}(z) = (1+b)^\mu [\Phi(z,\mu,b) - b^{-\mu}]
\]

where \( z \in U \). For further details, we refer the interested reader to the earlier work [13].
Definition 3. (see [1]) Let \( \mathbb{Q} \) be the set of all functions \( q \) that are analytic and univalent on \( \mathbb{U} \setminus E(q) \), where
\[
\mathbb{Q} = \left\{ q : \xi \in \partial U : \lim_{\zeta \to \xi} q(z) = \infty \right\},
\] (8)
and are such that \( \min |q'(\xi)| = \rho > 0 \) for \( \xi \in \partial U \setminus E(q) \). Further, let the subclass of \( \mathbb{Q} \) for which \( q(0) = a \) be denoted by \( \mathbb{Q}(a) \) with
\[
\mathbb{Q}(0) = \mathbb{Q}_0 \quad \text{and} \quad \mathbb{Q}(1) = \mathbb{Q}_1.
\] (9)

The subordination methodology is applied to appropriate classes of admissible functions. The following class of admissible functions is given by Antonino and Miller [1].

Definition 4. (see [1]) Let \( \Omega \) be a set in \( \mathbb{C} \). Also let \( q \in \mathbb{Q} \) and \( n \in \mathbb{N} \setminus \{ 1 \} \), \( \mathbb{N} \) being the set of positive integers. The class \( \mathbb{C}_q[\Omega, q] \) of admissible functions consists of those functions \( \psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C} \), which satisfy the following admissibility conditions:
\[
\psi(r, s, t, u; z) \notin \Omega
\]
whenever
\[
(8)
\]
and
\[
(9)
\]
where \( \zeta \in \mathbb{U} \), \( \zeta \in \partial U \setminus E(q) \) and \( k \geq n \).

Lemma 1 below is the foundation result in the theory of third-order differential subordination.

Lemma 1. (see [1]) Let \( p \in \mathbb{H}[a, n] \) with \( n \geq 2 \) and \( q \in \mathbb{Q}(a) \) satisfying the following conditions:
\[
\mathbb{H} \left( \frac{\zeta q''(\zeta)}{q'(\zeta)} \right) \geq 0 \quad \text{and} \quad \left| \frac{zq'(z)}{q'(\zeta)} \right| \leq k,
\]
where \( \zeta \in \mathbb{U} \), \( \zeta \in \partial U \setminus E(q) \) and \( k \geq n \). If \( \Omega \) is a set in \( \mathbb{C} \), \( \psi \in \mathbb{C}_q[\Omega, q] \) and
\[
\psi \left( p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z \right) \subset \Omega,
\]
then
\[
p(z) \prec \psi(z) \quad (z \in \mathbb{U}).
\]

Definition 5. (see [23]) Let \( \Omega \) be a set in \( \mathbb{C} \). Also let \( q \in \mathbb{H}[a, n] \) and \( q'(z) \neq 0 \). The class \( \mathbb{C}_q[\Omega, q] \) of admissible functions consists of those functions \( \psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C} \) that satisfy the following admissibility conditions:
\[
\psi(r, s, t, u; \xi) \in \Omega
\]
whenever
\[
r = q(z), \ s = \frac{zq'(z)}{q'(\zeta)} \quad \text{and} \quad \left| \frac{zq'(z)}{q'(\zeta)} \right| \leq m,
\]
where \( \zeta \in \mathbb{U} \), \( \zeta \in \partial U \) and \( m \geq n \).

Lemma 2. (see [23]) Let \( p \in \mathbb{H}[a, n] \) with \( \psi \in \mathbb{C}_q[\Omega, q] \). If the function
\[
\psi \left( p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z \right)
\]
is univalent in \( \mathbb{U} \) and \( p \in \mathbb{Q}(a) \) satisfying the following conditions:
\[
\mathbb{H} \left( \frac{zq''(z)}{q'(z)} \right) \geq 0 \quad \text{and} \quad \left| \frac{zq'(z)}{q'(\zeta)} \right| \leq m,
\]
where \( \zeta \in \mathbb{U} \), \( \zeta \in \partial U \) and \( m \geq n \), then
\[
\Omega \subseteq \left\{ \psi \left( p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z \right) : z \in \mathbb{U} \right\}
\]
implies that
\[
q(z) \prec p(z) \quad (z \in \mathbb{U}).
\]

The notion of the third-order differential subordination can be found in the work of Ponnuasamy and Juneja [14]. The recent works by Tang et al. (see, for example, [22] and [23]; see also [3]) on the third-order differential subordination attracted many researchers in this field. For example, see [5, 9, 10, 12, 14, 15, 21, 22, 23]. In the present paper, we investigate suitable classes of admissible functions associated with the Srivastava-Attiya operator \( J_{a,b} f(z) \) and derive sufficient conditions on the normalized analytic function \( f \) such that Sandwich-type subordination of the following form holds true:
\[
h_1(z) \prec \vartheta(f) \prec h_2(z) \quad (z \in \mathbb{U}),
\]
where \( h_1, h_2 \) are univalent in \( \mathbb{U} \) and \( \vartheta \) is a suitable operator.

2 Results Related to the Third-Order Subordination

In this section, we start with a given set \( \Omega \) and a given function \( q \) and we determine a set of admissible operators \( \psi \) so that (2) holds true. For this purpose, we introduce the following new class of admissible functions which will be required to prove the main third-order differential subordination theorems for the operator \( J_{a,b} f(z) \) defined by (5).
Definition 6. Let $\Omega$ be a set in $\mathbb{C}$ and $q \in \mathbb{Q}_0 \cap \mathcal{H}_0$. The class $\Theta_j[\Omega,q]$ of admissible function consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{U} \rightarrow \mathbb{C}$ that satisfy the following admissibility conditions:

$$\phi(\alpha, \beta, \gamma, \delta; z) \notin \Omega$$

whenever

$$\alpha = q(\zeta), \beta = \frac{kq^2(\zeta) + bq(\zeta)}{b + 1},$$

and

$$\Re \left( \frac{\phi(b + 1)^2 - b\alpha\beta}{\beta_{(b + 1) - b\alpha} - 2b} \right) \geq k \Re \left( \frac{\zeta q'(\zeta)}{q'(\zeta) + 1} \right),$$

where $z \in \mathbb{U}$.

Theorem 1. Let $\phi \in \Theta_j[\Omega,q]$. If the function $f \in \mathcal{S}$ and $q \in \mathbb{Q}_0$ satisfy the following conditions:

$$\Re \left( \frac{\zeta q'(\zeta)}{q'(\zeta)} \right) \geq 0, \quad \left| J_{\mu,b,f}(z) \right| \leq k,$$  \hspace{1cm} (10)

then

$$J_{\mu+1,b,f}(z) \prec q(z) \quad (z \in \mathbb{U}).$$  \hspace{1cm} (11)

Proof. Define the analytic function $p(z)$ in $\mathbb{U}$ by

$$p(z) = J_{\mu+1,b,f}(z).$$  \hspace{1cm} (12)

From equation (7) and (12), we have

$$J_{\mu,b,f}(z) = \frac{zp'(z) + bp(z)}{b + 1}.$$  \hspace{1cm} (13)

By a similar argument, we get

$$J_{\mu-1,b,f}(z) = \frac{z^2p''(z) + (2b + 1)zp'(z) + b^2p(z)}{(b + 1)^2}.$$  \hspace{1cm} (14)

and

$$J_{\mu-2,b,f}(z) = \frac{z^3p'''(z) + (3b + 3)z^2p''(z) + (3b^2 + 3b + 1)z^2p'(z) + b^3p(z)}{(b + 1)^3}.$$  \hspace{1cm} (15)

Define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$\alpha(r,s,t,u) = r, \quad \beta(r,s,t,u) = \frac{s + br}{b + 1},$$

$$\gamma(r,s,t,u) = \frac{t + (2b + 1)s + b^2r}{(b + 1)^2}.$$  \hspace{1cm} (16)

and

$$\delta(r,s,t,u) = \frac{u + (3b + 3)t + (3b^2 + 3b + 1)s + b^3r}{(b + 1)^3}.$$  \hspace{1cm} (17)

Let

$$\psi(r,s,t,u) = \phi(\alpha, \beta, \gamma, \delta; z).$$  \hspace{1cm} (18)

The proof will make use of Lemma 1. Using the equations (12) to (15), and from the equation (18), we have

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \in \Omega.$$  \hspace{1cm} (19)

Hence, clearly, (11) becomes

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \in \Omega.$$  \hspace{1cm} (20)

We note that

$$\frac{t}{s} + 1 = \frac{\gamma(b + 1)^2 - b^2\alpha}{(b + 1)(b - \alpha)} - 2b$$

and

$$\frac{u}{s} = \frac{\delta(b + 1)^3 - \gamma(b + 1)^2(3b + 3) + b^2\alpha(3 + 2b)}{(b + 1)(b - \alpha) + b + 1}.$$  \hspace{1cm} (21)

Thus, clearly, the admissibility condition for $\phi \in \Theta_j[\Omega,q]$ in Definition 6 is equivalent to the admissibility condition for $\psi \in \mathcal{U}_j[\Omega,q]$ as given in Definition 4 with $n = 2$. Therefore, by using (10) and Lemma 1, we have

$$J_{\mu+1,b,f}(z) \prec q(z).$$

This completes the proof of Theorem 1.

Our next result is a consequence of Theorem 1 for the case when the behavior of $q(z)$ on $\partial \mathbb{U}$ is not known.

Corollary 1. Let $\Omega \subset \mathbb{C}$ and let the function $q$ be univalent in $\mathbb{U}$ with $q(0) = 0$. Let $\phi \in \Theta_j[\Omega,q_\rho]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If the function $f \in \mathcal{S}$ and $q_\rho$ satisfies the following conditions:

$$\Re \left( \frac{\zeta q'(\zeta)}{q'(\zeta)} \right) \geq 0, \quad \left| J_{\mu,b,f}(z) \right| \leq k \quad (z \in \mathbb{U}; k \geq 2; \zeta \in \partial \mathbb{U} \setminus E(q_\rho))$$

and

$$\phi(J_{\mu+1,b,f}(z), J_{\mu,b,f}(z), J_{\mu-1,b,f}(z), J_{\mu-2,b,f}(z); z) \in \Omega,$$

then

$$J_{\mu+1,b,f}(z) \prec q(z) \quad (z \in \mathbb{U}).$$

Proof. By applying Theorem 1, we get

$$J_{\mu+1,f}(z) \prec q_\rho(z) \quad (z \in \mathbb{U}).$$

The result asserted by Corollary 1 is now deduced from the following subordination property

$$q_\rho(z) \prec q(z) \quad (z \in \mathbb{U}).$$

This completes the proof of Corollary 1.
If \( \Omega \neq \mathbb{C} \) is a simply-connected domain, then \( \Omega = h(\mathbb{U}) \) for some conformal mapping \( h(z) \) of \( \mathbb{U} \) onto \( \Omega \). In this case, the class \( \Theta_j[h(\mathbb{U}), q] \) is written as \( \Theta_j[h, q] \). This leads to the following immediate consequence of Theorem 1.

**Theorem 2** Let \( \phi \in \Theta_j[h, q] \). If the function \( f \in \mathscr{A} \) and \( q \in \mathbb{Q}_0 \) satisfy the following conditions:

\[
\Re \left( \frac{\zeta q''(\zeta)}{q'(\zeta)} \right) \geq 0, \quad \left| J_{\mu,b} \frac{f(z)}{q'(\zeta)} \right| \leq k, \quad (20)
\]

and

\[
\phi \left( J_{\mu+1,b} f(z), J_{\mu,b} f(z), J_{\mu-1,b} f(z), J_{\mu-2,b} f(z); z \right) < h(z), \quad (z \in \mathbb{U}).
\]

then

\[
J_{\mu+1,b} f(z) < q(z) \quad (z \in \mathbb{U}).
\]

The next result is an immediate consequence of Corollary 1.

**Corollary 2** Let \( \Omega \subset \mathbb{C} \) and let the function \( q \) be univalent in \( \mathbb{U} \) with \( q(0) = 0 \). Also let \( \phi \in \Theta_j[h, q_0] \) for some \( \rho \in (0, 1) \), where \( q_0(z) = q(\rho z) \). If the function \( f \in \mathscr{A} \) and \( q_0 \) satisfy the following conditions:

\[
\Re \left( \frac{\zeta q''(\zeta)}{q'(\zeta)} \right) \geq 0, \quad \left| J_{\mu,b} \frac{f(z)}{q'(\zeta)} \right| \leq k \quad (z \in \mathbb{U}; k \geq 2; \zeta \in \partial \mathbb{U} \setminus E(q_0)),
\]

and

\[
\phi \left( J_{\mu+1,b} f(z), J_{\mu,b} f(z), J_{\mu-1,b} f(z), J_{\mu-2,b} f(z); z \right) < h(z),
\]

then

\[
J_{\mu+1,b} f(z) < q(z) \quad (z \in \mathbb{U}).
\]

The following result yields the best dominant of the differential subordination (21).

**Theorem 3** Let the function \( h \) be univalent in \( \mathbb{U} \). Also let \( \phi : \mathbb{C}^4 \times \mathbb{U} \longrightarrow \mathbb{C} \) and \( \psi \) be given by (18). Suppose that the following differential equation:

\[
\psi(q(z), qz'(z), z^2 q''(z), z^3 q'''(z); z) = h(z), \quad (22)
\]

has a solution \( q(z) \) with \( q(0) = 0 \), which satisfies the condition (10). If the function \( f \in \mathscr{A} \) satisfies the condition (21) and if

\[
\phi \left( J_{\mu+1,b} f(z), J_{\mu,b} f(z), J_{\mu-1,b} f(z), J_{\mu-2,b} f(z); z \right)
\]

is analytic in \( \mathbb{U} \), then

\[
J_{\mu+1,b} f(z) < q(z) \quad (z \in \mathbb{U})
\]

and \( q(z) \) is the best dominant.

**Proof.** From Theorem 1, we see that \( q \) is a dominant of (21). Since \( q \) satisfies (22), it is also a solution of (21). Therefore, \( q \) will be dominated by all dominants. Hence \( q \) is the best dominant. This completes the proof of Theorem 3.
Proof. Let 
\[ \phi(\alpha, \beta, \gamma, \delta; z) = \beta - \alpha \quad \text{and} \quad \Omega = h(U), \]
where
\[ h(z) = \frac{Mz}{|b + 1|} \quad (M > 0). \]

Use Corollary 3, we need to show that \( \phi \in \Theta|/[\Omega, M] \), that is, that the admissibility condition (23) is satisfied. This follows readily, since it is seen that
\[ |\phi(y, w, x, y; z)| = \left| \frac{(k - 1)Mz^\theta}{b + 1} \right| \geq \frac{M}{|b + 1|} \]
whenever \( z \in U, \theta \in \mathbb{R} \) and \( k \geq 2 \). The required result now follows from Corollary 3. This completes the proof of Corollary 5.

**Definition 8.** Let \( \Omega \) be a set in \( \mathbb{C} \) and \( q \in \mathbb{Q}_1 \cap H_j \). The class \( \Theta_{1, j}[\Omega, q] \) of admissible functions consists of those functions \( \phi : \mathbb{C}^4 \times U \rightarrow \mathbb{C} \) that satisfy the following admissibility condition:
\[ \phi(\alpha, \beta, \gamma, \delta; z) \notin \Omega \]
whenever
\[ \alpha = q(\zeta), \quad \beta = \frac{k\zeta q'(\zeta) + (b + 1)q(\zeta)}{b + 1}, \]
\[ \Re \left( \frac{(b + 1)(\gamma - \alpha)}{\beta - \alpha} - 2(1 + b) \right) \geq k \Re \left( \frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right) \]
and
\[ \Re \left( \frac{(1 + b)^2 - 3\gamma(b + 2)(b + 1) + 3\alpha(b + 2)(b + 1) - (1 + b)^2}{\beta - \alpha} + 3b^2 + 12b + 11 \right) \]
\[ \geq k \Re \left( \frac{\zeta q''(\zeta)}{q'(\zeta)} \right). \]

where \( z \in U, \zeta \in \partial U \setminus E(q) \) and \( k \geq 2 \).

**Theorem 4** Let \( \phi \in \Theta_{1, j}[\Omega, q] \). If the function \( f \in \mathbb{A}_d \) and \( q \in \mathbb{Q}_1 \) satisfy the following conditions:
\[ \Re \left( \frac{\zeta q''(\zeta)}{q'(\zeta)} \right) \geq 0, \quad \left| \frac{J_{\mu, b}f(z)}{z^{q'(\zeta)}} \right| \leq k \]
and
\[ \left\{ \phi \left( \frac{J_{\mu, 1, b}f(z)}{z}, \frac{J_{\mu, b}f(z)}{z}, \frac{J_{\mu, 1, b}f(z)}{z}, \frac{J_{\mu, b}f(z)}{z}; z \in U \right) \right\} < \Omega, \]
then
\[ J_{\mu + 1, b}f(z) \propto q(z) \quad (z \in U). \]

**Proof.** Define the analytic function \( p(z) \) in \( U \) by
\[ p(z) = \frac{J_{\mu, 1, b}f(z)}{z}. \]
From the equations (7) and (26), we have
\[ \frac{J_{\mu, b}f(z)}{z} = \frac{z p'(z) + (b + 1)p(z)}{b + 1}. \]

By a similar argument, we get
\[ \frac{J_{\mu - 1, b}f(z)}{z} = \frac{z^2 p''(z) + z p'(z)(3 + 2b) + p(z)(1 + b)^2}{(b + 1)^2} \]
and
\[ \frac{J_{\mu - 2, b}f(z)}{z} = \frac{z^3 p'''(z) + 3(b + 2)z^2 p''(z) + (3b^2 + 9b + 7)z p'(z) + p(z)(b + 1)^3}{(b + 1)^3}. \]

We now define the transformation from \( \mathbb{C}^4 \) to \( \mathbb{C} \) by
\[ \alpha(r, s, t, u) = r, \quad \beta(r, s, t, u) = \frac{s + (b + 1)r}{(b + 1)}, \]
\[ \gamma(r, s, t, u) = \frac{t + (3 + 2b)s + (b + 1)^2r}{(b + 1)^2}, \]
and
\[ \delta(r, s, t, u) = \frac{u + 3(b + 2)r + (3b^2 + 9b + 7)s + (b + 1)^3r}{(b + 1)^3}. \]

Let
\[ \psi(r, s, t, u, \phi(\alpha, \beta, \gamma, \delta; z)) = \phi \left( \frac{r + (b + 1)x}{(b + 1)^x}, \frac{r + (3 + 2b)s + (b + 1)^2r}{(b + 1)^2}, \frac{u + 3(b + 2)r + (3b^2 + 9b + 7)s + (b + 1)^3r}{(b + 1)^3}; z \right). \]
Hence the equation (25) becomes
\[ \psi(p(z), z p'(z), z^2 p''(z), z^3 p'''(z); z) \in \Omega. \]
We also note that
\[ \frac{t}{s} + 1 = \frac{(b + 1)(\gamma - \alpha)}{\beta - \alpha} - 2(1 + b) \]
and
\[ \frac{u}{s} = \frac{(1 + b)^2 - 3\gamma(b + 2)(b + 1) + 3\alpha(b + 2)(b + 1) - (1 + b)^2}{\beta - \alpha} + 3b^2 + 12b + 11 \]
Thus, clearly, the admissibility condition for \( \phi \in \Theta_{1, j}[\Omega, q] \) in Definition 8 is equivalent to the admissibility condition for \( \psi \in \Psi_3[\Omega, q] \) as given in Definition 4 with \( n = 2 \). Therefore, by using (24) and Lemma 1, we have
\[ \frac{J_{\mu + 1, b}f(z)}{z} \propto q(z). \]
This completes the proof of Theorem 4.
If $\Omega \neq \mathbb{C}$ is a simply-connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of $U$ onto $\Omega$. In this case, the class $\Theta_{1}[h(U), q]$ is written as $\Theta_{1}[h, q]$. An immediate consequence of Theorem 4 is stated below.

**Theorem 5** Let $\phi \in \Theta_{1}[h, q]$. If the function $f \in \mathcal{A}$ and $q \in \mathbb{Q}$, satisfy the following conditions:

$$\Re \left( \frac{q''(\zeta)}{q'(\zeta)} \right) \geq 0, \quad \left| \frac{J_{\mu,h,f}(z)}{zq'(\zeta)} \right| \leq k \quad (34)$$

and

$$\phi \left( \frac{J_{\mu+1,h,f}(z)}{z}, \frac{J_{\mu,h,f}(z)}{z}, \frac{J_{\mu-1,h,f}(z)}{z}, \frac{J_{\mu-2,h,f}(z)}{z} ; z \right) \prec h(z), \quad (35)$$

then

$$\left| \frac{J_{\mu+1,h,f}(z)}{z} \right| < q(z) \quad (z \in U).$$

In view of Definition 8, and in the special case when $q(z) = Mz$ ($M > 0$), the class $\Theta_{1}[\Omega, q]$ of admissible functions, denoted by $\Theta_{1}[\Omega]$, is expressed as follows.

**Definition 9.** Let $\Omega$ be a set in $\mathbb{C}$ and $M > 0$. The class $\Theta_{1}[\Omega, M]$ of admissible functions consists of those functions $\phi : C^{4} \times U \longrightarrow \mathbb{C}$ such that

$$\phi \left( Me^{\theta}, \frac{(k+1+b)Me^{\theta}}{1+b}, \frac{L + [(3+2b)k + (b+1)^{2}]Me^{\theta}}{(b+1)^{2}}, \frac{N + 3(b+2)L + [(3b^2 + 9b + 7)k + (b+1)^{3}]Me^{\theta}}{(b+1)^{3}} ; z \right) \notin \Omega \quad (36)$$

whenever $z \in U$,

$$\Re \left( Le^{-\theta} \right) \geq (k-1)kn$$

and

$$\Re \left( Ne^{-\theta} \right) \geq 0 \quad (\forall \theta \in \mathbb{R}; k \geq 2).$$

**Corollary 6** Let $\phi \in \Theta_{1}[\Omega]$. If the function $f \in \mathcal{A}$ satisfies the following conditions:

$$\left| \frac{J_{\mu,h,f}(z)}{z} \right| \leq kM \quad (z \in U; k \geq 2; M > 0)$$

and

$$\phi \left( \frac{J_{\mu+1,h,f}(z)}{z}, \frac{J_{\mu,h,f}(z)}{z}, \frac{J_{\mu-1,h,f}(z)}{z}, \frac{J_{\mu-2,h,f}(z)}{z} ; z \right) \in \Omega,$$

then

$$\left| \frac{J_{\mu+1,h,f}(z)}{z} \right| < M.$$
By a similar argument, we get
\[
\frac{J_{\mu-2,b,f}(z)}{J_{\mu-1,b,f}(z)} = \frac{B}{b+1}
\]  
(41) and
\[
\frac{J_{\mu-3,b,f}(z)}{J_{\mu-2,b,f}(z)} = \frac{1}{b+1} \left[ B + B^{-1}(C + A^{-1}D - A^{-2}C^2) \right],
\]  
(42)
where
\[
B := (b+1)p(z) + \frac{z^p(z)}{p(z)} + \frac{2z^p(z)}{p(z)} \left( \frac{z^p(z)}{p(z)} \right)^2 + (b+1)z^p(z). \]
\[
C := \frac{z^p(z)}{p(z)} + \frac{z^p(z)}{p(z)} - \left( \frac{z^p(z)}{p(z)} \right)^2 + (b+1)z^p(z) \]
and
\[
D := \frac{3z^p(z)}{p(z)} + \frac{z^p(z)}{p(z)} + \frac{z^p(z)}{p(z)} - 3 \left( \frac{z^p(z)}{p(z)} \right)^2 - \frac{3z^p(z)}{p(z)} \left( \frac{z^p(z)}{p(z)} \right)^3 + 2 \left( \frac{z^p(z)}{p(z)} \right)^3 + (b+1)z^p(z). \]

We now define the transformation from \( \mathbb{C}^4 \) to \( \mathbb{C} \) by
\[
\alpha(r,s,t,u) = r, \quad \beta(r,s,t,u) = \frac{1}{b+1} \left( \frac{s}{r} + (b+1)r \right), \quad \gamma(r,s,t,u) = \frac{1}{b+1} \left( \frac{s}{r} + (b+1)r \right), \quad \delta(r,s,t,u) = \frac{1}{b+1} \left( F + F^{-1}(L + E^{-1}H - E^{-2}L^2) \right),
\]  
(44)
where
\[
L := (1+b)s + \frac{t}{r} + \frac{s}{r} - \left( \frac{s}{r} \right)^2 \]
and
\[
H := \frac{3t}{r} + \frac{u}{r} + \frac{s}{r} + 3 \left( \frac{s}{r} \right)^2 - 3 \left( \frac{st}{r^2} \right) + 2 \left( \frac{s}{r} \right)^3 + (1+b)s + (1+b)r. \]

Let
\[
\psi(r,s,t,u) = \phi(\alpha, \beta, \gamma, \delta; z) = \phi \left( r, \frac{E}{b+1}, \frac{F}{b+1}, \frac{F + F^{-1}(L + E^{-1}H - E^{-2}L^2)}{b+1} \right). \]
(45)
The proof will make use of Lemma 1. Using the equations (39) to (42), and from (45), we have
\[
\psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z) = \phi \left( \frac{J_{\mu,b,f}(z)}{J_{\mu+1,b,f}(z)}, \frac{J_{\mu-1,b,f}(z)}{J_{\mu+1,b,f}(z)}, \frac{J_{\mu-2,b,f}(z)}{J_{\mu+1,b,f}(z)}, \frac{J_{\mu-3,b,f}(z)}{J_{\mu+1,b,f}(z)}; z \right). \]
(46)

Hence the (38) becomes
\[
\psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z) \in \Omega.
\]
We note that
\[
\frac{t}{s} + 1 = \frac{(1+b)(\beta \gamma + 2\alpha^2 - 3\alpha \beta)}{(\beta - \alpha)}
\]
and
\[
\frac{u}{s} = \left[ (\delta - \gamma)(1+b)^2 \beta \gamma - (1+b)^2 (\gamma - \beta) \beta (1-\gamma + 3\alpha) - 3(b+1)(\beta - \gamma) \beta + 2(\beta - \alpha) + 3(1+b)\alpha(\beta - \alpha) + (\beta - \alpha)^2 (1+b)(\beta - \alpha) (1+b) - 3 - 4(1+b) \alpha \right] ; (\beta - \alpha)^{-1},
\]
Thus, clearly, the admissibility condition for \( \phi \in \Theta_{12}[\Omega, q] \) in Definition 10 is equivalent to the admissibility condition for \( \psi \in \Psi_{2}[\Omega, q] \) as given in Definition 4 with \( n = 2 \). Therefore, by using (37) and Lemma 1, we have
\[
\frac{J_{\mu,b,f}(z)}{J_{\mu+1,b,f}(z)} < q(z) \quad (z \in \mathbb{U}). \]
(47)
This completes the proof of Theorem 6.

If \( \Omega \neq \mathbb{C} \) is a simply-connected domain, then \( \Omega = h(\mathbb{U}) \) for some conformal mapping \( h(z) \) of \( \mathbb{U} \) onto \( \Omega \). In this case, the class \( \Theta_{12}[h(\mathbb{U}), q] \) is written simply as \( \Theta_{12}[h, q] \). An immediate consequence of Theorem 6 is now stated below without proof.

Theorem 7 Let \( \phi \in \Theta_{12}[h, q] \). If the function \( f \in \mathcal{A} \) and \( q \in \mathcal{Q}_1 \) satisfy the conditions (37) and
\[
\phi \left( \frac{J_{\mu,b,f}(z)}{J_{\mu+1,b,f}(z)}, \frac{J_{\mu-1,b,f}(z)}{J_{\mu+1,b,f}(z)}, \frac{J_{\mu-2,b,f}(z)}{J_{\mu+1,b,f}(z)}, \frac{J_{\mu-3,b,f}(z)}{J_{\mu+1,b,f}(z)} \right) \prec h(z), \]
then
\[
\frac{J_{\mu+1,b,f}(z)}{J_{\mu+1,b,f}(z)} \prec q(z) \quad (z \in \mathbb{U}). \]
(48)

3 Results Related to the Third-Order Superordination

In this section, we investigate and prove several theorems involving the third-order differential superordination for the operator \( J_{\mu,b,f}(z) \) defined in (6). For the purpose, we consider the following class of admissible functions.

Definition 11. Let \( \Omega \) be a set in \( \mathbb{C} \) and let \( q \in \mathcal{K}_0 \) with \( q'(z) \neq 0 \). The class \( \Theta_{12}[\Omega, q] \) of admissible functions consists of those functions \( \phi : \mathbb{C}^4 \times \mathcal{W} \to \mathbb{C} \) that satisfy the following admissibility conditions:
\[
\phi(\alpha, \beta, \gamma, \delta, \zeta) \in \Omega
\]
whenever
\[ \alpha = q(z), \beta = \frac{zf^{(z)} + mbq(z)}{m(b + 1)}, \]
\[ \Re \left( \frac{y(b + 1)^2 - b^2 \alpha}{(\beta(b + 1) - b \alpha)} - 2b \right) \leq \frac{1}{m} \Re \left( \frac{zf^{(z)}}{q(z)} + 1 \right) \]
and
\[ \Re \left( \frac{y(b + 1)^2 - b^2 \alpha}{(\beta(b + 1) - b \alpha)} + \frac{2b(3 + 2b) + 3b^2 + 6b + 2}{(b^2 + 1) \beta(b + 1) - b \alpha} \right) \leq \frac{1}{m} \Re \left( \frac{zf^{(z)}}{q(z)} \right), \]
where \( z \in U, \zeta \in \partial U \) and \( m \geq 2 \).

**Theorem 8** Let \( \phi \in \Theta^J[U, q] \). If the function \( f \in \mathcal{A} \), with \( J_{\mu_1+b_1}f(z) \in \mathbb{Q}_0 \), and if \( q \in \mathcal{H}_0 \) with \( q'(z) \neq 0 \), satisfying the following conditions:
\[ \Re \left( \frac{zf^{(z)}}{q'(z)} \right) \geq 0, \quad \left| J_{\mu_1+b_1}f(z) \right| \leq m \]
and the function
\[ \phi(J_{\mu_1+b_1}f(z), J_{\mu_2+b_2}f(z), J_{\mu_3+b_3}f(z), J_{\mu_4+b_4}f(z); z), \]
is univalent in \( U \), then
\[ \Omega \subset \{ \phi(J_{\mu_1+b_1}f(z), J_{\mu_2+b_2}f(z), J_{\mu_3+b_3}f(z), J_{\mu_4+b_4}f(z); z) : z \in U \}, \]
implies that
\[ q(z) \prec J_{\mu_1+b_1}f(z) \quad (z \in U). \]

**Proof.** Let the function \( p(z) \) be defined by (12) and \( \psi \) by (18). Since \( \phi \in \Theta^J[U, q] \), from (19) and (50), we have
\[ \Omega \subset \{ \psi(p(z), z\psi'(z), z^2 \psi''(z), z^3 \psi'''(z); z : z \in U \}. \]
From (16) and (17), we see that the admissibility condition for \( \phi \in \Theta^J[U, q] \) in Definition 11 is equivalent to the admissibility condition for \( \psi \in \Psi^J[U, q] \) as given in Definition 5 with \( n = 2 \). Hence \( \psi \in \Psi^J[U, q] \) and, by using (50) and Lemma 2, we find that
\[ q(z) \prec J_{\mu_1+b_1}f(z) \quad (z \in U). \]
This completes the proof of Theorem 8.

If \( \Omega \neq \mathbb{C} \) is a simply-connected domain, then \( \Omega = h(U) \) for some conformal mapping \( h(z) \) of \( U \) onto \( \Omega \). In this case, the class \( \Theta^J[h(U), q] \) is written simply as \( \Theta^J[h, q] \). Theorem 9 follows as an immediate consequence of Theorem 8.

**Theorem 9** Let \( \phi \in \Theta^J[h, q] \) and let \( h \) be analytic in \( U \). If the function \( f \in \mathcal{A} \) and \( J_{\mu+b}f(z) \in \mathbb{Q}_0 \), and if \( q \in \mathcal{H}_0 \) with \( q'(z) \neq 0 \), satisfying the conditions (49) and the function
\[ \phi(J_{\mu_1+b_1}f(z), J_{\mu_2+b_2}f(z), J_{\mu_3+b_3}f(z), J_{\mu_4+b_4}f(z); z), \]
is univalent in \( U \), then
\[ h(z) \prec \phi(J_{\mu_1+b_1}f(z), J_{\mu_2+b_2}f(z), J_{\mu_3+b_3}f(z), J_{\mu_4+b_4}f(z); z) \]
implies that
\[ q(z) \prec J_{\mu_1+b_1}f(z) \quad (z \in U). \]

Theorems 8 and 9 can only be used to obtain subordination for the third-order differential superordination of the form (50) or (51). The following theorem gives the existence of the best subordinant of (51) for a suitable \( \phi \).

**Theorem 10** Let the function \( h \) be univalent in \( U \) and let \( \phi : \mathbb{C}^4 \times \mathbb{W} \rightarrow \mathbb{C} \) and let \( \psi \) be given by (18). Suppose that the following differential equation:
\[ \psi(q(z), zq'(z), z^2 q''(z), z^3 q'''(z); z) = h(z) \]
has a solution \( q(z) \in \mathbb{Q}_0 \). If the function \( f \in \mathcal{A} \), with \( J_{\mu+b}f(z) \in \mathbb{Q}_0 \) and if \( q \in \mathcal{H}_0 \) with \( q'(z) \neq 0 \), satisfying the conditions (49) and
\[ \phi(J_{\mu+b}f(z), J_{\mu+b}f(z), J_{\mu+b}f(z), J_{\mu+b}f(z); z), \]
is analytic in \( U \), then
\[ h(z) \prec \phi(J_{\mu+b}f(z), J_{\mu+b}f(z), J_{\mu+b}f(z), J_{\mu+b}f(z); z) \]
implies that
\[ q(z) \prec J_{\mu+b}f(z) \quad (z \in U) \]
and \( q(z) \) is the best dominant.

**Proof.** By applying Theorem 8 and Theorem 9, we deduce that \( q \) is a subordinant of (51). Since \( q \) satisfies (52), it is also a solution of (51) and, therefore, \( q \) will be subordinated by all subordinants. Hence \( q \) is the best subordinant. This completes the proof of Theorem 10.

**Definition 12.** Let \( \Omega \) be a set in \( \mathbb{C} \) and let \( q \in \mathcal{H}_0 \) with \( q'(z) \neq 0 \). The class \( \Theta^J[U, q] \) of admissible functions consists of those functions \( \phi : \mathbb{C}^4 \times \mathbb{W} \rightarrow \mathbb{C} \) that satisfy the following admissibility condition:
\[ \phi(\alpha, \beta, \gamma, \delta; z) \in \Omega \]
whenever
\[ \alpha = q(z), \quad \beta = \frac{zf^{(z)} + mbq(z)}{m(b + 1)}, \]
\[ \Re \left( \frac{(b + 1)(\gamma - \alpha)}{\beta - \alpha} - 2(1 + b) \right) \leq \frac{1}{m} \Re \left( \frac{zf^{(z)}}{q'(z)} + 1 \right) \]
and
\[ \Re \left( \frac{b(1 + b^2 - 3b(b + 2)(b + 1) + 3a(b + 2)(b + 1) - (1 + b)^2 \alpha + 3b^2 + 12b + 11)}{\beta - \alpha} \right) \leq \frac{1}{m} \Re \left( \frac{zf^{(z)}}{q'(z)} \right), \]
where \( z \in U, \zeta \in \partial U \) and \( m \geq 2 \).
Theorem 11 Let \( \phi \in \Theta_{11}(\Omega, q) \). If the function \( f \in \mathcal{A} \), with \( \frac{J_{\mu,bf(z)}}{z} \in \mathbb{Q}_1 \), and if \( q \in \mathcal{H}_1 \) with \( q'(z) \neq 0 \), satisfying the following conditions:

\[
\Re \left( \frac{zf''(z)}{q'(z)} \right) \geq 0, \quad \left| \frac{J_{\mu,bf(z)}}{zf'(z)} \right| \leq m
\]  

(53)

and the function

\[
\phi \left( \frac{J_{\mu+1,bf(z)}}{z}, \frac{J_{\mu,bf(z)}}{z}, \frac{J_{\mu-1,bf(z)}}{z}, \frac{J_{\mu-2,bf(z)}}{z}; z \right),
\]

is univalent in \( \mathbb{U} \), then

\[
\Omega \subset \left\{ \phi \left( \frac{J_{\mu+1,bf(z)}}{z}, \frac{J_{\mu,bf(z)}}{z}, \frac{J_{\mu-1,bf(z)}}{z}, \frac{J_{\mu-2,bf(z)}}{z}; z \right) : z \in \mathbb{U} \right\}
\]  

(54)

implies that

\[
q(z) \prec \frac{J_{\mu+1,bf(z)}}{z} \quad (z \in \mathbb{U}).
\]

Proof. Let the function \( p(z) \) be defined by (26) and \( \psi \) by (32). Since \( \phi \in \Theta_{11}(\Omega, q) \), we find from (33) and (54) that

\[
\Omega \subset \{ \psi \left( p(z), zp'(z), z^2 p''(z), z^3 p'''(z) ; z \right) : z \in \mathbb{U} \}.
\]

From the equations (30) and (31), we see that the admissible condition for \( \phi \in \Theta_{11}(\Omega, q) \) in Definition 12 is equivalent to the admissible condition for \( \psi \) as given in Definition 5 with \( n = 2 \). Hence \( \psi \in \Psi_2(\Omega, q) \) and, by using (53) and Lemma 2, we have

\[
q(z) \prec \frac{J_{\mu+1,bf(z)}}{z} \quad (z \in \mathbb{U}).
\]

This completes the proof of Theorem 11.

If \( \Omega \neq \mathbb{C} \) is a simply-connected domain, then \( \Omega = h(\mathbb{U}) \) for some conformal mapping \( h(z) \) of \( \mathbb{U} \) onto \( \Omega \). In this case, the class \( \Theta_{11}(h(\mathbb{U}), q) \) is written simply as \( \Theta_{11}[h, q] \). Theorem 12 follows as an immediate consequence of Theorem 11.

Theorem 12 Let \( \phi \in \Theta_{11}[h, q] \) and let \( h \) be analytic in \( \mathbb{U} \). If the function \( f \in \mathcal{A} \), with \( q \in \mathcal{H}_1 \) and \( q'(z) \neq 0 \), satisfying the conditions (53) and the function

\[
\phi \left( \frac{J_{\mu+1,bf(z)}}{z}, \frac{J_{\mu,bf(z)}}{z}, \frac{J_{\mu-1,bf(z)}}{z}, \frac{J_{\mu-2,bf(z)}}{z}; z \right),
\]

is univalent in \( \mathbb{U} \), then

\[
h(z) \prec \phi \left( \frac{J_{\mu+1,bf(z)}}{z}, \frac{J_{\mu,bf(z)}}{z}, \frac{J_{\mu-1,bf(z)}}{z}, \frac{J_{\mu-2,bf(z)}}{z}; z \right),
\]

implies that

\[
q(z) \prec \frac{J_{\mu+1,bf(z)}}{z} \quad (z \in \mathbb{U}).
\]

Proof. Let the function \( p(z) \) be defined by (39) and \( \psi \) by (45). Since \( \phi \in \Theta_{11}[\Omega, q] \), we find from (46) and (56) that

\[
\Omega \subset \{ \psi \left( p(z), zp'(z), z^2 p''(z), z^3 p'''(z) ; z \right) : z \in \mathbb{U} \}.
\]

From the equations (43) and (44), we see that the admissible condition for \( \phi \in \Theta_{11}[\Omega, q] \) in Definition 13 is equivalent to the admissible condition for \( \psi \) as given in Definition 5 with \( n = 2 \). Hence \( \psi \in \Psi_2(\Omega, q) \) and, by using (55) and Lemma 2, we have

\[
q(z) \prec \frac{J_{\mu,bf(z)}}{J_{\mu+1,bf(z)}} \quad (z \in \mathbb{U}).
\]

This completes the proof of Theorem 13.
Theorem 14  Let $\phi \in \Theta_{J,2}[h,q]$. If the function $f \in \mathcal{S}$ and $J_{\mu,b} f(z) \in \mathcal{Q}_1$, with $q \in \mathcal{H}_1$ and $q'(z) \neq 0$, satisfying the conditions (55) and the function
\[
\phi \left( \frac{J_{\mu,b} f(z)}{J_{\mu+1,b} f(z)}, \frac{J_{\mu-1,b} f(z)}{J_{\mu,b} f(z)}, \frac{J_{\mu-2,b} f(z)}{J_{\mu-1,b} f(z)}, \frac{J_{\mu-3,b} f(z)}{J_{\mu-2,b} f(z)} \right) \rightarrow \Phi(z),
\]
is univalent in $U$, then
\[
h(z) \prec \phi \left( \frac{J_{\mu,b} f(z)}{J_{\mu+1,b} f(z)}, \frac{J_{\mu-1,b} f(z)}{J_{\mu,b} f(z)}, \frac{J_{\mu-2,b} f(z)}{J_{\mu-1,b} f(z)}, \frac{J_{\mu-3,b} f(z)}{J_{\mu-2,b} f(z)} \right) \qquad (57)
\]
implies that
\[
q(z) \prec \frac{J_{\mu,b} f(z)}{J_{\mu+1,b} f(z)} \quad (z \in U).
\]

4 A Set of Sandwich-Type Results

By combining Theorems 2 and 9, we obtain the following sandwich-type theorem.

Theorem 15  Let $h_1$ and $q_1$ be analytic functions in $U$. Also let $h_2$ be univalent function in $U$ and $q_2 \in \mathcal{Q}_0$ with $q_1(0) = q_2(0) = 0$ and $\phi \in \Theta_{J,1}[h_2,q_2] \cap \Theta_{J,2}[h_1,q_1]$. If the function $f \in \mathcal{S}$, with $J_{\mu+1,b} f(z) \in \mathcal{Q}_0 \cap \mathcal{H}_0$, and the function
\[
\phi \left( J_{\mu+1,b} f(z), J_{\mu,b} f(z), J_{\mu-1,b} f(z), J_{\mu-2,b} f(z) \right),
\]
is univalent in $U$, and if the conditions (10) and (49) are satisfied, then
\[
h_1(z) \prec \phi \left( J_{\mu+1,b} f(z), J_{\mu,b} f(z), J_{\mu-1,b} f(z), J_{\mu-2,b} f(z) \right) \prec h_2(z)
\]
implies that
\[
q_1(z) \prec J_{\mu+1,b} f(z) \prec q_2(z) \quad (z \in U). \quad (58)
\]

If, on the other hand, we combine Theorems 5 and 12, we obtain the following sandwich-type theorem.

Theorem 16  Let $h_1$ and $q_1$ be analytic functions in $U$. Also let $h_2$ be univalent function in $U$ and $q_2 \in \mathcal{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Theta_{J,1}[h_2,q_2] \cap \Theta_{J,1}[h_1,q_1]$. If the function $f \in \mathcal{S}$, with $J_{\mu+1,b} f(z) \in \mathcal{Q}_1 \cap \mathcal{H}_1$, and the function
\[
\phi \left( J_{\mu+1,b} f(z), J_{\mu,b} f(z), J_{\mu-1,b} f(z), J_{\mu-2,b} f(z) \right) \rightarrow \Phi(z),
\]
is univalent in $U$, and the conditions (24) and (53) are satisfied, then
\[
h_1(z) \prec \phi \left( J_{\mu+1,b} f(z), J_{\mu,b} f(z), J_{\mu-1,b} f(z), J_{\mu-2,b} f(z) \right) \prec h_2(z)
\]
implies that
\[
q_1(z) \prec J_{\mu+1,b} f(z) \prec q_2(z) \quad (z \in U). \quad (59)
\]

Finally, by combining Theorems 7 and 14, we obtain the following sandwich-type theorem.

Theorem 17  Let $h_1$ and $q_1$ be analytic functions in $U$. Also let $h_2$ be univalent functions in $U$ and $q_2 \in \mathcal{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Theta_{J,1}[h_2,q_2] \cap \Theta_{J,2}[h_1,q_1]$. If the function $f \in \mathcal{S}$, with $J_{\mu+1,b} f(z) \in \mathcal{Q}_1 \cap \mathcal{H}_1$, and the function
\[
\phi \left( J_{\mu+1,b} f(z), J_{\mu,b} f(z), J_{\mu-1,b} f(z), J_{\mu-2,b} f(z) \right),
\]
is univalent in $U$, and the conditions (37) and (55) are satisfied, then
\[
h(z) \prec \phi \left( J_{\mu,b} f(z), J_{\mu-1,b} f(z), J_{\mu-2,b} f(z), J_{\mu-3,b} f(z) \right) \prec h_2(z)
\]
implies that
\[
q_1(z) \prec J_{\mu+1,b} f(z) \prec q_2(z) \quad (z \in U). \quad (60)
\]

5 Perspective

In our present investigation, we have made use of the linear operator introduced and studied by Srivastava and Attiya [16], to systematically investigate several suitable classes of admissible functions. We have presented the dual properties of the third-order differential subordinations. As consequences of some of our main results, various sandwich-type theorems are established for a class of univalent analytic functions involving the celebrated Srivastava-Attiya transform. We have also indicated relevant connections of the new results presented in this article with those that were considered in earlier works.

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For this first-named author’s biographical details, professional qualifications as well as scholarly accomplishments (including the lists of his most recent publications such as Journal Articles, Books, Monographs and Edited Volumes, Book Chapters, Encyclopedia Chapters, Papers in Conference Proceedings, Forewords to Books and Journals, *et cetera*), the interested reader should look into the following Web Site:

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