Applied Mathematics & Information Sciences
An International Journal

Series Solutions for Nonlinear Partial Differential Equations with Slip Boundary Conditions for non-Newtonian MHD Fluid in Porous Space

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Received: 21 Jan. 2012; Revised 27 Aug. 2012; Accepted 28 Aug. 2012
Published online: 1 Jan. 2013

Abstract: The fully developed flow of an incompressible, thermodynamically compatible non-Newtonian magnetohydrodynamics (MHD) fluid in a pipe with porous space and partial slip is studied in this paper. Two illustrative models of viscosity namely (i) Constant model and (ii) Variable model are considered. Series solutions for nonlinear coupled partial differential equations are first developed and then convergence of the obtained series solutions has been discussed explicitly. The recurrence formulae for finding the coefficients are also given in each case. Finally the role of pertinent parameters is illustrated graphically.

Keywords: Nonlinear coupled partial differential equations, slip boundary conditions, series solution, convergence, MHD non-Newtonian fluid, porosity.

1. Introduction

The flows of non-Newtonian fluids are encountered in various industrial and technological applications [1-7]. A number of industrially important fluids such as molten plastics, polymers, pulps, foods and slurries display non-Newtonian fluid behavior. Moreover, the analysis of heat transfer due to the flow of non-Newtonian fluids has been an active research topic nowadays. The thermal boundary layer flows of non-Newtonian fluids have practical importance in various engineering systems such as the design of thrust bearing and radial diffusers, transpiration cooling, drag reduction, thermal recovery of oil, etc. Furthermore blood flow (with shear rate below $100S^{-1}$) in the coronary arteries represents a mathematical model of non-Newtonian MHD fluid in a porous medium. It is also well accepted now that slip effects may appear for two types of fluids (i.e., rare field gases [8] and fluids having much more elastic character). It is noticed through experiment observations [9, 10] that the occurrence of slippage is possible in the non-Newtonian fluids such as polymer solution and molten polymer as well. The fluids that exhibit slip effect have many applications, for instance, the polishing of artificial heart valves and internal cavities [11]. In addition, porous media is used to transport and store energy in many industrial applications, such as heat pipe, solid matrix heat exchangers, electronic cooling, and chemical reactors. An important characteristic for the combination of the fluid and the porous medium is the tortuosity which represents the hindrance to flow diffusion imposed by local boundaries or local viscosity [12].

Despite the overwhelming importance and frequent occurrence of non-Newtonian fluid behavior in industry and technology, Massoudi and Christie [13] numerically examined the pipe flow of non-Newtonian fluid for viscosity models. Hayat et al. [14] presented the homotopy solution of the problem considered in ref. [13] up to second order deformation. A very little efforts [15, 16] have been reported yet in the literature to present series solution for non Newtonian magnetohydrodynamics (MHD) fluid with slip and porosity parameters simultaneously.

Motivated by these facts, the aim of present investigations is to examine the effects of slip, MHD and porosity parameters for steady flow of an in compressible non-Newtonian fluid in a pipe. It is well known that the non-

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linear equations are usually solved by numerical methods. Here we intend to solve the relevant nonlinear coupled equations for velocity and temperature profiles by using homotopy analysis method (HAM) [17-22]. Convergence of the obtained solutions is explicitly shown. The effects of the various parameters of interest on the velocity and temperature are pointed out. The heat transfer analysis is also examined. The paper is organized as follows. Section 2 contains the problem description. In Section 3 solution of the problem is found using HAM. Section 3.1 contains the case-I for constant viscosity whereas, Section 3.2 is devoted to obtain the solution for variable viscosity. Convergence is carefully analyzed in Section 4. Results and discussion are given in Section 5 and finally our conclusion is presented in Section 6.

2. Formulation of the problem

In the present analysis, we consider the steady unidirectional flow of third grade fluid and heat transfer through a porous pipe. The fluid is electrically conducting under the application of a constant magnetic field. The flow is induced by a constant pressure gradient. The velocity field is of the form

$$\mathbf{V} = [0, 0, v(r)]\,. \tag{1}$$

The Cauchy stress tensor $\mathbf{T}$ in third grade fluid is written as

$$\begin{aligned}\mathbf{T} = &-p_1\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3\\ &+\beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(tr\mathbf{A}_1^2)\mathbf{A}_1,\end{aligned} \tag{2}$$

with p_1 as the hydrostatic pressure, μ the dynamic viscosity, $\mathbf{I}$ the identity tensor and $\alpha_i (i = 1, 2)$ and $\beta_j (j = 1-3)$ the material constants. The Rivlin-Ericksen tensors are

$$\mathbf{A}_1 = \nabla\mathbf{V} + (\nabla\mathbf{V})^t, \tag{3}$$

$$\mathbf{A}_n = \frac{d\mathbf{A}_{n-1}}{dt} + \mathbf{A}_{n-1}\nabla\mathbf{V} + (\nabla\mathbf{V})^t A_{n-1}, \quad n > 1, \tag{4}$$

in which ∇ is the gradient operator. Moreover, thermodynamics imposes the following constraints [23]

$$\begin{aligned}&\mu \geq 0,\ \alpha_1 \geq 0,\ |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3},\\ &\beta_1 = \beta_2 = 0,\ \beta_3 \geq 0.\end{aligned} \tag{5}$$

Keeping in mind the constitutive equation, the following law is proposed [24]:

$$R_z = -\frac{\phi}{k}\left[\mu + \Lambda\left(\frac{dv}{dr}\right)^2\right]v, \tag{6}$$

where R_z is the $z-$components of Darcy's resistance $\mathbf{R}$.

The governing momentum and energy equations after employing Eqs. (5) and (6) can be expressed as

$$\begin{aligned}&\frac{1}{r}\frac{d}{dr}\left(r\mu\frac{dv}{dr}\right) + \frac{2\beta_3}{r}\frac{d}{dr}\left[r\left(\frac{dv}{dr}\right)^3\right] - R_z\\ &= -\frac{\partial\widehat{p}}{\partial z} + \sigma B_0^2\, v,\end{aligned} \tag{7}$$

$$\mu\left(\frac{dv}{dr}\right)^2 + 2\beta_3\left(\frac{dv}{dr}\right)^4 + k\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right)\right] = 0. \tag{8}$$

The corresponding slip conditions are

$$\begin{aligned}&v(R) - \gamma\left[\frac{dv}{dr}(R) + \frac{\Lambda}{\mu}\left(\frac{dv}{dr}(R)\right)^3\right] = 0;\\ &\theta(R) = 0, \qquad \frac{dv}{dr}(0) = \frac{d\theta}{dr}(0) = 0.\end{aligned} \tag{9}$$

Let us introduce the following non-dimensional parameters

$$\begin{aligned}&\Lambda = \frac{2\beta_3 v_0^2}{\mu_0 R^2},\ c = \frac{\frac{\partial p_1}{\partial z}R^2}{\mu_0 v_0},\ r = \frac{\overline{r}}{R},\\ &v = \frac{\overline{v}}{v_0},\ \mu = \frac{\overline{\mu}}{\mu_0},\ \theta = \frac{\overline{\theta} - \theta_0}{\theta_1 - \theta_0},\\ &\Gamma = \frac{\mu_0^2 v_0}{k(\theta_1 - \theta_0)},\ p = \frac{\phi}{k_1 R^2},\ \gamma = \frac{\overline{\gamma}}{R},\\ &M^2 = \frac{\sigma B_0^2 R^2}{\mu_0},\ \Gamma = \frac{\mu_0^2 v_0}{k(\theta_1 - \theta_0)}.\end{aligned} \tag{10}$$

Here ϕ is porosity, k_1 is the permeability, σ is the electrical conductivity, B_0 is the magnetic field strength, k is the thermal conductivity. and R, v_0, μ_0, θ_0, $\overline{\theta}$, θ_1 Λ, p, Γ, M, c denote the radius, reference velocity, reference viscosity, reference temperature, pipe temperature, fluid temperature, third grade parameter, porosity parameter, viscous dissipation parameter, MHD parameter, pressure gradient respectively. The modified pressure $\widehat{p}$ is given by

$$\widehat{p} = p_1 - \alpha_2\left(\frac{dv}{dr}\right)^2. \tag{11}$$

Substituting Eq. (2) in the balance of linear momentum and using the non-dimensional quantities given in Eq. (10), the dimensionless form of governing equations from Eqs. (7) to (9), after dropping bars for simplicity, lead to the following non-dimensional coupled equations

$$\begin{aligned}&\frac{d\mu}{dr}\frac{dv}{dr} + \frac{\mu}{r}\frac{dv}{dr} + \mu\frac{d^2v}{dr^2}\\ &+\frac{\Lambda}{r}\left(\frac{dv}{dr}\right)^3 + 3\Lambda\left(\frac{dv}{dr}\right)^2\frac{d^2v}{dr^2}\\ &-p\left[\mu + \Lambda\left(\frac{dv}{dr}\right)^2\right]v - M^2 v = c,\end{aligned} \tag{12}$$

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} + \Gamma\left(\frac{dv}{dr}\right)^2\left[\mu(r) + \Lambda\left(\frac{dv}{dr}\right)^2\right] = 0, \tag{13}$$

$$\begin{aligned}&v(1) - \gamma\left[\frac{dv}{dr}(1) + \frac{\Lambda}{\mu}\left(\frac{dv}{dr}(1)\right)^3\right] = 0;\\ &\theta(1) = 0, \qquad \frac{dv}{dr}(0) = \frac{d\theta}{dr}(0) = 0.\end{aligned} \tag{14}$$

3. Solution of the problem

In this section the HAM solutions will be determined for the velocity and temperature by using constant and variable viscosity models.

3.1. Case 1: Constant viscosity model

For constant viscosity model take

$$\mu = 1. \tag{15}$$

For HAM solution we select

$$v_0(r) = \frac{cr^2 + 2cr + 2c^3\Lambda\gamma - c}{2},$$
$$\theta_0 = \frac{c^2\Gamma(1-r^4)}{64}, \tag{16}$$

as initial approximations of v and θ respectively, which satisfy the corresponding boundary conditions. We use the method of higher order differential mapping, [25] to choose the linear operator $_1$ which is defined by

$$_1 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}, \tag{17}$$

such that

$$_1(C_1 + C_2 \ln r) = 0, \tag{18}$$

where C_1 and C_2 are the arbitrary constants.

The $zeroth-$order deformation problems are of the form:

$$(1-q)_1[v^*(r,q) - v_0(r)]$$
$$= q\hbar\mathcal{N}_1[v^*(r,q), \theta^*(r,q)], \tag{19}$$
$$(1-q)_1[\theta^*(r,q) - \theta_0(r)]$$
$$= q\hbar\mathcal{N}_2[v^*(r,q), \theta^*(r,q)], \tag{20}$$

$$v^*(r,q) - \gamma\left[\frac{\partial v^*(r,q)}{\partial r} + \Lambda\left(\frac{\partial v^*(r,q)}{\partial r}\right)^2\right]\Bigg|_{r=1} = 0,$$

$$\left.\frac{\partial v^*(r,q)}{\partial r}\right|_{r=0} = 0, \tag{21}$$

$$\theta^*(r,q)|_{r=1} = 0, \left.\frac{\partial \theta^*(r,q)}{\partial r}\right|_{r=0} = 0,$$

where the nonlinear operators $\mathcal{N}_1$ and $\mathcal{N}_2$ are

$$\mathcal{N}_1[v^*(r,q), \theta^*(r,q)] = \frac{1}{r}\frac{dv^*}{dr} + \frac{d^2v^*}{dr^2}$$

$$+\frac{\Lambda}{r}\left(\frac{dv^*}{dr}\right)^3 + 3\Lambda\left(\frac{dv^*}{dr}\right)^2\frac{d^2v^*}{dr^2}$$

$$-p\left[1 + \Lambda\left(\frac{dv^*}{dr}\right)^2\right]v^* - M^2v - c, \tag{22}$$

$$\mathcal{N}_2[v^*(r,q), \theta^*(r,q)] = \frac{1}{r}\frac{d\theta^*}{dr}$$

$$+\frac{d^2\theta^*}{dr^2} + \Gamma\left(\frac{dv^*}{dr}\right)^2 + \Gamma\Lambda\left(\frac{dv^*}{dr}\right)^4. \tag{23}$$

If $\hbar$ is convergence parameter and $0 \le q \le 1$ is an embedding parameter then for $q = 0$ and $q = 1$, we have

$$v^*(r,0) = v_0(r),\ \theta^*(r,0) = \theta_0(r),$$
$$v^*(r,1) = v(r),\ \theta^*(r,1) = \theta(r). \tag{24}$$

When q increases from 0 to 1, $v^*(r,q), \theta^*(r,q)$ varies from $v_0(r)$, $\theta_0(r)$ to $v(r)$, $\theta(r)$, respectively. By Taylor's theorem and Eq. (24) we have

$$v^*(r,q) = v_0(r) +_{m=1}^{\infty}\left(\frac{1}{m!}\left.\frac{\partial^m v^*(r,q)}{\partial q^m}\right|_{q=0}\right)q^m,$$
$$\theta^*(r,q) = \theta_0(r) +_{m=1}^{\infty}\left(\frac{1}{m!}\left.\frac{\partial^m \theta^*(r,q)}{\partial q^m}\right|_{p=0}\right)q^m. \tag{25}$$

The convergence of the series given in Eq. (25) depends upon the choice of $\hbar$, such that series converges at $q = 1$, then Eq. (25) becomes

$$\left.\begin{aligned} v(r) &= v_0(r) +_{m=1}^{\infty} \tfrac{1}{m!}\left.\tfrac{\partial^m v^*(r,q)}{\partial q^m}\right|_{q=0} \\ \theta(r) &= \theta_0(r) +_{m=1}^{\infty} \tfrac{1}{m!}\left.\tfrac{\partial^m \theta^*(r,q)}{\partial q^m}\right|_{q=0} \end{aligned}\right\}., \tag{26}$$

The mth order deformation problems are given by

$$_1[v_m(r) - \chi_m v_{m-1}(r)] = \hbar\Re 1_m(r), \tag{27}$$

$$_1[\theta_m(r) - \chi_m \theta_{m-1}(r)] = \hbar\Re 2_m(r), \tag{28}$$

$$v_m(1) - \gamma\left[\frac{\partial v_m(1)}{\partial r} + \Lambda\left(\frac{\partial v_m(1)}{\partial r}\right)^3\right]$$

$$= \theta_m(1) = 0,\ v_m'(0) = \theta_m'(0) = 0, \tag{29}$$

where

$$\Re 1_m(r) = \frac{1}{r}\frac{dv_{m-1}}{dr} + \frac{d^2v_{m-1}}{dr^2}$$

$$+\frac{\Lambda}{r}{}_{k=0}^{m-1}{}_{i=0}^{k}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-1}}{dr}\frac{dv_i}{dr}$$

$$+3\Lambda_{k=0}^{m-1}{}_{i=0}^{k}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-1}}{dr}\frac{d^2v_i}{dr^2}$$

$$-(1-\chi_m)c - qv_{m-1} - p\Lambda_{k=0}^{m-1}\sum_{i=0}^{k}\left(\frac{dv_{m-1-k}}{dr}\right)$$

$$\times\frac{dv_{k-1}}{dr}v_i - M^2v_{m-1}, \tag{30}$$

$$\Re 2_m(r) = \frac{1}{r}\frac{d\theta_{m-1}}{dr} + \frac{d^2\theta_{m-1}}{dr^2}$$

$$+\Gamma\sum_{k=0}^{m-1}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_k}{dr}+$$
$$\Lambda\Gamma\sum_{k=0}^{m-1}\sum_{j=0}^{k}{}_{i=0}^{j}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-j}}{dr}\frac{dv_{j-i}}{dr}\frac{dv_i}{dr} \tag{31}$$

are recurrence formulae, in which

$$\chi_m=\begin{cases}0, & m\le 1,\\ 1, & m>1.\end{cases}$$

3.2. Case II: Variable viscosity model

In this case we choose

$$v_0(r)=\frac{cr^2+2cr+2c^3\Lambda\gamma-c}{2}, \tag{32}$$

$$\mu=r,\ \ \theta_0=\frac{c^4\Gamma(1-r^2)}{64} \tag{33}$$

and the operator

$${}_2=\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}, \tag{34}$$

such that

$${}_2(C_3+\frac{C_4}{r})=0, \tag{35}$$

where C_3 and C_4 are the arbitrary constants.

The $zeroth-$ and $mth-$order order deformation problems in this case are developed as

$$(1-q)_2[v^*(r,q)-v_0(r)]$$
$$=q\hbar\mathcal{N}_3[v^*(r,q),\theta^*(r,q)], \tag{36}$$
$$(1-q)_1[\theta^*(r,q)-\theta_0(r)]$$
$$=q\hbar\mathcal{N}_4[v^*(r,q),\theta^*(r,q)], \tag{37}$$
$$v^*(r,q)-\gamma\left[\frac{\partial v^*(r,q)}{\partial r}+\frac{\Lambda}{r}\left(\frac{\partial v^*(r,q)}{\partial r}\right)^3\right]\Bigg|_{r=1}=0,$$
$$\frac{\partial v^*(r,q)}{\partial r}\Bigg|_{r=0}=0,$$
$$\theta^*(r,q)|_{r=1}=0,\ \frac{\partial\theta^*(r,q)}{\partial r}\Bigg|_{r=0}=0, \tag{38}$$
$${}_2[v_m(r)-\chi_m v_{m-1}(r)]=\hbar\Re 3_m(r), \tag{39}$$
$${}_2[\theta_m(r)-\chi_m\theta_{m-1}(r)]=\hbar\Re 4_m(r), \tag{40}$$
$$v_m(1)-\gamma\left[\frac{\partial v_m(1)}{\partial r}+\frac{\Lambda}{r}\left(\frac{\partial v_m(1)}{\partial r}\right)^2\right]=0,$$
$$\theta_m(1)=0,\ v_m'(0)=\theta_m'(0)=0, \tag{41}$$

where

$$\mathcal{N}_3[v^*(r,q),\theta^*(r,q)]=\frac{2}{r}\frac{dv^*}{dr}+\frac{d^2v^*}{dr^2}$$
$$+\frac{\Lambda}{r^2}\left(\frac{dv^*}{dr}\right)^3+\frac{3\Lambda}{r}\left(\frac{dv^*}{dr}\right)^2\frac{d^2v^*}{dr^2}$$
$$-p\left[1+\frac{\Lambda}{r}\left(\frac{dv^*}{dr}\right)^2\right]v^*-\frac{M^2v^*}{r}-\frac{c}{r}, \tag{42}$$

$$\mathcal{N}_4[v^*(r,q),\theta^*(r,q)]=\frac{1}{r}\frac{d\theta^*}{dr}+\frac{d^2\theta^*}{dr^2}$$
$$+\Gamma\left(\frac{dv^*}{dr}\right)^2+\Gamma\Lambda\left(\frac{dv^*}{dr}\right)^4+\Gamma r\left(\frac{dv^*}{dr}\right)^2 \tag{43}$$

and the corresponding recurrence formulae are

$$\Re 3_m(r)=2r\frac{dv_{m-1}}{dr}+r^2\frac{d^2v_{m-1}}{dr^2}$$
$$+\Lambda\sum_{k=0}^{m-1}\sum_{i=0}^{k}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-1}}{dr}\frac{dv_i}{dr}$$
$$+3\Lambda r\sum_{k=0}^{m-1}\sum_{i=0}^{k}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-1}}{dr}\frac{d^2v_i}{dr^2}$$
$$-(1-\chi_m)cr-M^2rv_{m-1}-qr^2v_{m-1}$$
$$-p\Lambda r\sum_{k=0}^{m-1}\sum_{i=0}^{k}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-1}}{dr}v_i,$$
$$\Re 4_m(r)=\frac{1}{r}\frac{d\theta_{m-1}}{dr}+\frac{d^2\theta_{m-1}}{dr^2}$$
$$+\Gamma r\sum_{k=0}^{m-1}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_k}{dr}$$
$$+\Lambda\Gamma\sum_{k=0}^{m-1}\sum_{j=0}^{k}\sum_{i=0}^{j}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-j}}{dr}\frac{dv_{j-i}}{dr}\frac{dv_i}{dr}. \tag{44}$$

4. Convergence of the solution

The most important aspect of series solution is to discuss the convergence of solution. In homotopy analysis method the convergence of series is ensured by using a auxiliary parameter $\hbar$. By means of the so$-$called $\hbar-$curve, it is straightforward to choose an appropriate range for $\hbar$ which ensures the convergence of the solution series. As pointed out by Liao [26], the appropriate region for $\hbar$ is a horizontal line segment. Figures 1 and 2 provide the $\hbar-$curves for constant viscosity model. The admissible values of velocity are $-1.8\le\hbar\le 0$ and for temperature are $-1.8\le\hbar\le -0.2$. Figures 3 and 4 represent the $\hbar-$curves for variable viscosity model. The admissible ranges for both velocity and temperature profiles are $-1.8\le\hbar\le -0.6$ and $-1.4\le\hbar\le -0.5$, respectively.

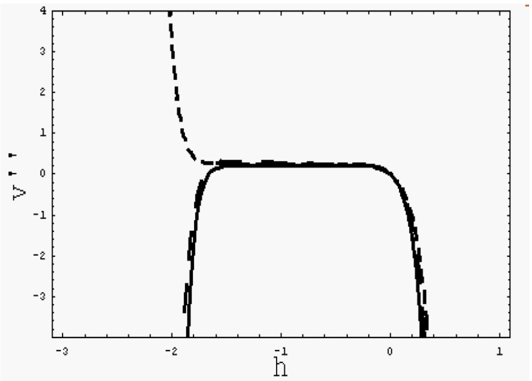

Figure 1 $\hbar-$curve for velocity profile in case of constant viscosity.

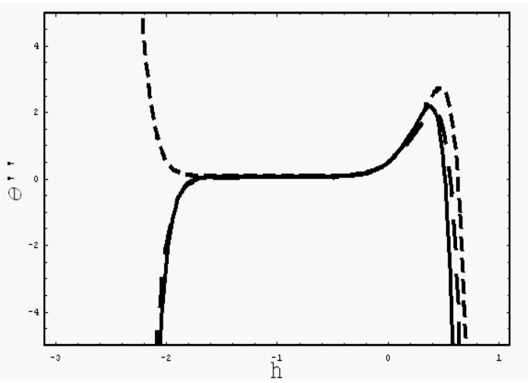

Figure 2 $\hbar-$curve for temperature in case of constant viscosity.

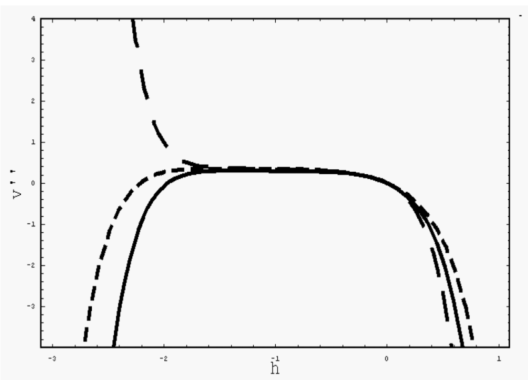

Figure 3 $\hbar-$curve for velocity profile in case of variable viscosity.

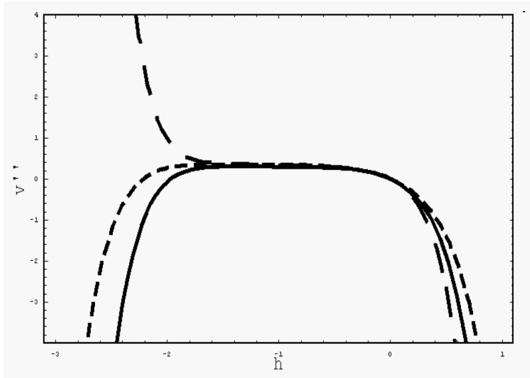

Figure 4 $\hbar-$curve for temperature in case of constant viscosity.

5. Results and discussion

Body Math To explore the effects of emerging parameters on the velocity and temperature profiles figures 5 to 14 have been displayed. The investigation of the effects of slip parameter γ and MHD parameter M and porosity parameter p for constant viscosity model are shown in figures 5 to 9. Figure 5 illustrates the variation of γ on the velocity profile for constant viscosity. The values of the γ are chosen 0, 0.03, 0.05 and 0.08. It is found that the velocity decreases with an increase in the values of γ. Figure 6 has been prepared to explain the variation of M on the velocity distribution. The chosen values of M are 0, 1, 2 and 3. Here, it is revealed that velocity decrease when large values of M have been taken into account. Obviously, the boundary layer thickness decreases by increasing the MHD parameter M. Figure 7 brings out the influence of porosity parameter p on the velocity distribution. The selected values of p are -1,-2 1 and 2. It can easily be observed that velocity decreases by enhancing injection $(p > 0)$. This is in accordance with the fact that suction causes reduction in the boundary layer and as a result injection controls the boundary layer thickness. For the case of blowing, it is well known that in the case of Newtonian fluids, there is no solution to the Navier-Stokes equations for blowing. However, a solution to the equations of motion in the case of fluid injected into the domain is possible in the case of non-Newtonian fluids and the results established here are in keeping with the results of Rajagopal and Gupta [27]. As expected, the blowing causes thickening of the velocity boundary layer and this boundary layer thickness is greater when compared to the case of suction. In order to see the variations of M and p on temperature distribution figures 8 and 9 are plotted. The figures elucidate that the temperature decreases by increasing M and p. The figures 10 to 14 have been prepared to explain the effects of slip parameter γ and MHD parameter M and porosity parameter p for variable viscosity model. It is found that the velocity and temperature profiles behave in same manner in both constant and variable viscosity models. But the velocity and temperature profiles in case of constant viscosity model are smaller than for variable viscosity model. This is not surprising: it is well known that in variable viscosity model the velocity and temperature distributions are qualitatively larger than the constant model.

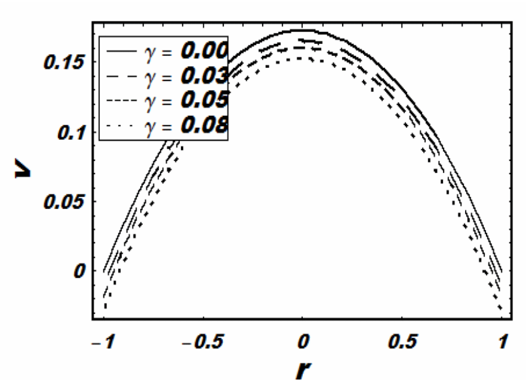

Figure 5 Profiles of velocity for various values of slip for constant viscosity model when $M = 1$ and $p = 1$ are fixed.

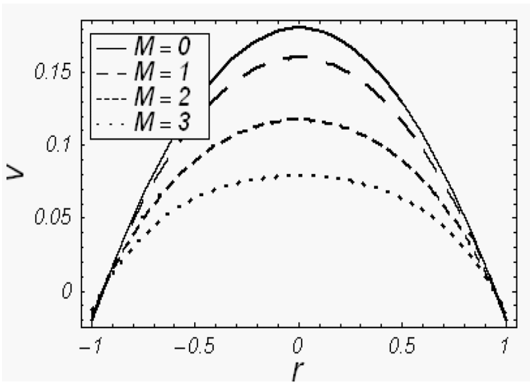

Figure 6 Profiles of velocity for various values of MHD for constant viscosity model when $\gamma = 0.05$ and $p = 1$ are fixed.

6. Conclusion

In this paper, the effects of MHD, porosity and slip parameter on a constant and variable viscosity for steady flow in a pipe are investigated. The series solutions have been developed and their convergence is carefully analyzed. These solutions are not only valid for small but also

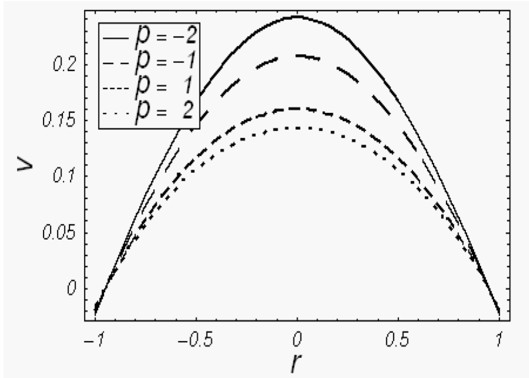

Figure 7 Profiles of velocity for various values of suction and blowing for constant viscosity model when $\gamma = 0.05$ and $M = 1$ are fixed.

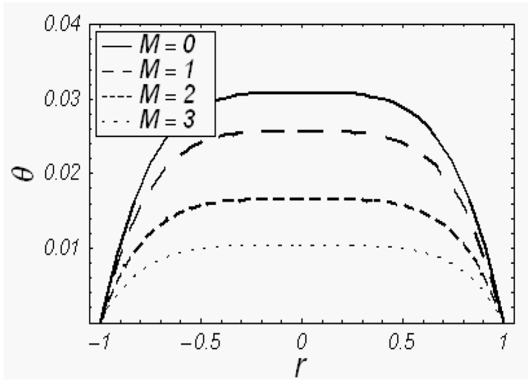

Figure 8 Profiles of temperature for various values of MHD for constant viscosity model when $p = 1$ is fixed.

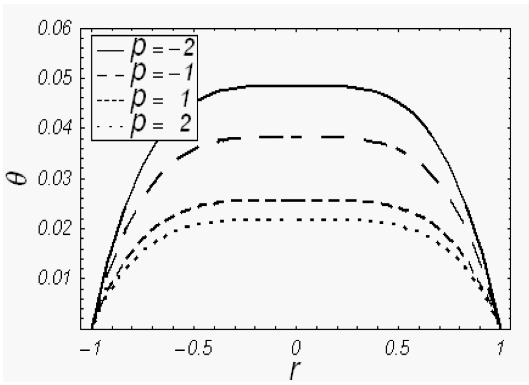

Figure 9 Profiles of temperature for various values of slip parameter for constant viscosity model when $M = 1$ is fixed.

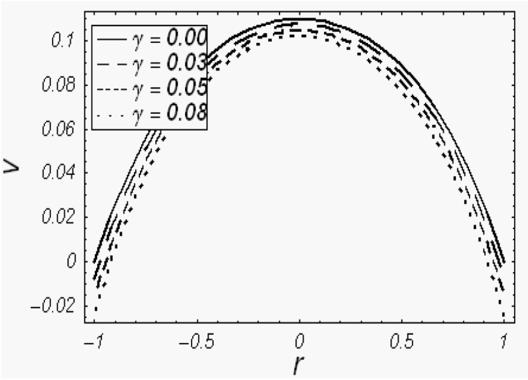

Figure 10 Profiles of velocity for various values of slip for variable viscosity model when $M = 1$ and $p = 1$ are fixed.

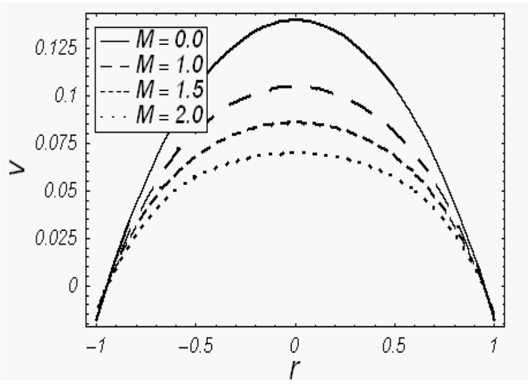

Figure 11 Profiles of velocity for various values of MHD for variable viscosity model when $\gamma = 0.05$ and $p = 1$ are fixed.

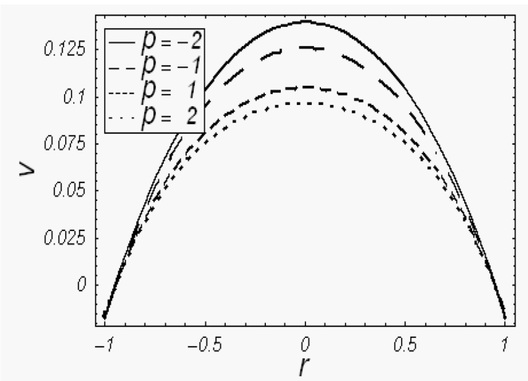

Figure 12 Profiles of velocity for various values of suction and blowing for variable viscosity model when $\gamma = 0.05$ and $M = 1$ are fixed.

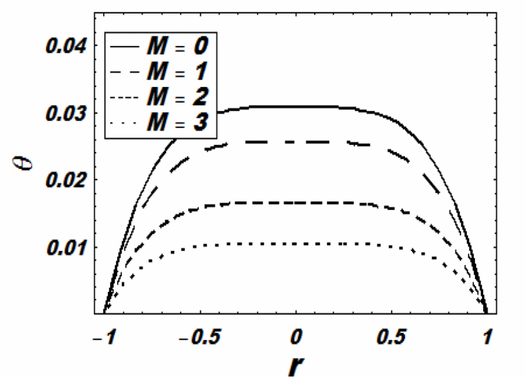

Figure 13 Profiles of temperature for various values of MHD for variable viscosity model when $p = 1$ is fixed.

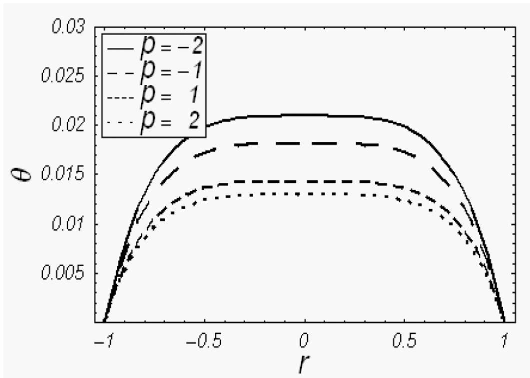

Figure 14 Profiles of temperature for various values of slip parameter for constant viscosity model when $M = 1$ and $p = 1$ are fixed.

for large values of all the emerging parameters. The solutions valid for the no-slip condition for all values of the non-Newtonian parameters can be derived as special cases of the present analysis at $\gamma = 0$. The space occupying the fluid is porous. The reduced system of coupled non-linear differential equations is solved using homotopy analysis method (HAM). The effects of emerging parameters have been seen and discussed through graphs. To the best of our knowledge, no such analysis is available in the literature which can describe the MHD, slip and porosity effects on velocity and temperature profiles for constant and variable viscosity simultaneous. The following observations have been made:

–The velocity decreases by increasing the values of γ.
–The velocity and temperature decrease with an increase in the values of M.
–Increasing the values of the suction velocity profile decreases $(p < 0)$.

–The increase in injection velocity increases $(p > 0)$.
–The velocity and temperature distributions are qualitatively larger than when compare with the constant model.
–The results obtained are compared with those reported earlier. It is interesting to note that when $\gamma = p = M = 0$ then one recovers the case [14]. The problem reduces to the case [15] for $\gamma = M = 0$. The case of [16] can also be recovered by $\gamma = p = 0$. This provides the useful check

Acknowledgment: R. Ellahi thanks to CIES USA to honored him by Fullbright Fellowship for the year 2011-2012.

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