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A Rigorous Approach for Calculating the Chemical Potential of an Ultrarelativistic Spherical Degenerate Fermi Gas Interacting with Nuclear Matter

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Abstract: We present an accurate analytical formulation to determine the chemical potential of an ultrarelativistic spherical degenerate Fermi gas interacting with nuclear matter by examining rigorously the three terms of the general mathematical expression for the chemical-potential energy namely, the Fermi energy, the optical-potential energy, and a third term which is negligible if the Fermi gas is dilute. In fact, we derive the above expression as a function of distance and absolute temperature. The Fermi velocity is also determined. In addition, essential aspects of the optical potential are discussed.

Keywords: Ultrarelativistic Fermi gas, Chemical potential, Fermi energy, Fermi velocity, Optical potential

1 Introduction

It is well-known that both non-relativistic and relativistic Fermi gases have a markedly importance in various areas of Physics; consider, for example, electron and neutron gases.[1]-[8] Relativistic Fermi gases occupy a significant place in Relativistic Quantum Statistical Mechanics. In particular, finding the chemical potential of an ultrarelativistic degenerate Fermi gas is a useful task given the relevance of the concept of chemical potential in several branches of Physics. First of all, the concept in question has a great importance by itself in Statistical Mechanics. In addition, this importance holds in non-relativistic applications to Condensed Matter Physics, Molecular Physics and Chemical Physics, and relativistic applications to Astrophysics, and Nuclear Physics although also non-relativistic cases can be found in Nuclear Physics (see, for instance, refs. [1]-[4], [6]). On the other hand, we may note that the chemical potential of relativistic systems has been scarcely investigated in the current literature. This fact may be emphasized for the ultrarelativistic domain where significant research efforts are needed.

Starting from the notion of chemical potential of a non-relativistic degenerate Fermi gas, one can infer how is, quantitatively, the relativistic version of the chemical potential of the aforementioned gas as we will see later. The chemical-potential energy of a Fermi gas can be written as sum of three terms, namely, the Fermi energy, the optical-potential energy, and a third term which is negligible if the gas is dilute. In particular, the second term, i.e., the optical-potential energy presents a notorious interest. The optical potential arises from the many-body theory and, unfortunately, a number of its more salient features have not been well understood in an appreciable part of the current literature. One can find issues relative to the optical potential in Bose condensation, superfluidity, superconductivity, neutron stars, etc. so that, currently, one speaks of nuclear-optical potential. Indeed, Nuclear Astrophysics is a significant domain in which the importance of the concept of optical potential is manifest (see, for example, refs. [1]-[4], [6]). At this point, we can mention neutron-neutron scattering, which leads to a large attractive optical potential in neutron stars, compressing a star with gravitational field.[4] Determining the optical potential for both non-relativistic and relativistic neutron-nucleus scattering is really a useful task. One of the main ingredients necessary for this determination is the scattering amplitude. On the other hand, when one refers to coherent elastic neutron-neutron scattering amplitude, note that this amplitude comes from

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short-range strong interactions;[2,4] however, the range of the optical potential is large.

In the following, we shall give a mathematical relationship for the chemical-potential energy of an ultrarelativistic degenerate Fermi gas interacting with nuclear matter as sum of the three terms mentioned above. The relationship in question will be expressed in terms of distance and absolute temperature. The main aspects related to the Fermi energy and optical-potential energy will be discussed for the ultrarelativistic domain. In particular, the optical potential will be treated by emphasizing its dependence on the scattering amplitude, which depends upon energy in the relativistic domain. As we will see, this energy dependence becomes distance dependence due to the distance dependence of the corresponding Fermi wavenumber. In addition, we will define the scattering-amplitude operator whose expectation value will be determined.

2 Theoretical formulation

Assuming a non-relativistic degenerate spherical Fermi gas interacting with nuclear matter, which may be regarded as reservoir, the chemical-potential energy of the gas, in terms of radial distance at zero absolute temperature, reads (see, for example, ref. [4]:

$$\mu(r;T=0)=\frac{\hbar^2k_F^2(r)}{2m_0}\left[1+\frac{4bk_F(r)}{3\pi}+\frac{4(11-2\ln 2)b^2k_F^2(r)}{15\pi^2}\right], \tag{1}$$

where $\hbar$ is the reduced Planck constant, $k_F(r)$ is Fermi wavenumber, m_0 is the fermion rest-mass, e.g., the free-electron mass, and b is the scattering amplitude which now is independent of energy.

In addition, we have:

$$k_F(r)=\left[3\pi^2n(r)\right]^{1/3}, \tag{2}$$

where $n(r)$ stands for fermion spatial density.

The first factor on the right-hand side of (1) is the non-relativistic Fermi energy at zero absolute temperature. In order to pass to the relativistic domain, we will replace the above factor by the following expression:

$$E_F(r;T=0)=c\sqrt{p_F^2(r)+m_0^2c^2}, \tag{3}$$

where $p_F(r)$ denotes the magnitude of the Fermi momentum and c is the speed of light in vacuum.

The ultrarelativistic Fermi energy at zero absolute temperature comes from relation (3) with $p_F(r)=\hbar k_F(r)$ and neglecting $m_0^2c^2$, so we get:

$$E_F(r;T=0)\simeq\hbar ck_F(r). \tag{4}$$

The second factor (which, as one sees, is the sum of three terms) on the right-hand side of (1) holds in the relativistic case[2,4,7], so it is clear that the chemical-potential energy at zero absolute temperature is obtained by multiplying the right-hand side of (4) by the aforementioned sum of three terms on the right-hand side of (1). The scattering amplitude depends now on energy (Fermi energy) so, looking at the right-hand side of (4), one concludes that the above amplitude depends upon distance; then we consider the function $b(r)$. On the other hand, we want to determine the chemical potential for any value of temperature absolute T, so we consider the Sommerfeld's formula:

$$E_F(r;T)\simeq E_F(r;T=0)\left[1-\frac{\pi^2k_B^2T^2}{12E_F^2(r;T=0)}\right], \tag{5}$$

where k_B is the Boltzmann constant. Formula (5) applies if and only if the subtrahend of the second factor on the right-hand side of (5) is strictly smaller than unity, which leads to $T<2\sqrt{3}E_F(r;T=0)/\pi k_B$.

We also consider the Fermi-Dirac (or Saxon-Woods) distribution for the gas as follows:

$$n(r)=\frac{2n(R)}{1+\exp\left(\frac{r-R}{a}\right)}, \tag{6}$$

where R is the radius of the (spherical) gas and a is the thickness of its surface.

Then, multiplying actually (4) by the second factor of (1) and by using formulae (2), (5) and (6), one gets the following three new terms of the chemical-potential energy:

$$E_F(r;T)\simeq\frac{hc}{2\pi}\left[\frac{6\pi^2n(R)}{1+\exp\left(\frac{r-R}{a}\right)}\right]^{1/3}-\frac{\pi^3k_B^2T^2}{6hc}\left[\frac{6\pi^2n(R)}{1+\exp\left(\frac{r-R}{a}\right)}\right]^{-1/3}, \tag{7}$$

$$V(r;T)\simeq 2hcb(r)\left[\frac{2n(R)}{\pi\sqrt{3}\left[1+\exp\left(\frac{r-R}{a}\right)\right]}\right]^{2/3}-\frac{2\pi^2k_B^2T^2b(r)}{9hc}, \tag{8}$$

$$U(r;T)\simeq\frac{4(11-2\ln 2)b^2(r)}{15}\left[\frac{3hcn(R)}{\pi\left[1+\exp\left(\frac{r-R}{a}\right)\right]}-\frac{\pi k_B^2T^2}{6hc}\left[\frac{6\pi^2n(R)}{1+\exp\left(\frac{r-R}{a}\right)}\right]^{1/3}\right]. \tag{9}$$

Formula (8) gives the optical-potential energy whereas formula (9) corresponds to the part of the chemical-potential energy proportional to the cube of the Fermi wavenumber (see eqs.(1) and (4)). With respect to expression (8), we want to note that if $b(r)>0$, then the optical potential is repulsive while it is attractive if $b(r)<0$.

Now we wish to obtain an expression for the ultrarelativistic Fermi velocity at $T = 0^{\circ}K$. To get this end, we consider first that $mv_F = \hbar k_F$, where m denotes relativistic mass and v_F is the magnitude of the Fermi velocity. The above formula becomes:

$$\frac{m_0 v_F(r)}{\sqrt{1-\frac{v_F^2(r)}{c^2}}} = \hbar k_F(r)\,. \tag{10}$$

From eq.(10) it follows:

$$v_F(r) = \frac{\hbar c k_F(r)}{\sqrt{m_0^2c^2+\hbar^2 k_F^2(r)}}\,. \tag{11}$$

In order to derive from eq.(11) an approximate formula for the ultrarelativistic case $\hbar k_F(r) \ggg m_0 c$, we expand (11) as a function of $m_0 c/\hbar k_F(r)$ in Taylor series around the origin up to the second order, so that, after simple calculations, one obtains:

$$v_F(r) \simeq c\left[1-\frac{m_0^2c^2}{2\hbar^2 k_F^2(r)}\right]\,. \tag{12}$$

Formula (12) in conjunction with formulae (2) and (6) yields:

$$v_F(r) \simeq c\left[1-\frac{m_0^2c^2}{2\hbar^2}\left[\frac{6\pi^2 n(R)}{1+\exp\left(\frac{r-R}{a}\right)}\right]^{-2/3}\right]\,. \tag{13}$$

From relation (13) it follows that $v_F \to c$ as $r \to \infty$. In this respect, for $r \gg R$, neglecting unity in the Saxon-Woods expression inserted into (13), we get:

$$v_F(r) \simeq c\left[1-\frac{m_0^2c^2}{2\hbar^2}[6\pi^2 n(R)]^{-2/3} \cdot \exp\left[\frac{2(r-R)}{3a}\right]\right]\,. \tag{14}$$

For determining the ultrarelativistic Fermi velocity, now we proceed by doing the following steps: (1) taking $1 - v_F/c = x$, so the left-hand side of (10) becomes $m_0 c(1-x)/(\sqrt{x}\sqrt{2-x})$; (2) expanding $(1-x)/\sqrt{2-x}$ in Taylor series around $x = 0$ up to the first order; (3) keeping $\sqrt{x}$. Then, by executing steps (1), (2) and (3), and taking into account the right-hand side of (10), we arrive at:

$$\frac{m_0 c}{\sqrt{2\left[1-\frac{v_F(r)}{c}\right]}} - \frac{3m_0 c}{4\sqrt{2}}\sqrt{1-\frac{v_F(r)}{c}} \simeq \hbar k_F(r)\,. \tag{15}$$

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The second term on the left-hand side of (15) may be neglected so, from the new expression (15), the reader can appreciate that $v_F(r)$ coincides with formula (12).

Next we will tackle a linear operator, namely, the scattering-amplitude operator $\hat{b}$, such that its expectation value reads $\bar{b} = \langle \psi | \hat{b} | \psi \rangle$. Therefore, we have:

$$\bar{b} = \int_{\Omega} |\psi(r)|^2 b(r) d\Omega\,, \tag{16}$$

where $\psi(r)$ represents wavefunction associated with the time-independent relativistic Schrödinger equation, and Ω is the spherical region which contains the Fermi gas (for the sake of brevity of notation, we denote the above spatial region and its volume by the same symbol). Given that $\int_{\Omega} n(r) d\Omega = N$, where N is the number of particles, and $\int_{\Omega} |\psi(r)|^2 d\Omega = 1$, then one has that $n(r) = N|\psi(r)|^2$, so, by formulas (16) and (6) and taking into account that $d\Omega = 4\pi r^2 dr$, we get:

$$\bar{b} = \frac{8\pi n(R)}{N}\int_0^R \frac{r^2 b(r)}{1+\exp\left(\frac{r-R}{a}\right)} dr\,. \tag{17}$$

3 Discussion and conclusions

Investigating the physical mechanisms underlying ultrarelativistic degenerate Fermi gases interacting with atomic nuclei is an exciting task. Indeed, studying the behaviour of ultrarelativistic fermionic particles is manifestly relevant for several areas of Physics. In the context of the above behaviour, finding the chemical potential of the aforementioned gases is really useful from both the theoretical and practical points of view. It is well-known that elucidating the physics of Fermi gases is crucial in fields as, for instance, nanophysics[8] and, particularly, nanoscale superconductivity.[9,10] The optical potential, a concept arising from the many-body theory, appears as a very important piece of the chemical potential (see relations (1) and (8)). In this respect, we have derived formulas (7), (8), (9), (13) and (17) as main results of our formulation. Relations (7), (8) and (9) present an approximate quadratic dependence on T coming from expression (5). Moreover, formulae (8) and (9) exhibit a significant dependence upon $b(R)$. At this point, it is interesting to remark that the Fermi gas in question is dilute if $|b(r)|k_F(r) \ll 1$ and very dilute if $|b(r)|k_F(r) \lll 1$. Consequently, if the gas is dilute, then $U(r;T)$ may be neglected (see formulae (1) and (9)) so that the chemical potential is approximately equal to the Fermi level plus the optical potential whereas, if the gas is very dilute, then both $U(r;T)$ and $V(r;T)$ may be neglected (see formulae (1), (8) and (9)), so now the chemical potential equals approximately the Fermi level.

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