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Statistical Inference and Prediction for The Inverse Weibull Distribution based On Record Data

M. M. Mohie El-Din[1], *Fathy H. Riad*[2] *and Mohamed A. El-Sayed*[3,4,*]

[1] Department of Mathematics, Faculty of Science, Al-Azhar University, Egypt
[2] Department of Mathematics, Faculty of Science, Minia University, Egypt
[3] Math Dept., Faculty of Science in Qena, South Valley Univ., Egypt
[4] CS Dept.,Computers and IT College, Taif Univ., KSA

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Abstract: In this paper, we obtained based on record value, the maximum likelihood, minimum variance unbiased and Bayes estimators of the two parameters of the inverse Weibull distribution are computed and compared. A Bayesian prediction interval for the s^{th} future record is obtained in a closed form. Based on simulated record values, numerical computations and comparisons between the different estimators are given.

Keywords: inverse Weibull distribution, Record values, Bayesian inference Prediction.

1 Introduction

Record values arise naturally in many real life applications involving data relating to sport, weather and life testing studies. Many authors have been studied record values and associated statistics, for example, see Chandler [9], Nagaraja [19], Ahsanullah ([1],[2]), Arnold and Balakrishnan [3], Arnold, et.al. ([4],[5]), Balakrishnan , Chan ([6],[7]), Raqab [22], Sultan [23] ,and Preda et al [21].
The inverse Weibull distribution plays an important role in many applications, including the dynamic components of diesel engine and several data set such as the times to breakdown of an insulating fluid subject to the action of a constant tension, see Nelson [20]. Calabria and Pulcinia [8] provide an interpretation of the inverse Weibull distribution in the context of the load strength relationship for a component. Maswadah [14] has fitted the inverse Weibull distribution to the flood data reported in Dumonceaux and Antle [10]. For more details on the inverse Weibull distribution, see, for example Johnson et al. [12], Marušić et al. [13], Murthy et al. [18], Mohie El-Din et al. [15] , [16] and [17].
The inverse Weibull model was developed by Erto [11]. The probability density function (pdf) of the random variable X having a three-parameter inverse Weibull distribution with location parameter $\alpha \geq 0$, scale parameter $\eta > 0$ and shape parameter $\beta > 0$ is given by [11], [13]:

$$f(x;\alpha,\beta,\eta) = \begin{cases} \frac{\beta}{\eta}(\frac{\eta}{x-\alpha})^{\beta+1} e^{-(\frac{\eta}{x-\alpha})^{\beta}}, & x > \alpha, \quad \eta,\beta > 0, \\ 0, & x \leq \alpha, \end{cases} \tag{1}$$

If $\alpha = 0$, the resulting distribution is called the two-parameter inverse Weibull distribution. The cumulative distribution cdf of the inverse Weibull distribution as follows:

$$F(x;\alpha,\beta,\eta) = e^{-(\frac{\eta}{x-\alpha})^{\beta}}, \quad x > \alpha, \quad \eta,\beta > 0. \tag{2}$$

Assuming that we have m lower record values, $X_{L(1)}, X_{L(2)}, \cdots, X_{L(m)}$, be the first m lower record values from the inverse Weibull distribution.

* Corresponding author e-mail: mas06@fayoum.edu.eg

The pdf of $X_{L_{(m)}}$ is given by

$$f_{x_{L(m)}}(x) = \frac{1}{\Gamma(m)}\{-\ln[F(x)]\}^{m-1} f(x)$$
$$= \frac{\beta}{\eta\Gamma(m)}\left(\frac{\eta}{x-\alpha}\right)^{\beta m+1} e^{-\left(\frac{\eta}{x-\alpha}\right)^{\beta}} \quad (3)$$

2 Maximum likelihood estimation

Point Estimation

Let $X_{L(1)}, X_{L(2)}, \cdots, X_{L(m)}$ be m lower record values each of which has the inverse Weibull distribution whose the pdf and cdf are, respectively, given by (1) and (2). Based on those lower record values, for simplicity of notation, we will use x_i instead of $x_{L(i)}$. The likelihood function may then be written is

$$L(\eta;\underline{X}) = f(x_m)\prod_{i=1}^{m-1} h(X_i|\eta),$$
$$-\infty < X_m < X_{m-1} < \ldots < X_1 < \infty \quad (4)$$

where

$$\underline{X} = (X_1, X_2, \cdots, X_m), \quad and \quad h(X_i|\eta) = \frac{f(X_i|\eta)}{F(X_i|\eta)}$$

hence

$$L(\eta;\underline{X}) = \beta^m \eta^{m\beta} e^{-\eta^{\beta}(x_m-\alpha)^{-\beta}} \prod_{i=1}^{m}(x_i-\alpha)^{-\beta-1} \quad (5)$$

we obtain the log-likelihood function

$$\pounds = \log L = m\log\beta + m\beta\log\eta - \eta^{\beta}(x_m-\alpha)^{-\beta}$$
$$-(\beta+1)\sum_{i=1}^{m}\log(x_i-\alpha) \quad (6)$$

we obtain the estimators of η when β and α are known by differentiating (6) with respect to η and β and equating to zero, in this case we have

$$\frac{\partial\pounds}{\partial\eta} = \frac{m\beta}{\eta} - \beta\eta^{\beta-1}(x_m-\alpha)^{-\beta} = 0$$

$$\widehat{\eta} = m^{\frac{1}{\beta}}(x_m-\alpha) = m^{\frac{1}{\beta}} T_m^{\frac{-1}{\beta}} \quad (7)$$

where $T_m = (x_m-\alpha)^{-\beta}$.

From (5), one can see that the statistic T_m is sufficient and complete for the parameter η, and is distributed as gamma (m,η) with pdf

$$f(T_m|\eta) = \frac{\eta^{m\beta}}{\Gamma(m)} T_m^{m-1} e^{-\eta^{\beta}T_m}, \qquad T_m > 0. \quad (8)$$

We study this case, when β and α are known and η is unknown.

Lemma: *Let $X_{L(i)}$ $\forall i = 1,2,\ldots,m$ be the $i\underline{th}$ record values of the inverse Weibull distribution, then*

$$E\left[T_i^{-\frac{\omega}{\beta}}|\eta\right] = \frac{\Gamma\left(i-\frac{\omega}{\beta}\right)}{\Gamma(i)}\eta^{\omega} \quad (9)$$

where $T_i = (x_i-\alpha)^{-\beta}$.

Proof : We starting with the pdf of the $i\underline{th}$ record value, we derived the general expected value

$$E\left[T_i^{-\frac{\omega}{\beta}}|\eta\right] = \int_0^{\infty} T_i^{-\frac{\omega}{\beta}} f(T_i|\eta)\,dT_i$$

by using the pdf of the $\underline{ith}$ record values of the inverse Weibull distribution form (3), we obtain

$$E\left[T_i^{-\frac{\omega}{\beta}}|\eta\right] = \frac{\eta^{i\beta}}{\Gamma(i)}\int_0^{\infty} T_i^{i-\frac{\omega}{\beta}-1} e^{-\eta^{\beta} T_i} dT_i$$

we assume that $y = \eta^{\beta} T_i$ and integration then, we obtain,

$$E\left[T_i^{-\frac{\omega}{\beta}}|\eta\right] = \frac{\Gamma\left(i-\frac{\omega}{\beta}\right)}{\Gamma(i)}\eta^{\omega}$$

the lemma is proved.

From (7) and (9), we obtain the expected value and variance of the estimate $\widehat{\eta}$ is given by

$$E(\widehat{\eta}^{\omega}|\eta) = m^{\frac{\omega}{\beta}} E[T^{-\frac{\omega}{\beta}}]$$

then, if $i = m$ and $\omega = 1$ we obtain

$$E(\widehat{\eta}|\eta) = \frac{m^{\frac{1}{\beta}}\Gamma\left(m-\frac{1}{\beta}\right)}{\Gamma(m)}\eta \tag{10}$$

if $i = m$ and $\omega = 1,2$ we obtain

$$Var(\widehat{\eta}|\eta) = m^{\frac{2}{\beta}}\left[\frac{\Gamma\left(m-\frac{2}{\beta}\right)}{\Gamma(m)} - \left(\frac{\Gamma\left(m-\frac{1}{\beta}\right)}{\Gamma(m)}\right)^2\right]\eta^2. \tag{11}$$

We observe that the estimate $\widehat{\eta}$ is biased from (10) but we can transform it to unbiased $\widetilde{\eta}$, as follows, if we suppose

$$\widetilde{\eta} = \frac{\Gamma(m)}{m^{\frac{1}{\beta}}\Gamma\left(m-\frac{1}{\beta}\right)}\eta.$$

Then, the expected value and variance of $\widetilde{\eta}$ are

$$E(\widetilde{\eta}|\eta) = \eta, \quad Var(\widetilde{\eta}|\eta) = \eta^2\left[\frac{\Gamma(m)\Gamma\left(m-\frac{2}{\beta}\right)}{\Gamma^2\left(m-\frac{1}{\beta}\right)} - 1\right].$$

The mean squared error of the estimate $\widehat{\eta}$ is given by

$$E(\widehat{\eta}|\eta-\eta)^2 = E\{Var(\widehat{\eta}|\eta) - [E(\widehat{\eta}|\eta)-\eta]^2\} \tag{12}$$

from (10) and (11) in (12) we obtain the mean squared error of the estimate $\widehat{\eta}$.

3 Bayesian Inference

Point Estimation
Assuming that the parameter η is a realization of a random variable α which has the gamma conjugate prior distribution of the form

$$\pi(\eta) = \frac{\beta b^n}{\Gamma(n)}\eta^{n\beta-1}e^{-b\eta^{\beta}}, \quad \eta > 0, \quad (n, b > 0). \tag{13}$$

Combining (5) and (13), the posterior density is a gamma distribution with parameters $(m+n, (n+T_m)\eta)$ of the form

$$\pi^*(\eta|\underline{X}) = \frac{\pi(\eta)L}{\int_0^{\infty}\pi(\eta)Ld\eta}$$
$$= \frac{\beta(b+T_m)^{m+n}}{\Gamma(m+n)}\eta^{m\beta+n\beta-1}e^{-(b+T_m)\eta^{\beta}}. \tag{14}$$

Assuming a squared error loss function, the Bayes estimate of η is its posterior mean obtain by

$$\widehat{\eta}_B = \frac{\Gamma\left(m+n+\frac{1}{\beta}\right)}{\Gamma(m+n)(b+T_m)^{\frac{1}{\beta}}}. \tag{15}$$

Combining (13) and (8), the marginal density function of T_m is

$$f(T_m) = \frac{b^n}{B(m,n)} \frac{T_m^{m-1}}{(b+T_m)^{m+n}}, \qquad T_m > 0, \tag{16}$$

which is Beta (m,n) density of the second kind (see Johnson, Kotz and Balakrishnan [12], from which one can obtain

$$\begin{aligned} E(\widehat{\eta}_B) &= \frac{\Gamma\left(n+\frac{1}{\beta}\right)}{b^{\frac{1}{\beta}}\Gamma(n)} \\ Var(\widehat{\eta}_B) &= \frac{\Gamma^2\left(m+n+\frac{1}{\beta}\right)\Gamma\left(n+\frac{2}{\beta}\right)}{b^{\frac{2}{\beta}}\Gamma(m+n)\Gamma\left(m+n+\frac{2}{\beta}\right)\Gamma(n)} \\ &\quad -\frac{\Gamma^2\left(n+\frac{1}{\beta}\right)}{b^{\frac{2}{\beta}}\Gamma^2(n)} \end{aligned} \tag{17}$$

The mean squared error of the estimate $\widehat{\eta}_B$ is given by

$$E(\widehat{\eta}_B - \eta)^2 = E\{Var(\widehat{\eta}_B) - [E(\widehat{\eta}_B) - \eta]^2\} \tag{18}$$

from (17) in (18) we obtain the mean squared error of the estimate $\widehat{\eta}_B$.

4 Bayesian Prediction

Assume that $X_{L(1)}, X_{L(2)}, \cdots, X_{L(m)}$ are m lower record values each of which has the inverse Weibull distribution whose pdf is given by (1). Based on these lower record values, we would like to predict the sth lower record, $s > m$. Let $Y = X_{L(s)} = X_s$ be the sth lower record, the conditional pdf of Y for given $x_{L(m)} = x_m$ and $\eta > 0$ is

$$f(y|x_m;\eta) = \frac{[G(y) - G(x_m)]^{s-m-1}}{\Gamma(s-m)} \frac{f_x(y)}{F_x(x_m)} \tag{19}$$

where

$$\begin{aligned} G(y) &= -\log F_x(y) = \left(\frac{Y-\alpha}{\eta}\right)^{-\beta} \\ &= \eta^\beta (Y-\alpha)^{-\beta} = \eta^\beta T_s \\ G(x_m) &= \left(\frac{x_m-\alpha}{\eta}\right)^{-\beta} = \eta^\beta (x_m-\alpha)^{-\beta} = \eta^\beta T_m. \end{aligned} \tag{20}$$

Applying (1) and (2) in (19) we obtain

$$\begin{aligned} f(y|x_m) &= \frac{\beta\eta^{s\beta-m\beta}}{(y-\alpha)^{\beta+1}\Gamma(s-m)} [T_s - T_m]^{s-m-1} e^{-\eta^\beta(T_s-T_m)} \\ & 0 < y < x_m < \infty. \end{aligned} \tag{21}$$

Combining the posterior density function (14) and (21) and integrating at η we obtain the Bayes predictive density

$$\begin{aligned} f(y|\underline{X}) &= \int_0^\infty f(y|x_m;\eta)\pi^*(\eta|\underline{X})d\eta \\ &= \Upsilon\beta\left[\frac{b+T_m}{b+T_s}\right]^{m+n+1}\left[\frac{T_s-T_m}{b+T_s}\right]^{s-m-1}(y-\alpha)^{-\beta-1}, \\ & 0 < y < x_m < \infty. \end{aligned} \tag{22}$$

where $\Upsilon = 1/\{(b+T_m)B(s-m, m+n)\}$,
Thus, the Bayesian prediction bounds for $Y = X_s$ is obtained by evaluation

$$
\begin{aligned}
Pr(Y \geq t|\underline{X}) &= \int_t^{x_m} f(y|\underline{X})dy \\
&= \Upsilon\beta \int_t^{x_m} \left[\frac{b+T_m}{b+T_s}\right]^{m+n+1}. \\
&\quad \left[\frac{T_s-T_m}{b+T_s}\right]^{s-m-1} (y-\alpha)^{-\beta-1}dy. \qquad (23)
\end{aligned}
$$

Upon using the transformation

$$w_t = \frac{T_t - T_m}{b+T_s}$$

the above integral is equal

$$
\begin{aligned}
Pr(Y \geq t|\underline{X}) &= \Upsilon \int_0^{w_t} w^{s-m-1}(1-w)^{m+n-1}dw \\
&= F(w_t) \qquad (24)
\end{aligned}
$$

where

$$w_t = \frac{T_t - T_m}{b - T_t}, \qquad T_t = (t-\alpha)^{-\beta} \qquad (25)$$

and $F(.)$ is the Beta cdf with parameters $(s-m, m+n)$.
The $(1-\phi)100\%$ predictive interval for the sth lower record is given by
$Pr(L(\underline{X}) \leq Y \leq U\underline{X}) = 1-\phi$. Thus, applying (24), we obtain the lower and upper prediction bounds of $Y = X_s$, analytically in the forms

$$
\begin{aligned}
U(\underline{X}) &= \alpha + \left[\frac{T_m + b\delta_1(s)}{1-\delta_1(s)}\right]^{-\frac{1}{\beta}} \\
\text{and} \quad L(\underline{X}) &= \alpha + \left[\frac{T_m + b\delta_2(s)}{1-\delta_2(s)}\right]^{-\frac{1}{\beta}} \qquad (26)
\end{aligned}
$$

where

$$\delta_1 = F^{-1}(\frac{\phi}{2}) \qquad \delta_2 = F^{-1}(1-\frac{\phi}{2}). \qquad (27)$$

For the special case, when predicting the next lower $Y = X_{m+1} = X_s, s = m+1$. from (24) reduce to

$$
\begin{aligned}
Pr(x_{m+1} \geq t|\underline{X}) &= (n+m)\int_0^{w_t} (1-w)^{m+n-1}dw \\
&= 1-(1-w_t)^{m+n}, \qquad (28)
\end{aligned}
$$

where w_t is given by (25). By using (28), the lower and upper prediction bounds for the next record, x_{m+1} are obtained by

$$
\begin{aligned}
U_1(\underline{X}) &= \alpha + \left[\frac{T_m + b(1-\xi_1)}{\xi_1}\right]^{\frac{1}{\beta}} \\
\text{and} \quad L_1(\underline{X}) &= \alpha + \left[\frac{T_m + b(1-\xi_2)}{\xi_2}\right]^{\frac{1}{\beta}} \qquad (29)
\end{aligned}
$$

where

$$\xi_1 = (1-\frac{\phi}{2})^{\frac{1}{m+n}} \quad \text{and} \quad \xi_2 = (\frac{\phi}{2})^{\frac{1}{m+n}} \qquad (30)$$

5 Numerical Illustration

In order to illustrate the usefulness of the inferences discussed in the previous section, four simulated record values of sizes $m = 4, 5, 6$ and 7 from the inverse Weibull distribution are obtained.

Table 1: The MSEs of MLE's of $\widehat{\eta}$ and the Bayes risk of $\widehat{\eta}_B$

η	m	$MSE(\widehat{\eta})$	$MSE(\widehat{\eta}_B)$	(L,U)
1.0	4	0.127766	0.048935	(0.695566,1.593930)
	5	0.092555	0.055794	(0.755629 , 1.745365)
	6	0.073115	0.048761	(0.801408 , 1.790953)
	7	0.060909	0.038636	(0.839466, 1.696166)
1.2	4	0.179583	0.084054	(0.580566,1.466531)
	5	0.128880	0.096419	(0.640162,1.505239)
	6	0.090886	0.083002	(0.684357, 1.546739)
	7	0.083309	0.063295	(0.820400,1.651211)

We calculate the mean square error of the MLE of the estimate $\widehat{\eta}$ and the Bayes risk of $\widehat{\eta}_B$, and compared from them. We obtained the %95 Bayesian predictive interval of x_{m+1} from (29)

6 Conclusion

From previous the table, we observe that $MSE(\widehat{\eta}_B) < MSE(\widehat{\eta})$ the Bayes estimate is the efficient estimate of η is more efficient than the MLE. Although the number of generated records m is relatively small, all the estimators either point or interval become better, by being closer to the population parameter value of ω as m increase. The prediction interval for the next record is always include its generated value $X_{L_{(m+1)}}$, and become better as m increases.

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