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Scattering of a Schrodinger's Particle from a Schwarzschild Black Hole in the Far Zone

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Abstract: In this work the problem of the scattering of a relatively slow Schrodinger's particle from a Schwarzschild black hole in the far zone (i.e. for $r \gg$ Schwarzschild radius(r_S)) has been considered. A calculation of the scattering amplitude corresponding to a low energy and k-independent s-wave value has been found and equal to one-half the Schwarzschild radius and a further calculation of the corresponding scattering length and phase-shift for the same zone limit has also been considered.

Keywords: Generalized Schrodinger's Equation, Schwarzschild Metric, Scattering Cross section, Black Hole.

1 Introduction

One of the elegant and highly motivating applications of quantum mechanics is the theory of scattering of waves and particles. It has been extensively proved that almost everything we know about particles and nuclei has been discovered in scattering experiments beginning from the work of Ernest Rutherford in 1911 on $\boldsymbol{\alpha}$-particle scattering experiments up till nowadays large particle colliders and accelerators done in almost all nuclear and high energy physics experiments all over the world.

Unlike the traditional approaches dealt with in most standard references on non-gravitational scattering problems (such as those presented in [1, 2] for the ordinary treatment of non-gravitational quantum scattering), the novelty of the current work done is based on the formal incorporation of the gravitational interacting field in the ordinary Schrodinger's equation (i.e. a use of Schrodinger's equation in curved space-time [3]), and as a result the emergence of a new (and a non-trivial) gravitational interacting term (corresponding to the space-time metric used) has appeared and considered as a source term in the analysis of the problem, though the rest of the calculations has been done in nearly the same sort of way presented in most references on quantum mechanics in flat space-time [4] but with the extra assumption of being in the very far zone (i.e. for $\boldsymbol{r} \gg \boldsymbol{r_S}$) which ended up with an interesting physical result such as a k-independent s-wave scattering amplitude of $\frac{1}{2}\boldsymbol{r_S}$ and corresponding to a low energy scattering domain.

Although the Green's function used in the present work has been very oversimplified, it still hold much to say about the physics of the scattering problem in the far-zone limit considered at hand.

2 Formulation of the Scattering Problem

Starting with a generalized version of the Schrodinger's equation [3], we have

$$\frac{\hbar^2}{2m}\left(\frac{1}{\sqrt{|\tilde{g}|}}\partial_i\left(\sqrt{|\tilde{g}|}\tilde{g}^{ij}\partial_j\psi\right)\right) + V_{NG}\psi = i\hbar\,\partial_0\psi \qquad (1)$$

where $\tilde{g}_{ij}$ represents the spatial sector of the full spacetime metric $g_{\mu\nu}$, and V_{NG} corresponds to all non-gravitational interaction terms. Therefore, resorting to the full spacetime metric of the Schwarzschild solution for the empty spacetime outside a spherical body of mass M [5], we have

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$$g_{\mu\nu}=\begin{pmatrix}1-\frac{2MG}{c^2r} & 0 & 0 & 0\\ 0 & -\left(1-\frac{2MG}{c^2r}\right)^{-1} & 0 & 0\\ 0 & 0 & -r^2 & 0\\ 0 & 0 & 0 & -r^2\sin^2\theta\end{pmatrix} \tag{2}$$

from which we get

$$\tilde{g}_{ij}=\begin{pmatrix}-\left(1-\frac{2MG}{c^2r}\right)^{-1} & 0 & 0\\ 0 & -r^2 & 0\\ 0 & 0 & -r^2\sin^2\theta\end{pmatrix} \tag{3}$$

where the Schwarzschild radius r_S is defined to be

$$r_S\equiv\frac{2MG}{c^2} \tag{4}$$

A calculation now of the corresponding Laplace-Beltrami operator involved in the generalized Schrodinger's equation (1) gives

$$\begin{aligned}\nabla_i\nabla^i &= \frac{1}{\sqrt{|\tilde{g}|}}\partial_i\left(\sqrt{|\tilde{g}|}\tilde{g}^{ij}\partial_j\right)\\ &=\frac{\sqrt{1-\frac{2MG}{c^2r}}}{r^2\sin\theta}\left\{\partial_r\left(-r^2\sin\theta\sqrt{1-\frac{2MG}{c^2r}}\partial_r\right)+\partial_\theta\left(\frac{-\sin\theta}{\sqrt{1-\frac{2MG}{c^2r}}}\partial_\theta\right)\right.\\ &\left.+\partial_\phi\left(\frac{-1}{\sqrt{1-\frac{2MG}{c^2r}}\sin\theta}\partial_\phi\right)\right\}\end{aligned} \tag{5}$$

$$\begin{aligned}&=\frac{\sqrt{1-\frac{2MG}{c^2r}}}{r^2\sin\theta}\left\{\frac{1}{\sqrt{1-\frac{2MG}{c^2r}}}\left(-2r+\frac{3MG}{c^2}\right)\sin\theta\,\partial_r-r^2\sin\theta\sqrt{1-\frac{2MG}{c^2r}}\partial_r^2\right.\\ &-\frac{\cos\theta}{\sqrt{1-\frac{2MG}{c^2r}}}\partial_\theta-\frac{\sin\theta}{\sqrt{1-\frac{2MG}{c^2r}}}\partial_\theta^2\\ &\left.-\frac{1}{\sin\theta\sqrt{1-\frac{2MG}{c^2r}}}\partial_\phi^2\right\}\end{aligned} \tag{6}$$

$$=\left(\frac{3MG}{c^2r^2}-\frac{2}{r}\right)\partial_r-\left(1-\frac{2MG}{c^2r}\right)\partial_r^2-\frac{\cot\theta}{r^2}\partial_\theta-\frac{1}{r^2}\partial_\theta^2-\frac{1}{r^2\sin^2\theta}\partial_\phi^2 \tag{7}$$

Recalling that the Laplacian operator in spherical coordinates is given by

$$\nabla^2=\partial_r^2+\frac{2}{r}\partial_r+\frac{\cot\theta}{r^2}\partial_\theta+\frac{1}{r^2}\partial_\theta^2+\frac{1}{r^2\sin^2\theta}\partial_\phi^2 \tag{8}$$

we end up with

$$\nabla_i\nabla^i=\frac{3MG}{c^2r^2}\partial_r+\frac{2MG}{c^2r}\partial_r^2-\nabla^2 \tag{9}$$

Using this in the time-independent version of the Schrodinger's equation (1) we get the corresponding inhomogeneous Helmholtz equation

$$\nabla^2\psi+k^2\psi=\frac{3MG}{c^2r^2}\partial_r\psi+\frac{2MG}{c^2r}\partial_r^2\psi \tag{10}$$

which has the formal solution

$$\psi=\psi_0+\int G_+(\bar{r},\bar{r}')\left(\frac{3MG}{c^2r'^2}\partial_{r'}\psi+\frac{2MG}{c^2r'}\partial_{r'}^2\psi\right)d^3r' \tag{11}$$

where $k^2=\frac{2mE}{\hbar^2}$, and G_+ is defined to be the retarded Green's function for an outgoing spherical wave emitted from $\bar{r}'$ and given by

$$G_+(\bar{r},\bar{r}')=\frac{-e^{ik|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|} \tag{12}$$

[4], where ψ_0 here represents an incoming plane-wave solution of the corresponding homogeneous equation, i.e.

$$\psi_0=e^{i\bar{k}.\bar{r}} \tag{13}$$

Since we are assumed to be working in the far zone (i.e. for $r\gg r_S$), it is legitimate to consider the following approximation for the Green's function

$$G_+(\bar{r},\bar{r}')\approx\frac{-e^{ikr}}{4\pi r}\quad(\text{for } r\gg r_S) \tag{14}$$

Therefore, our approximate solution for the scattering problem in the far zone becomes

$$\psi=e^{i\bar{k}.\bar{r}}-\frac{e^{ikr}}{4\pi r}\int\left(\frac{3MG}{c^2r'^2}\partial_{r'}\psi+\frac{2MG}{c^2r'}\partial_{r'}^2\psi\right)d^3r' \tag{15}$$

from which we get for the scattering amplitude

$$f_k=\frac{-1}{4\pi}\int\left(\frac{3MG}{c^2r'^2}\partial_{r'}\psi+\frac{2MG}{c^2r'}\partial_{r'}^2\psi\right)d^3r' \tag{16}$$

3 Born Approximations and Scattering Cross Section

So far everything has been exact except for the far zone approximation of the Green's function. Suppose now that the incoming plane wave is not substantially altered by the black-hole. Therefore, as a first approximation to the problem, we can invoke the Born approximation and use

$$\psi\approx\psi_0 \tag{17}$$

inside the integral from which we get for the scattering amplitude (with the further assumption that $\bar{k}=k\hat{z}$)

$$\begin{aligned}f_k&=\frac{-1}{4\pi}\int\left(\frac{3MG}{c^2r'^2}(ik\cos\theta')\right.\\ &\left.+\frac{2MG}{c^2r'}(-k^2\cos^2\theta')\right)e^{ikr'\cos\theta'}r'^2dr'd\cos\theta'\,d\phi'\end{aligned} \tag{18}$$

This integral can be easily evaluated by the use of a Laplace transform technique (in the $s\longrightarrow 0$ limit),

$$\begin{aligned}f_k&=\frac{-1}{2}\lim_{s\to 0}\int_{-1}^{1}\int_{0}^{\infty}e^{-sr'}\left(\frac{3MG}{c^2r'^2}(ikx')\right.\\ &\left.+\frac{2MG}{c^2r'}(-k^2x'^2)\right)e^{ikr'x'}r'^2dr'dx'\end{aligned} \tag{19}$$

$$\begin{aligned}&=\frac{-1}{2}\lim_{s\to 0}\left\{ik\frac{3MG}{c^2}\int_{-1}^{1}x'\int_{0}^{\infty}e^{-sr'}(\cos kr'x'+i\sin kr'x')dr'dx'\right.\\ &-k^2\frac{2MG}{c^2}\int_{-1}^{1}x'^2\int_{0}^{\infty}e^{-sr'}(r'\cos kr'x'\\ &\left.+ir'\sin kr'x')\,dr'dx'\right\}\end{aligned} \tag{20}$$

$$\begin{aligned}&=\frac{-1}{2}\lim_{s\to 0}\left\{ik\frac{3MG}{c^2}\int_{-1}^{1}x'\left(\frac{s}{s^2+k^2x'^2}+i\frac{kx'}{s^2+k^2x'^2}\right)dx'\right.\\ &-k^2\frac{2MG}{c^2}\int_{-1}^{1}x'^2\left(\frac{s^2-k^2x'^2}{\left(s^2+k^2x'^2\right)^2}\right.\\ &\left.\left.+i\frac{2skx'}{\left(s^2+k^2x'^2\right)^2}\right)dx'\right\}=\frac{MG}{c^2}=\frac{1}{2}r_S\end{aligned} \tag{21}$$

Therefore (and as we will discuss below) our calculated scattering amplitude corresponds to an isotropic k-independent low energy s-wave value of exactly 1/2 the Schwarzschild radius from which we get for the total scattering cross section

$$\sigma=\int|f_k|^2d\Omega=\pi r_S^2 \tag{22}$$

which very much resembles the classical result of the hard-sphere scattering problem. According to the formal definition of the low energy scattering length a, we have

$$a = -\lim_{k\to 0} f_k \tag{23}$$

from which we get for our low energy limit

$$a = -\frac{1}{2} r_S \tag{24}$$

(with the minus sign being appropriate for an attractive black-hole gravitational potential [1]).

4 Partial waves Analysis

Due to the fact that $[\hat{H},\ \hat{L}^2] = 0$ in our case in addition to the asymptotic behavior of the interaction part of the Hamiltonian and the azimuthal symmetry of the problem, it is legitimate (and necessary) to use the well-known elastic partial wave expansion of the scattering amplitude given by

$$f_k = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin\delta_l\, P_l(\cos\theta) \tag{25}$$

from which we deduce that for our low energy result, the only non-vanishing partial phase shift is obtained from

$$f_k = \frac{1}{2} r_S = \frac{1}{k} e^{i\delta_0}\sin\delta_0 \tag{26}$$

This will give the two simultaneous equations (working well in the $kr_S \longrightarrow 0$ limit)

$$\cos 2\delta_0 = 1, \quad \sin 2\delta_0 = kr_S \tag{27}$$

which finally gives for the zero-component phase shift

$$\delta_0 = \frac{1}{2}\tan^{-1}(kr_S) \approx \frac{kr_S}{2} \tag{28}$$

with all other components being zero.

5 Conclusions

As we can see, the use of the generalized form of the Schrodinger's equation in a curved background (1) has been used to calculate different parameters for the scattering of a relatively slow particle from a Schwarzschild black hole in the far zone limit which confirmed a corresponding low energy (or s-wave) and k-independent value of the elastic scattering amplitude of 1/2 the Schwarzschild radius along with the corresponding low energy scattering length and phase shift involved.

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