Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/100209

Quasi-2-Normed Spaces and Some Fixed Point Theorems

Kristaq Kikina, Luljeta Gjoni and Kostaq Hila[*]

Department of Mathematics and Computer Science, Faculty of Natural Sciences, University of Gjirokastra, Gjirokastra 6001, Albania

Received: 9 Aug. 2015, Revised: 27 Oct. 2015, Accepted: 28 Oct. 2015
Published online: 1 Mar. 2016

Abstract: In [9], it was introduced the notion of quasi-2-normed spaces. In that paper and the others following it, no example of a quasi-2-normed space not being a 2-normed space is provided. In this paper it is shown the existence of quasi-2-normed spaces and also, there are provided theorems that extend in quasi-2-normed spaces some well-known theorems for almost contractions.

Keywords: Almost contractions, 2-normed spaces, quasi-2-normed spaces, real linear spaces

1 Introduction

Gähler [5] initiated the Theory of 2-normed and 2-Banach spaces. These new spaces have subsequently been studied by several mathematicians (for example [1], [2], [3], [6], [7], [8]). In 2006, Park introduced the concepts of quasi-2-normed spaces and quasi-$(2;p)$-normed spaces [9]. In the up today current literature, it is not mentioned the existence of those spaces. We emphasize the fact that the examples given in [7] for quasi-2-normed spaces are not rigorous, those spaces own to the classes of 2-normed spaces.

In this paper, we construct and provide some examples which solve the problem of the existence of quasi-2-normed spaces.

Berinde [11] introduced a large class of contractive mappings, initially called weak contractions, but for which Berinde and Pacurar [13] later adopted the more suggestive term of almost contractions. Kikina et al. [12] obtained some theorems for almost contractions in generalized metric spaces.

In this paper, our main aim is to obtain some theorems for almost contractions in quasi-2-normed spaces.

Let us recall some definitions and results.

Definition 1.[5] *Let X be a real linear space of dimension greater than 1 and let $\|\cdot,\cdot\|$ be a real valued function on $X\times X$ satisfying the following four conditions:*
($2N_1$) $\|x,y\| = 0$ if and only if x and y are linearly dependent in X,
($2N_2$) $\|x,y\| = \|y,x\|$ for all $x,y\in X$
($2N_3$) $\|x,\alpha y\| = |\alpha|\cdot\|x,y\|$ for every real number α;
($2N_4$) $\|x,y+z\| \le \|x,y\| + \|x,z\|$ for all $x,y,z\in X$
The function $\|\cdot,\cdot\|$ is called a 2-norm on X and the pair $(X,\|\cdot,\cdot\|)$ is called a linear 2-normed space.

So a 2-norm $\|x,y\|$ always satisfies $\|x,y+\alpha x\| = \|x,y\|$, for all $x,y\in X$ and all scalars α.

We cite some examples of 2-normed spaces from the current literature.

*Example 1.*Let $X = R^3$. Define $\|x,y\| = \max\{|x_1y_2-x_2y_1|,|x_1y_3-x_3y_1|,|x_2y_3-x_3y_2|\}$, where $x=(x_1,x_2,x_3)$, $y=(y_1,y_2,y_3)\in R^3$. Then $\|x,y\|$ is a 2-norm on R^3 (see [4]).

*Example 2.*Let P_n denotes the set of real polynomials of degree less than or equal to n, on the interval [0, 1]. By considering usual addition and scalar multiplication, P_n is a linear vector space. Let $\{x_1,x_2,\ldots,x_{2n}\}$ be distinct fixed points in [0, 1] and define the 2-norm on P_n as $\|f,g\| = \sum_{k=1}^{2n}|f(x_k)g'(x_k)-f'(x_k)g(x_k)|$. Then $(P_n,\|f,g\|)$ is a 2-normed space (see [1]).

Definition 2.[9] *Let X be a linear space. A quasi-2-normed is a real valued function on $X\times X$ satisfying three conditions of Definition 1: ($2N_1$), ($2N_2$), ($2N_3$) and the condition ($2N_4^\bullet$): There is a constant $s\ge 1$ such that $\|x+y,z\| \le s\|x,z\| + s\|y,z\|$ for all $x,y,z\in X$.*

The pair $(X,\|\cdot,\cdot\|)$ is called quasi-2-normed space if $\|\cdot,\cdot\|$ is a quasi-2-norm on X. The smallest possible s is called the modulus of concavity of $\|\cdot,\cdot\|$.

[*] Corresponding author e-mail: kostaq_hila@yahoo.com

A quasi-2-norm $\|\cdot,\cdot\|$ is called a quasi-$(2;p)$-norm $(0 < p \leq 1)$ if $\|x+y,z\|^p \leq \|x,z\|^p + \|y,z\|^p$ for all $x,y,z \in X$.

Every 2-normed space is a special case of quasi-2-normed spaces (for $s = 1$). In the following section we provide some examples of quasi-2-normed spaces which are not 2-normed spaces.

Definition 3. *A sequence $\{x_n\}$ in a quasi-2-normed space $(X,\|\cdot,\cdot\|)$ is said to be a Cauchy sequence if $\lim\limits_{m,n\to\infty} \|x_m - x_n, u\| = 0$ for all u in X.*

Definition 4. *A sequence $\{x_n\}$ in a quasi-2-normed space $(X,\|\cdot,\cdot\|)$ is said to be convergent if there is a point x in X such that $\lim\limits_{n\to\infty} \|x_n - x, y\| = 0$ for all y in X. If $\{x_n\}$ converges to x, we write $\{x_n\} \to x$ as $n \to \infty$.*

Definition 5. *A linear quasi-2-normed space $(X,\|\cdot,\cdot\|)$ is said to be complete if every Cauchy sequence is convergent to an element of X.*

Definition 6. *A complete quasi-2-normed space is called a quasi-2-Banach space.*

Definition 7. *Let $T : X \to X$ be a mapping where $(X,\|\cdot,\cdot\|)$ is a quasi-2-normed space. For each $x \in X$, let $O(x) = \{x, Tx, T^2x, \ldots\}$ which will be called the orbit of T at x. $(X,\|\cdot,\cdot\|)$ is called T-orbitally complete if and only if every Cauchy sequence which is contained in $O(x)$ converges to a point in X.*

Definition 8. *[11] Let (X,d) be a metric space. A map $T : X \to X$ is called an almost contraction if there exist a constant $\delta \in (0,1)$ and some $L \geq 0$ such that*

$$d(Tx,Ty) \leq \delta d(x,y) + Ld(y,Tx) \text{ for all } x,y \in X.$$

In order to give a more generalizing character to the main results of this paper, we will use the following class of implicit functions:

Definition 9. *[12] The set of all upper semi-continuous functions with 5 variables $f : R_+^5 \to R$ satisfying the properties:*

(a). f is non decreasing in respect with each variable.
(b). $f(t,t,t,t,t) \leq t$, $t \in R_+$

will be noted $\mathbb{F}_5$ and every such function will be called a $\mathbb{F}_5$-function.

Some examples of $\mathbb{F}_5$-function are as follows:

1. $f(t_1,t_2,t_3,t_4,t_5) = \max\{t_1,t_2,t_3,t_4,t_5\}$
2. $f(t_1,t_2,t_3,t_4,t_5) = [\max\{t_1t_2, t_2t_3, t_3t_4, t_4t_5, t_5t_1\}]^{1/2}$
3. $f(t_1,t_2,t_3,t_4,t_5) = [\max\{t_1^p, t_2^p, t_3^p, t_4^p, t_5^p\}]^{1/p}$, $p > 0$
4. $f(t_1,t_2,t_3,t_4,t_5) = (a_1t_1^p + a_2t_2^p + a_3t_3^p + a_4t_4^p + a_5t_5^p)^{\frac{1}{p}}$, where $p > 0$ and $0 \leq a_i, \sum_{i=1}^5 a_i \leq 1$
5. $f(t_1,t_2,t_3,t_4,t_5) = \frac{t_1+t_2+t_3}{3}$ or $f(t_1,t_2,t_3,t_4,t_5) = \frac{t_1+t_2}{2}$ etc.

We state the following lemma which we will use for the proof of the main theorem.

Lemma 1. *Let $(X,\|\cdot,\cdot\|)$ be a quasi-2-normed space with the coefficients $s \geq 1$ and $\{x_n\}$ is a sequence in X. If $\|x_n - x_{n+1}, u\| \leq c^n l$, for all $u \in X$ and $n \in N$, where $0 \leq c < \frac{1}{s} \leq 1$, $l \geq 0$, then $\{x_n\}$ is a Cauchy sequence.*

Proof.

$$\begin{aligned}
&\|x_n - x_{n+m}, u\| \leq s\|x_n - x_{n+1}, u\| + s\|x_{n+1} - x_{n+m}, u\|) \\
&\leq s\|x_n - x_{n+1}, u\| + s^2\|x_{n+1} - x_{n+2}, u\| \\
&+ s^2\|x_{n+2} - x_{n+m}, u\| \leq \ldots \\
&\leq s\|x_n - x_{n+1}, u\| + s^2\|x_{n+1} - x_{n+2}, u\| \\
&+ s^3\|x_{n+2} - x_{n+3}, u\| + \ldots \\
&+ s^{m-2}\|x_{n+m-3} - x_{n+m-2}, u\| + s^{m-1}\|x_{n+m-2} - x_{n+m-1}, u\| \\
&+ s^{m-1}\|x_{n+m-1} - x_{n+m}, u\| \\
&\leq sc^n l + s^2c^{n+1}l + s^3c^{n+2}l + \ldots + s^{m-1}c^{n+m-2}l + s^m c^{n+m-1}l \\
&\leq sc^n l\frac{1-(sc)^m}{1-sc} \leq sc^n l\frac{1-(sc)^m}{1-sc} < \frac{sc^n l}{1-sc}
\end{aligned}$$

And so $\lim\limits_{n\to\infty} \|x_n - x_{n+m}, u\| = 0$. It implies that $\{x_n\}$ is a Cauchy sequence in X.

2 Some quasi-2-normed spaces

Example 3. Let $X = R^3$ and $x = x_1i + x_2j + x_3k$, $y = y_1i + y_2j + y_3k \in R^3$. Define

$$\|x,y\| = s\left|x_{i_0}y_{i_0+1} - x_{i_0+1}y_{i_0}\right| + \sum_{i\neq i_0}^{3} |x_iy_{i+1} - x_{i+1}y_i|$$

where $\left|x_{i_0}y_{i_0+1} - x_{i_0+1}y_{i_0}\right| = \min\{|x_iy_{i+1} - x_{i+1}y_i| : 1 \leq i \leq 3\}$, $x_4 = x_1, y_4 = y_1$ and $s > 1$. Then $(R^3, \|x,y\|)$ is a quasi-2-normed space.

Proof. The conditions (2N$_1$), $(2N_2)$ and $(2N_3)$ are satisfied and this is evident. Let us prove the condition $(2N_4^{\bullet})$: If

$$\left|x_{i_0}(y_{i_0+1} + z_{i_0+1}) - x_{i_0+1}(y_{i_0} + z_{i_0})\right| = \\
\min\{|x_i(y_{i+1} + z_{i+1}) - x_{i+1}(y_i + z_i)| : 1 \leq i \leq 3\},$$

we have

$$\|x,y+z\| = s\left|x_{i_0}(y_{i_0+1}+z_{i_0+1}) - x_{i_0+1}(y_{i_0}+z_{i_0})\right| + \sum_{i\neq i_0}^{3} |x_i(y_{i+1}+z_{i+1}) - x_{i+1}(y_i+z_i)|$$
$$\leq s(\left|x_{i_0}(y_{i_0+1}+z_{i_0+1}) - x_{i_0+1}(y_{i_0}+z_{i_0})\right| + \sum_{i\neq i_0}^{3} |x_i(y_{i+1}+z_{i+1}) - x_{i+1}(y_i+z_i)|)$$
$$\leq s\sum_{i=1}^{3} |x_i(y_{i+1}+z_{i+1}) - x_{i+1}(y_i+z_i)|$$
$$\leq s\sum_{i=1}^{3} |x_iy_{i+1} - x_{i+1}y_i| + s\sum_{i=1}^{3} |x_iz_{i+1} - x_{i+1}z_i|$$
$$\leq s\|x,y\| + s\|x,z\|$$

Thus, $(R^3, \|x,y\|)$ is a quasi-2-normed space.

At last, let us show that $(R^3, \|x,y\|)$ defined as above, is not a 2-normed space.

For $x = (0,1,-1)$; $y = (0,2,1)$ and $z = (1,0,0)$ we have

$$\|x,y+z\| = \|(0,1,-1),(1,2,1)\| = s\cdot 1+3+1 = s+4$$
$$\|x,y\| = \|(0,1,-1),(0,2,1)\| = s\cdot 0+3+0 = 3$$
$$\|x,z\| = \|(0,1,-1),(1,0,0)\| = 1+s\cdot 0+1 = 2$$

and $\|x,y+z\| = s+4 > \|x,y\| + \|x,z\| = 3+2 = 5$.

That is, the condition $(2N_4)$ is not satisfied. Therefore, for every $s > 1$, the quasi-2-normed space $(R^3, \|x,y\|)$ is not a 2-normed space.

*Example 4.*Let P_2 denotes the set of real polynomials of degree 2, on the interval [0, 1]. By considering usual addition and scalar multiplication, P_2 is a linear vector space. Let $\{x_1,x_2,x_3,x_4\}$ be distinct fixed points in [0, 1]. Define the quasi-2-norm on P_2 as

$$\|f,g\| = s\left|f(x_{i_0})g'(x_{i_0}) - f'(x_{i_0})g(x_{i_0})\right| + \sum_{i\neq i_0}^{4} \left|f(x_i)g'(x_i) - f'(x_i)g(x_i)\right|,$$

where

$$\left|f(x_{i_0})g'(x_{i_0}) - f'(x_{i_0})g(x_{i_0})\right| = \min\{\left|f(x_i)g'(x_i) - f'(x_i)g(x_i)\right| : 1\leq i\leq 4\} \text{ and } s > 1.$$

Then $(P_2, \|f,g\|)$ is a quasi-2-normed space.

*Proof.*The conditions $(2N_1)$, $(2N_2)$ and $(2N_3)$ are satisfied and this is evident. Let us prove the condition $(2N_4^{\bullet})$: If

$$\left|f(x_{i_0})g'(x_{i_0}) - f'(x_{i_0})g(x_{i_0})\right| = \min\{\left|f(x_i)g'(x_i) - f'(x_i)g(x_i)\right| : 1\leq i\leq 4\} \text{ we have}$$

$$\|f,g+h\| = s\left|f(x_{i_0})(g+h)'(x_{i_0}) - f'(x_{i_0})(g+h)(x_{i_0})\right| + \sum_{i\neq i_0}^{4} \left|f(x_i)(g+h)'(x_i) - f'(x_i)(g+h)(x_i)\right|$$
$$\leq s(\left|f(x_{i_0})(g+h)'(x_{i_0}) - f'(x_{i_0})(g+h)(x_{i_0})\right| + \sum_{i\neq i_0}^{4} \left|f(x_i)(g+h)'(x_i) - f'(x_i)(g+h)(x_i)\right|)$$
$$= s(\sum_{i=1}^{4} \left|f(x_i)(g+h)'(x_i) - f'(x_i)(g+h)(x_i)\right|)$$
$$\leq s(\sum_{i=1}^{4} \left|f(x_i)g'(x_i) - f'(x_i)g(x_i)\right| + \sum_{i=1}^{4} \left|f(x_i)h'(x_i) - f'(x_i)h(x_i)\right|) \leq s\|x,y\| + s\|x,z\|$$

Thus, $(P_2, \|f,g\|)$ is a quasi-2-normed space.

At last, let us show that $(P_2, \|f,g\|)$ defined as above, is not a 2-normed space.

Let us consider the case $x_1 = 1, x_2 = \frac{1}{2}, x_3 = \frac{1}{3}$ and $x_4 = 0$. For $f = x, g = x^2$ and $h = (x-1)^2$, we have

$$\left|f(x_1)(g+h)'(x_1) - f'(x_1)(g+h)(x_1)\right| = |2\cdot 0 - 2\cdot 1| = 2$$
$$\left|f(x_2)(g+h)'(x_2) - f'(x_2)(g+h)(x_2)\right| = \left|\tfrac{5}{4}\cdot(-1) - 1\cdot\tfrac{5}{4}\right| = \tfrac{5}{2}$$
$$\left|f(x_3)(g+h)'(x_3) - f'(x_3)(g+h)(x_3)\right| = \left|\tfrac{10}{9}\cdot(-\tfrac{4}{3}) - \tfrac{2}{3}\cdot\tfrac{13}{9}\right| = \tfrac{66}{27} = \tfrac{22}{9}$$
$$\left|f(x_4)(g+h)'(x_4) - f'(x_4)(g+h)(x_4)\right| = |1\cdot(-2) - 0\cdot 2| = 2$$

and $\|f,g+h\| = s\cdot 2 + \frac{5}{2} + \frac{22}{9} + 2 = s\cdot 2 + \frac{125}{18}$.

In similar way, we get

$$\left|f(x_1)g'(x_1) - f'(x_1)g(x_1)\right| = |2\cdot 0 - 2\cdot 0| = 0$$
$$\left|f(x_2)g'(x_2) - f'(x_2)g(x_2)\right| = \left|\tfrac{5}{4}\cdot(-1) - 1\cdot\tfrac{1}{4}\right| = \tfrac{3}{2}$$
$$\left|f(x_3)g'(x_3) - f'(x_3)g(x_3)\right| = \left|\tfrac{10}{9}\cdot(-\tfrac{4}{3}) - \tfrac{2}{3}\cdot\tfrac{4}{9}\right| = \tfrac{48}{27} = \tfrac{16}{9}$$
$$\left|f(x_4)g'(x_4) - f'(x_4)g(x_4)\right| = |1\cdot(-2) - 0\cdot 1| = 2$$

and $\|f,g\| = s\cdot 0 + \frac{3}{2} + \frac{16}{9} + 2 = \frac{95}{18}$.

Also, we have

$$\left|f(x_1)h'(x_1) - f'(x_1)h(x_1)\right| = |2\cdot 0 - 2\cdot 1| = 2$$
$$\left|f(x_2)h'(x_2) - f'(x_2)h(x_2)\right| = \left|\tfrac{5}{4}\cdot 0 - 1\cdot 1\right| = 1$$
$$\left|f(x_3)h'(x_3) - f'(x_3)h(x_3)\right| = \left|\tfrac{10}{9}\cdot 0 - \tfrac{2}{3}\cdot 1\right| = \tfrac{2}{3}$$
$$\left|f(x_4)h'(x_4) - f'(x_4)g(x_4)\right| = |1\cdot 0 - 0\cdot 1| = 0$$

and $\|f,h\| = 2 + 1 + \frac{2}{3} + s\cdot 0 = \frac{11}{3}$.

From the above results, we get:

$\|f,g+h\| = s\cdot 2 + \frac{125}{18} > \|f,g\| + \|f,h\| = \frac{95}{18} + \frac{11}{3} = \frac{161}{18}$.

Therefore, for every $s > 1$, the quasi-2-normed space $(P_2, \|f,g\|)$ is not a 2-normed space.

Remark 1. The Examples 3 and 4 can be generalized analogously for the case of R^n and P_n respectively.

Remark 2. The examples we provided above, show that for every $s > 1$, there exist quasi-2-normed spaces which are not 2-normed spaces.

3 Main theorems

Definition 10. *Let $(X,\|\cdot,\cdot\|)$ be a quasi-2-normed space with the coefficients $s\geq 1$ and $f\in\mathbb{F}_5$. A map $T:X\to X$ is called an almost f-contraction if there exist a constant $\delta\in[0,\frac{1}{s})$ and some $L\geq 0$ such that*

$$\|T(x)-T(y),u\|\leq \delta f(\|x-y,u\|,\|x-Tx,u\|,\|y-Ty,u\|,\left\|y-T^2x,u\right\|,\|y-Tx,u\|)+L\|y-Tx,u\| \quad (1)$$

for all $x,y,u\in X$.

Theorem 1. *Let $(X,\|\cdot,\cdot\|)$ be a quasi-2-normed space with the coefficients $s\geq 1$ and $T:X\to X$ an almost f-contraction. If $(X,\|\cdot,\cdot\|)$ is T-orbitally complete, then*

1. *$Fix(T)=\{x\in X:Tx=x\}\neq\emptyset$*
2. *For any $x_0\in X$, the Picard iteration $\{x_n\}$ converges to some $\alpha\in Fix(T)$.*

Proof. Let x_0 be an arbitrary point in X. Define the sequences $\{x_n\}$ as follows:

$$x_n=Tx_{n-1}=T^nx_0, n=1,2,...$$

Take $u\in X$. Denote

$$d_n(u)=\|x_n-x_{n+1},u\|,\ n=0,1,2,...$$

By the inequality (1) we get:

$$d_n(u)=\|x_n-x_{n+1},u\|=\left\|T^nx_0-T^{n+1}x_0,u\right\|$$
$$\leq \delta f(\left\|T^{n-1}x_0-T^nx_0,u\right\|,\left\|T^{n-1}x_0-T^nx_0,u\right\|,$$
$$\left\|T^nx_0-T^{n+1}x_0,u\right\|,\left\|T^nx_0-T^{n+1}x_0,u\right\|,$$
$$\|T^nx_0-T^nx_0,u\|)+L\|T^nx_0-T^nx_0,u\|$$
$$=\delta f(d_{n-1}(u),d_{n-1}(u),d_n(u),d_n(u),0)+0\leq \delta d_{n-1}(u)$$

And so, inductively, we obtain

$$d_n(u)\leq \delta^n d_0(u)=\delta^n l, n\in N \quad (2)$$

where $l=d_0(u)=\|x_0-x_1,u\|$.

Then, from (2) and Lemma 1 is derived that $\{x_n\}$ is a Cauchy sequence in X and hence is convergent in X. Let $\lim\limits_{n\to\infty}x_n=\lim\limits_{n\to\infty}T^nx_0=\alpha\in X$. The limit α is unique: Assume that $\alpha'\neq\alpha$ is also $\lim\limits_{n\to\infty}x_n$. Then by condition $(2N_4^{\bullet})$ of Definition 2, we obtain

$$\left\|\alpha-\alpha',u\right\|\leq k\|\alpha-x_n,u\|+k\left\|x_n-\alpha',u\right\|$$

Letting n tend to infinity we get $\|\alpha-\alpha',u\|=0$ for all $u\in X$ and so $\alpha=\alpha'$.

Let we prove now that α is a fixed point of T. Assume that $\alpha\neq T\alpha$. Then, by Definition 2, we obtain $\|\alpha-T\alpha,u\|\leq s\|\alpha-x_n,u\|+s\|x_n-T\alpha,u\|$

And so, if $n\to\infty$, we get

$$\|\alpha-T\alpha,u\|\leq s\overline{\lim_{n\to\infty}}\|x_n-T\alpha,u\| \quad (3)$$

From (1),

$$\|x_n-T\alpha,u\|=\|Tx_{n-1}-T\alpha,u\|$$
$$\leq \delta f(\|x_{n-1}-\alpha,u\|,\|x_{n-1}-Tx_{n-1},u\|,\|\alpha-T\alpha,u\|,$$
$$\left\|\alpha-T^2x_{n-1},u\right\|,\|\alpha-Tx_{n-1},u\|)+L\|\alpha-Tx_{n-1},u\|$$
$$=\delta f(\|x_{n-1}-\alpha,u\|,\|x_{n-1}-x_n,u\|,\|\alpha-T\alpha,u\|,$$
$$\|\alpha-x_{n+1},u\|,\|\alpha-x_n,u\|)+L\|\alpha-x_n,u\|.$$

Letting n tend to infinity we have

$$\overline{\lim_{n\to\infty}}\|x_n-T\alpha,u\|\leq \delta f(0,0,\|\alpha-T\alpha,u\|,0,0)$$
$$\leq \delta\|\alpha-T\alpha,u\|. \quad (4)$$

From (3) and (4),

$$\|\alpha-T\alpha,u\|\leq s\overline{\lim_{n\to\infty}}\|x_n-T\alpha,u\|\leq s\delta\|\alpha-T\alpha,u\|$$

Since $0\leq s\delta<1$ we have $\|\alpha-T\alpha,u\|=0$ for all $u\in X$. So α is a fixed point of T and this completes the proof.

Theorem 2. *Let $(X,\|\cdot,\cdot\|)$ be a quasi-2-normed space with the coefficients $s\geq 1$ and $T:X\to X$ be a mapping. If $(X,\|\cdot,\cdot\|)$ is T-orbitally complete, $Fix(T)\neq\emptyset$ and satisfies the following inequality:*

$$\|T(x)-T(y),u\|\leq \delta_1 f(\|x-y,u\|,\|x-Tx,u\|,\|y-Ty,u\|,\left\|y-T^2x,u\right\|,\|y-Tx,u\|)+L_1\|x-Tx,u\| \quad (5)$$

for all $x,y,u\in X$, where $\delta_1\in[0,\frac{1}{s})$ and $L_1\geq 0$, then T has a unique fixed point.

Proof. Let $\alpha\in Fix(T)$. Assume that $\alpha'\neq\alpha$ is also a fixed point of T. From (5),

$$\left\|\alpha-\alpha',u\right\|=\left\|T(\alpha)-T(\alpha'),u\right\|$$
$$\leq \delta_1 f(\left\|\alpha-\alpha',u\right\|,\|\alpha-T\alpha,u\|,\left\|\alpha'-T\alpha',u\right\|,$$
$$\left\|\alpha'-T^2\alpha,u\right\|,\left\|\alpha'-T\alpha,u\right\|)+L_1\|\alpha-T\alpha,u\|$$
$$=\delta_1 f(\left\|\alpha-\alpha',u\right\|,0,0,\left\|\alpha'-\alpha,u\right\|,\left\|\alpha'-\alpha,u\right\|)+L_1\cdot 0$$
$$\leq \delta_1\left\|\alpha-\alpha',u\right\|$$

Since $0<\delta_1<1$, we have $\alpha=\alpha'$. This completes the proof.

Theorem 3. *Let $(X,\|\cdot,\cdot\|)$ be a quasi-2-normed space with the coefficients $s\geq 1$ and $T:X\to X$ be a mapping. If $(X,\|\cdot,\cdot\|)$ is T-orbitally complete, $T:X\to X$ is an almost f-contraction and satisfies the inequality (5), then:*

1. *T has a unique fixed point, i.e. $Fix(T)=\{\alpha\}$.*
2. *For any $x_0\in X$, the Picard iteration $\{x_n\}$ converges to α.*

Proof. The conditions of Theorem 1 hold, and so $Fix(T)\neq\emptyset$. By Theorem 2 $Fix(T)=\{\alpha\}$ and for any $x_0\in X$, the Picard iteration $\{x_n\}$, $x_n=Tx_{n-1}, n\in N$, converges to α. This completes the proof of the theorem.

*Example 5.*Let $(X,\|.,.\|)$ be a quasi-2-normed space with the coefficient $s \geq 1$. Let $T : X \to X$ be the identity mapping such that $Tx = x$ for all $x \in X$.

Actually, the quasi-2-normed space $(X,\|.,.\|)$ is T-orbitally complete.

We verify the conditions of Theorem 3 in case $f(t_1,t_2,t_3,t_4,t_5) = t_1$. The inequality (1) takes the form

$$\|T(x)-T(y),u\| \leq \delta\,\|x-y,u\| + L\,\|y-Tx,u\| \qquad (6)$$

The above inequality takes the form

$$\|T(x)-T(y),u\| = \|x-y,u\| \leq \delta\,\|x-y,u\| + L\,\|y-x,u\|$$

and consequently it is satisfied for any $\delta \in [0,\frac{1}{s})$ and $L \geq 1$.

All the conditions of Theorem 3 are satisfied. The $Fix(T) = X$ and, for any $x \in X$, the Picard iteration $\{T^n x\}$ converges to x.

*Example 6.*Let $X = P_2$ be the set of real polynomials of degree 2 on the interval [0, 1]. Let $(X,\|.,.\|)$ be the quasi-2-normed space with the coefficients $s = \frac{3}{2} > 1$ of Example 4. Let $T : X \to X$ be a mapping such that $Tx = \frac{1}{2}x$.

We verify the conditions of Theorem 2 in case $f(t_1,t_2,t_3,t_4,t_5) = t_1$. The quasi-2-normed space $(X,\|.,.\|)$ is T-orbitally complete, since $T^n x = (\frac{1}{2})^n x$ and $\lim_{n\to 0} T^n x = 0 \in X$. The map T has at least one fixed point $(0 \in Fix(T) \neq \emptyset)$, the condition (5) takes the form:

$$\begin{aligned}\|T(x)-T(y),u\| &= \left\|\tfrac{1}{2}x-\tfrac{1}{2}y,u\right\| = \tfrac{1}{2}\,\|x-y,u\| \\ &\leq \delta_1\,\|x-y,u\| + L_1\left\|x-\tfrac{1}{2}x,u\right\|\end{aligned}$$

and consequently it is satisfied for any $\delta_1 \in [\frac{1}{2}, \frac{1}{s} = \frac{2}{3})$ and $L_1 \geq 0$. All the conditions of Theorem 2 are satisfied. The $Fix(T) = \{0\}$ and, for any $x \in X$, the Picard iteration $\{T^n x\}$ converges to 0.

4 Corollaries

For different expressions of f in the Theorems 1, 2 and 3 we get different Theorems. We give some of them:

1) If $f(t_1,t_2,t_3,t_4,t_5) = t_1$, then by Theorem 1 we obtain a fixed point Theorem that extends the well-known Berinde weak (almost) contraction principle (Theorem 1 in [11]) in a quasi-2-normed space.

2) By Theorem 1, with $L = 0$ it follows Theorem 2 with $L_1 = \{0\}$ and conversely. So, the case $L = 0$ or $L_1 = 0$ implies the existence and uniqueness of the fixed point.

3) If $f(t_1,t_2,t_3,t_4,t_5) = t_1$ and $L = 0$, then by Theorem 1 we obtain a fixed point Theorem that extends the well-known Banach's contraction principle in a quasi-2-normed space.

4) If $f(t_1,t_2,t_3,t_4,t_5) = \frac{t_2+t_3}{2}$ and $L = 0$, then by Theorem 1 we obtain a fixed point Theorem that extends the well-known Kannan contraction principle in a quasi-2-normed space.

5) If $f(t_1,t_2,t_3,t_4,t_5) = \max\{t_2,t_3\}$ and $L = 0$, then by Theorem 1 we obtain a fixed point Theorem that extends the well-known Bianchini Contraction principle [15] in a quasi-2-normed space.

Remark 3. For suitable forms of f we can obtain several other corollaries that extend well-known theorems of Rhoades classifications [14] in a quasi-2-normed space (or in a 2-normed space for $s = 1$).

Acknowledgement

The authors wish to express their warmest thanks to the referees for a careful reading of manuscript and for very useful suggestions and remarks that contributed to the improvement of initial version of the manuscript.

References

[1] M. Acikgöz, N. Aslan, N. Köskeroglu and S. Araci, p-Adic approach to linear 2-normed spaces, Mathematica Moravica **13**, No. 2, 7-22(2009).
[2] M. Acikgöz, A review on 2-normed structures, Int. J. Math. Anal. **1**, No. 4, 187-191(2007).
[3] P. Chuhan and N. Malviya, Fixed points of expansive type mappings in 2-Banach spaces, Int. J. Anal. Appl. **3**, No. 1, 60-67(2013).
[4] W.R. Freese and Y.J. Cho, Geometry of linear 2-normed spaces, Nova Science Publishers, Inc. New York, 2001.
[5] S. Gähler, Lineare 2-normierte Raume, Math. Nachr., **28**, No. 1-2, 1-43(1965).
[6] K. Iseki, Mathematics on two normed spaces, Bull. Korean Math. Soc. **13**, No. 2, 127-135(1976).
[7] M. Kir and M. Acikgöz, A study involving the completion of a quasi-2-normed space, Int. J. Anal. Volume 2013, Article ID 512372, 4 pages.
[8] S. Mantu, D. Debashis, G. Anamika and D. Lokenath, Asymptotic regularity and fixed point theorems on a 2-Banach space, Surveys in Mathematics and its Applications **7** (2012).
[9] C. Park, Generalized quasi-Banach spaces and quasi-$(2,p)$-normed spaces, J. Chungcheong Math. Soc. **19**, No. 2, 197-206(2006).
[10] M. Abrishami-Moghaddam, On the triangle inequality in quasi 2-normed spaces and a new characterization of 2-inner product spaces, Sci. J. Pure Appl. Sci. **3**, No. 6, 433-437(2014).
[11] V. Berinde, Approximating fixed point of weak contractions using the Picard iteration, Nonlinear Analysis Forum **9**, No. 1, 43-53(2004).
[12] L. Kikina, K. Kikina and I. Vardhami, Fixed point theorems for almost contractions in generalized metric spaces. Creat. Math. Inform. **23**, No. 1, 65-72(2014).
[13] V. Berinde and M. Pacurar, Fixed points and continuity of almost contractions, Fixed Point Theory **9**, No. 1, 23-34(2008).
[14] B.E. Rhoades, A comparison of various definitions of contractive mappings. Trans. Amer. Math. Soc. **226**, 256-290(1977).

[15] R.M.T. Bianchini, Su un problema di S. Reich riguardante la teoria dei punti fissi, Boll. Un. Mat. Ital. **5**, 103-108(1972).

Kristaq Kikina is professor of Mathematics at The Department of Mathematics and Computer Science, University of Gjirokastra, Albania. He received the PhD degree in Mathematics at University of Tirana, Albania. His main research interests include Fixed points theory (in particular Metric spaces, quasi-metric spaces, GMS and GQMS) He has published several research papers in various international reputed peer-reviewed journals

Luljeta Gjoni (Kikina) is professor of Mathematics at The Department of Mathematics and Computer Science, University of Gjirokastra, Albania. She received her M.Sc. in Mathematics at University of Tirana and PhD degree in Mathematics at Polytechnic University of Tirana, Albania. Her main research interests include Fixed points theory (in particular Metric spaces, quasi-metric spaces, GMS and GQMS) She has published several research papers in various international reputed peer-reviewed journals.

Kostaq Hila is professor of Mathematics at The Department of Mathematics and Computer Science, University of Gjirokastra, Albania. He received his M.Sc. and PhD degree in Mathematics at University of Tirana, Albania. His main research interests include algebraic structures theory (in particular algebraic theory of semigroups and ordered semigroups, LA-semigroups etc.), algebraic hyperstructures theory, fuzzy-rough-soft sets and applications. He has published several research papers in various international reputed peer-reviewed journals. He is a referee of several well-known international peer-reviewed journals.