Applications of First Integral Method to Some Complex Nonlinear Evolution Systems

Marwan Alquran*, Qutaibeh Katatbeh and Banan Al-Shrida

Department of Mathematics and Statistics, Jordan University of Science and Technology, Irbid 22110, Jordan

Received: 2 Jun. 2014, Revised: 31 Aug. 2014, Accepted: 2 Sep. 2014
Published online: 1 Mar. 2015

Abstract: The aim of the present paper is to study nonlinear system of partial differential equations (PDEs) involving both complex-valued functions and real-valued unknown functions. We shall extend the use of the first integral method "based on the theory of commutative algebra" to construct new solutions to the coupled Higgs field equations, the Davey-Sterwatson (DS) equations and the coupled Klein-Gordon-Zakharov equations. All the algebraic computations in this work are performed using Mathematica software.

Keywords: First integral method, Coupled Higgs field equations, Davey-Sterwatson equations, Coupled Klein-Gordon-Zakharov equations.

1 Introduction

Nonlinear partial differential equations (PDEs) appear in various scientific and engineering fields, such as fluid mechanics, plasma physics, optics, chemistry, biology, solid state physics, chemical physics and stochastic control with relevance to information sciences. In the past several decades, new exact solutions may help to find new phenomena. A variety of powerful methods, such as bilinear transformation, the tanh-sech method extended to construct new solutions to the coupled Higgs field equations, the Davey-Sterwatson (DS) equations and the coupled Klein-Gordon-Zakharov equations. All the algebraic computations in this work are performed using Mathematica software.

Our interest in the present work is in implementing the first integral method. The first integral method was first proposed by Feng [1] in solving Burgers-KdV equation. It is a direct algebraic method based on the commutative algebra. Recently, it was successfully used for constructing exact solutions to a variety of nonlinear problems see [2,3,4,5,6,7,8,9]. In this work, we consider the following mathematical models:

First, we study the Coupled Higgs field equations

\[ u_{tt} - u_{xx} - au + b|u|^2u - 2uv = 0 \]
\[ v_{tt} + v_{xx} - b|u|^2v_{xx} = 0 \]  
where \( a > 0 \) and \( b > 0 \). \( u = u(x,t) \) is a complex-valued function and \( v = v(x,t) \) is a real-valued function. Authors in [10] apply the functional variable method and obtained analytical solution to this system.

Second, we study The Davey-Sterwatson (DS) equations

\[ iu_t + \frac{1}{2}b^2(u_{xx} + b^2u_{yy}) + a|u|^2u - uv = 0, \]
\[ v_{xx} - b^2v_{yy} - 2a(|u|^2)_{xx} = 0, \]  
where \( a \) is a real constant, \( b^2 = \pm 1 \), \( u(x,y,t) \) is a complex valued function and \( v(x,y,t) \) is a real valued function. These equations were introduced in order to discuss the instability of uniform trains of weakly nonlinear water waves in two dimensional space. Yomba use the extended F-expansion method and general projective Riccati equations method to construct exact solutions to Davey-Sterwatson (DS) equations see [11,12]

Finally, we study the Coupled Klein-Gordon-Zakharov equations

\[ u_{tt} - c_0\nabla^2 u + f_0^2 u + \delta uv = 0 \]
\[ v_{tt} - c_0^2\nabla^2 v - b\nabla^2(|u|^2) = 0, \]  
where \( c_0, f_0, \) and \( b \) are constants, \( u(x,y,z,t) \) is a complex valued function and \( v(x,y,z,t) \) is a real valued function. General projective Riccati equations method in [11] is applied to construct exact solution for (3). Also the

* Corresponding author e-mail: marwan04@just.edu.jo
extended F-expansion method is used to solve the same equation [12].

Our goal in this work is implementing the first integral method with help of the symbolic computational Mathematica software to show its applicability in handling nonlinear equations, so that one can apply it to models of various types of nonlinear equations.

2 Analysis of the first integral method

In this section we go briefly over the procedure of the first integral method [1, 4, 9]. Consider the nonlinear PDE for a function $u$ of two variables, $x$ and $t$:

$$F(u, u_t, u_x, u_{tt}, u_{tx}, ...) = 0.$$  \hspace{1cm} (4)

introduce the wave variable $\xi = x - ct$ so that $u(x, t) = u(\xi)$. Based on this we obtain

$$\frac{\partial}{\partial t}(\cdot) = -c \frac{d}{d\xi}(\cdot), \quad \frac{\partial}{\partial x}(\cdot) = \frac{d}{d\xi}(\cdot),$$
$$\frac{\partial^2}{\partial t^2}(\cdot) = c^2 \frac{d^2}{d\xi^2}(\cdot), \quad \frac{\partial^2}{\partial x^2}(\cdot) = \frac{d^2}{d\xi^2}(\cdot).$$  \hspace{1cm} (5)

Using (5) changes the PDE in (4) to an ODE

$$G(u, u', u'', u''', ...) = 0,$$  \hspace{1cm} (6)

where the prime denotes the derivatives with respect to $\xi$. Next, we introduce a new independent variables

$$X(\xi) = u(\xi), \quad Y(\xi) = u'(\xi),$$  \hspace{1cm} (7)

This yields to a system of ODEs

$$X'(\xi) = Y(\xi), \quad Y'(\xi) = H(X(\xi), Y(\xi)),$$  \hspace{1cm} (8)

According to the qualitative theory of ordinary differential equations, if we can find integrals to (8), we can reduce (8) to a first-order ODE to be solved directly. But in general (it is really difficult for us to realize this because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first integrals, nor is there a logical way for telling us what these first integrals are) see [1].

Suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of equations in (8) and $q(X, Y) = \sum_{i=0}^{m} a_i(X)Y^i$ is an irreducible polynomial in complex domain $C[X, Y]$ such that

$$q[X(\xi), Y(\xi)] = \sum_{i=0}^{m} a_i(X)Y^i = 0,$$  \hspace{1cm} (9)

where $a_i(X), (i = 0, 1, 2, ..., m)$ are polynomials of $X$ and $a_m(X) \neq 0$. Equation (9) is called the first integral to (8). According to the division theorem, there exists a polynomial $g(X) + h(X)Y$ in a complex domain $C[X, Y]$ such that

$$\frac{dq}{d\xi} = \frac{\partial q}{\partial X} \frac{\partial X}{d\xi} + \frac{\partial q}{\partial Y} \frac{\partial Y}{d\xi} = [g(X) + h(X)Y] \sum_{i=0}^{m} a_i(X)Y^i.$$  \hspace{1cm} (10)

3 Coupled Higgs field equations

In this section we study the Coupled Higgs field equations

$$u_{tt} - u_{xx} - au + b|u|^2u - 2uv = 0$$
$$v_{tt} + v_{xx} - b|u|^2v = 0$$  \hspace{1cm} (11)

First, we make the following transformation:

$$u(x, t) = u(\xi) \exp(\eta), \quad v(x, t) = v(\xi)$$
$$\xi = k(x + \lambda t), \quad \eta = ax + \beta t.$$  \hspace{1cm} (12)

Substituting (12) into (11) we obtain

$$0 = bu^3 + (a - \alpha^2 + \beta^2 + 2\nu)u + k^2(\lambda^2 - 1)u''$$
$$+ i(-2(\alpha - \beta\lambda)u'),$$
$$0 = -2bu^2 - 2bvu' + (1 + \lambda^2)v'',$$  \hspace{1cm} (13)

where the prime denotes the derivation with respect to $\xi$. We divide the first equation of (13) into two parts imaginary part and real part as follow

$$Im : (-2(\alpha - \beta\lambda)u') = 0.$$  \hspace{1cm} (14)
$$Re : bu^3 + (a - \alpha^2 + \beta^2 + 2\nu)u + k^2(\lambda^2 - 1)u'' = 0.$$  \hspace{1cm} (15)

We only solve (15), instead of both (14) and (15), provided that

$$\alpha = \beta\lambda.$$  \hspace{1cm} (16)

integrating the second equation of (13) twice and setting the constant of integration to be zero. We find

$$v = \frac{bu^2}{1 + \lambda^2}.$$  \hspace{1cm} (17)

Substituting (17) into (15) we have

$$-(a - \alpha^2 + \beta^2)u + (b - \frac{2b}{1 + \lambda^2})u^3 + k^2(\lambda^2 - 1)u'' = 0.$$  \hspace{1cm} (18)

Using (7) we obtain

$$X'(\xi) = Y(\xi),$$  \hspace{1cm} (19)
$$Y'(\xi) = \frac{a - \alpha^2 + \beta^2}{k^2(\lambda^2 - 1)}X(\xi) - \frac{b}{k^2(\lambda^2 + 1)}X(\xi)^3.$$  \hspace{1cm} (20)

Suppose that $m = 1$ in (9), then

$$q[X, Y] = a_0(X) + a_1(x)Y = 0.$$
From (10) we obtain
\[
\frac{dq}{d\xi} = \frac{\partial g}{\partial X} \frac{\partial X}{\partial \xi} + \frac{\partial g}{\partial Y} \frac{\partial Y}{\partial \xi}
\]
\[
= [a_0'(X) + a_1'(X)Y']
\]
\[+ a_1(X) \left( \frac{a - \alpha^2 + \beta^2}{k^2(\lambda^2 - 1)} X - \frac{b}{k^2(\lambda^2 + 1)} X^3 \right)
\]
\[
- a_1'(X)Y^2 + a_0'(X)Y
\]
\[
+ \left( \frac{a - \alpha^2 + \beta^2}{k^2(\lambda^2 - 1)} X - \frac{b}{k^2(\lambda^2 + 1)} X^3 \right) a_1(X).
\]
(21)

and
\[
\frac{dq}{d\xi} = [g(X) + h(X)Y](a_0(X) + a_1(x)Y)
\]
\[
= h(X)a_1(X)Y^2 + [g(X)a_1(X) + h(X)a_0(X)]Y
\]
\[+ g(X)a_0(X).
\]
(22)

By equating the coefficients of \(Y^i\) \((i = 2, 1, 0)\) in (21) and

\[
a'_1(X) = h(X)a_1(X),
\]
(23)

\[
a'_0(X) = h(X)a_0(X) + a_1(x)g(X),
\]
(24)

\[
a_0(X)g(X) = \left[ \frac{a - \alpha^2 + \beta^2}{k^2(\lambda^2 - 1)} X - \frac{b}{k^2(\lambda^2 + 1)} X^3 \right] a_1(X).
\]
(25)

Since \(a_1(X)\) \((i = 0, 1)\) are polynomials, then from
(23) we deduce that \(a_1(X)\) is a constant and \(h(X) = 0\). For simplicity, take \(a_1(X) = 1\). Balancing the degrees of \(g(X)\) and \(a_0(X)\), we conclude that \(deg(g(X)) = 1\) only. Suppose that \(g(X) = A_1X + A_0\) and \(A_1 \neq 0\), then we find that \(a_0(X)\) is expressed as

\[
a_0(X) = \frac{A_2}{2} X^2 + A_1X + A_0.
\]
(26)

Substituting \(a_0(X), a_1(X)\) and \(g(X)\) into (25) and setting all coefficients of powers of \(X\) to be zeros, then we obtain the following system of nonlinear algebraic equations

\[
0 = A_0A_1
\]
\[
0 = \left( \frac{A_2}{2} + \frac{b}{k^2(1 + \lambda^2)} \right)
\]
\[
0 = \frac{3}{2} A_1 A_2
\]
\[
0 = A_1^2 + A_0A_2 - \frac{a^2 + \beta^2}{k^2(1 + \lambda^2)}
\]
(27)

Solving (27), we obtain

\[
A_0 = \pm \sqrt{2} \sqrt{bk(-1 + \lambda^2) \sqrt{1 + \lambda^2}}
\]
\[
A_1 = 0,
\]
\[
A_2 = \pm i \frac{\sqrt{b}}{k \sqrt{1 + \lambda^2}}.
\]
(28)

where \(\beta, \lambda\) and \(k\) are arbitrary. Using (26) in (9), we obtain

\[
Y(\xi) = -\frac{A_2}{2} X^2 - A_1X - A_0.
\]
(29)

Combining (29) with (19), we obtain the exact solution to (20)

\[
X(\xi) = -\frac{\sqrt{2} \sqrt{A_0} \tan\left( \frac{\sqrt{A_0} \sqrt{\xi - 2c_0}}{\sqrt{2}} \right)}{\sqrt{A_0^2}},
\]
(30)

where \(c_0\) is the integration constant. Therefore, the solutions of (11) are

\[
u(x, t) = \pm \tanh\left( \frac{-a + \beta^2(-1 + \lambda^2)(k(x + \lambda t) - 2c_0)}{\sqrt{2k\sqrt{1 + \lambda^2}}} \right)
\]
\[
\times i \exp(i\beta(\lambda x + \nu)) \frac{\sqrt{-a + \beta^2(-1 + \lambda^2)}}{\sqrt{b\sqrt{1 - \frac{2}{1 + \lambda^2}}}}.
\]
(31)

\[
v(x, t) = \frac{\mu}{\lambda^2 - 1} \tan^{2}\left( \frac{\sqrt{2}(k(x + \lambda t) - 2c_0)}{\sqrt{2k\sqrt{1 + \lambda^2}}} \right),
\]
(32)

where \(\mu = a - \beta^2(-1 + \lambda^2)\).

4 Davy–Sterewatson (DS) equations

Consider the Davy–Sterewatson (DS) equations

\[
iu_x + \frac{1}{2} b^2 (u_{xx} + b^2 u_{yy}) + a |u|^2u - uv = 0,
\]
\[
v_{xx} - b^2 v_{yy} - 2a(|u|^2)_{xx} = 0.
\]
(33)

Apply the following transformations:

\[
u(x, y, t) = u(\xi) \exp(i\eta), \quad v(x, y, t) = v(\xi),
\]
\[
\xi = k(x + ly + \lambda t), \quad \eta = \alpha x + \beta y + \pi.
\]
(34)

Substitution (34) into (33) yield the following system of ODEs

\[
0 = 2a\lambda^4 - (b^2\alpha^2 + b^4\beta^2 + 2\gamma + 2\gamma)u + b^2\lambda^2 (1 + b^2\lambda^2)u'p,
\]
\[
0 = -4a\lambda^2 + 2 \lambda u'' - (b^2\alpha^2 + b^4\beta^2 + 2\gamma)u,
\]
(35)

where \(\lambda = -b^2\alpha - b^4\beta\) and the prime denotes the derivation with respect to \(\xi\).

Integrating the second equation of (35) twice and setting the constant of integration to be zero. We find

\[
v = \frac{2a}{1 - b^2\lambda^2} u^2.
\]
(36)

Substituting (36) into the first equation of (35) we have

\[
0 = b^2k^2 (1 + b^2\lambda^2) u'' - (b^2\alpha^2 + b^4\beta^2 + 2\gamma)u
\]
\[
+ a(2 + \frac{4}{b^2\lambda^2 - 1})u^3.
\]
(37)

Using (7) we obtain

\[
X'(\xi) = Y(\xi),
\]
(38)
\[ Y' (\xi) = \frac{b^2 \alpha^2 + b^4 \beta^2 + 2 \gamma}{b^2 k^2 (1 + b^2 l^2)} X(\xi) \]
\[ - \frac{2a}{b^2 k^2 (1 + b^2 l^2)} X(\xi)^3. \]  
Suppose that \( m = 1 \) in (9), then
\[ q[X, Y] = a_0(X) + a_1(x) Y = 0. \]
From (10) we obtain
\[ \frac{dq}{d\xi} = \frac{\partial q}{\partial X} \frac{\partial X}{\partial \xi} + \frac{\partial q}{\partial Y} \frac{\partial Y}{\partial \xi} \]
\[ = a_1(X) \left[ \frac{b^2 \alpha^2 + b^4 \beta^2 + 2 \gamma}{b^2 k^2 (1 + b^2 l^2)} X - \frac{2a}{b^2 k^2 (1 + b^2 l^2)} X^3 \right] \]
\[ + a_0(X) \left[ \frac{b^2 \alpha^2 + b^4 \beta^2 + 2 \gamma}{b^2 k^2 (1 + b^2 l^2)} X - \frac{2a}{b^2 k^2 (1 + b^2 l^2)} X^3 \right] a_1(X) \]
\[ + a'_1(X) Y^2 + a'_0(X) Y. \]  
Solving (46), we obtain
\[ A_0 = \pm \frac{(-1 + b^2 l^2) \sqrt{\frac{a}{b^2 - b^4 l^2} (b^2 \alpha^2 + b^4 \beta^2 + 2 \gamma)}}{2ak(1 + b^2 l^2)}, \]
\[ A_1 = 0, \]
\[ A_2 = \pm \frac{2}{k} \sqrt{\frac{a}{b^2 - b^4 l^2}}, \]  
where \( \alpha, \beta, \gamma, l \) and \( k \) are arbitrary. Using (47) in (9), we obtain
\[ Y(\xi) = \frac{A_2}{2} X^2 - A_1 X - A_0. \]  
Combining (48) with (38), we obtain the exact solution to (39)
\[ X(\xi) = -\sqrt{2} \sqrt{A_0} \tan \left( \frac{\sqrt{A_0} \sqrt{\xi - 2c_0}}{\sqrt{2}} \right), \]
where \( c_0 \) is the integration constant. Therefore, the solutions of (33) are
\[ u(x, t) = \pm \tan \left( \frac{b^2 \alpha^2 + b^4 \beta^2 + 2 \gamma}{2b^2 k^2 + 2b^4 k^4 l^2} (\xi - 2c_0) \right) \]
\[ \times k \sqrt{\frac{b^2 - b^4 l^2}{b^2 k^2 (1 + b^2 l^2)}} \exp(i(\alpha x + \beta y + \gamma t)) \]  
\[ v(x, t) = \tan^2 \left( \frac{b^2 \alpha^2 + b^4 \beta^2 + 2 \gamma}{2b^2 k^2 + 2b^4 k^4 l^2} (\xi - 2c_0) \right) \]
\[ \times \frac{b^2 \alpha^2 + b^4 \beta^2 + 2 \gamma}{1 + b^2 l^2}, \]
where \( \lambda = -b^2 \alpha - b^4 \beta. \)

### 5 Coupled Klein–Gordon–Zakharov equations

Consider the Coupled Klein–Gordon–Zakharov equations
\[ u_{tt} - c_0 \nabla^2 u + f_0 \partial u + \delta uv = 0, \]
\[ v_{tt} - c_0^2 \nabla^2 v - b \nabla^2 (|u|^2) = 0, \]  
where \( c_0, f_0, \) and \( b \) are constants, \( u(x, y, z, t) \) is a complex valued function and \( v(x, y, z, t) \) is a real valued function. To solve (52), we apply the transformations
\[ u(x, y, z, t) = u(\xi) \exp(\eta t), \]
\[ v(x, y, z, t) = v(\xi), \]
\[ \xi = k(x + ly + n z + \lambda t), \]
\[ \eta = \alpha x + \beta y + \omega z + \gamma t. \]  
Substituting (53) into (52) yield the following system of ODEs
\[ 0 = (f_0^2 - \gamma^2 + c_0^2 (\alpha^2 + \beta^2 + \omega^2) + \delta \nu) u, \]
\[ -k^2 (\lambda^2 + c_0^2 (1 + n^2 + l^2)) u'', \]
\[ 0 = 2b(1 + n^2 + l^2)(u'^2 + uu') - (\lambda^2 - c_0^2 (1 + n^2 + l^2)) v'' \]  
(54)
where
\[ \alpha = -l\beta + \left(-1 + \frac{1}{c_0^2}\right)\gamma\lambda - n\omega. \]

and the prime denotes the derivation with respect to \( \xi \).

Integrating the second equation of (54) twice and setting the constant of integration to be zero. We find
\[ v = -\frac{b(1 + n^2 + l^2)}{-\lambda^2 + c_0^2(1 + n^2 + l^2)}u^2. \] (55)
Substituting (55) into the first equation of (54) we have
\[ 0 = (f_0^2 - \gamma^2 + c_0^2(\alpha^2 + \beta^2 + \omega^2)u + \frac{b\delta(1 + n^2 + l^2)u^3}{\lambda^2 - c_0^2(1 + n^2 + l^2)}) \\
- k^2(-\lambda^2 + c_0^2(1 + n^2 + l^2))u'' \] (56)

Using (7) we obtain
\[ X' (\xi) = Y (\xi), \] (57)
\[ Y' (\xi) = \frac{f_0^2 - \gamma^2 + c_0^2(\alpha^2 + \beta^2 + \omega^2)}{k^2(-\lambda^2 + c_0^2(1 + n^2 + l^2))}X(\xi) \\
- \frac{b\delta(1 + n^2 + l^2)}{k^2(-\lambda^2 + c_0^2(1 + n^2 + l^2))}X^3(\xi)^3. \] (58)
Suppose that \( m = 1 \) in (9), then
\[ q[X, Y] = a_0(X) + a_1(x)Y = 0. \]

From (10) we obtain
\[ \frac{dq}{d\xi} = \frac{\partial q}{\partial X} \frac{\partial X}{\partial \xi} + \frac{\partial q}{\partial Y} \frac{\partial Y}{\partial \xi} \]
\[ = \frac{f_0^2 - \gamma^2 + c_0^2(\alpha^2 + \beta^2 + \omega^2)}{k^2(-\lambda^2 + c_0^2(1 + n^2 + l^2))}Xa_1(X) \\
- \frac{b\delta(1 + n^2 + l^2)}{k^2(-\lambda^2 + c_0^2(1 + n^2 + l^2))}X^3a_1(X) \]
\[ + a_0'(X)Y + a_1'(X)Y^2. \] (59)

and
\[ \frac{dq}{d\xi} = [g(X) + h(X)Y](a_0(X) + a_1(X)Y) \]
\[ = h(X)a_1(X)Y^2 + [g(X)a_1(X) + h(X)a_0(X)]Y \]
\[ + g(X)a_0(X). \] (60)
By equating the coefficients of \( Y^i (i = 2, 1, 0) \) in (59) and (60) we obtain
\[ a_0'(X) = h(X)a_1(X), \] (61)
\[ a_0'(X) = h(X)a_0(X) + a_1(X)g(X), \] (62)
\[ a_0(X)g(X) = \left[ \frac{f_0^2 - \gamma^2 + c_0^2(\alpha^2 + \beta^2 + \omega^2)}{k^2(-\lambda^2 + c_0^2(1 + n^2 + l^2))} \right] a_1(X) \]
\[ - \left[ \frac{b\delta(1 + n^2 + l^2)}{k^2(-\lambda^2 + c_0^2(1 + n^2 + l^2))} \right] X^3a_1(X). \] (63)

Since \( a_i(X) (i = 0, 1) \) are polynomials, then from (61) we deduce that \( a_1(X) \) is a constant and \( h(X) = 0 \). For simplicity, take \( a_1(X) = 1 \). Balancing the degrees of \( g(X) \) and \( a_0(X) \), we conclude that \( deg(g(X)) = 1 \) only. Suppose that \( g(X) = A_1X + A_0 \) and \( A_1 \neq 0 \), then we find that \( a_0(X) \) is expressed as
\[ a_0(X) = \frac{A_2}{2}X^2 + A_1X + A_0. \] (64)
Substituting \( a_0(X), a_1(X) \) and \( g(X) \) into (63) and setting all coefficients of powers of \( X \) to be zeros, then we obtain the following system of nonlinear algebraic equations
\[ 0 = A_0a_1, \]
\[ 0 = A_1^2 + A_0A_2 - \frac{f_0^2 - \gamma^2 + c_0^2(\alpha^2 + \beta^2 + \omega^2)}{k^2(-\lambda^2 + c_0^2(1 + n^2 + l^2))}, \]
\[ 0 = \frac{3}{2}A_1A_2, \]
\[ 0 = \frac{A_1^2}{2} + \frac{b\delta(1 + n^2 + l^2)}{k^2(-\lambda^2 + c_0^2(1 + n^2 + l^2))^2}X^3 \] (65)
Solving (65), we obtain
\[ A_0 = \pm \frac{f_0^2 - \gamma^2 + c_0^2(\alpha^2 + \beta^2 + \omega^2)}{\sqrt{2\sqrt{bk}\sqrt{\delta}\sqrt{-1 - l^2 - n^2}}}, \]
\[ A_1 = 0, \]
\[ A_2 = \pm \frac{\sqrt{2\sqrt{bk}\sqrt{\delta}\sqrt{-1 - l^2 - n^2}}}{k\sqrt{\lambda^2 - c_0^2k^2(1 + n^2 + l^2)}} \] (66)
where \( \beta, \omega, \gamma, \lambda, n, l \) and \( k \) are arbitrary. Using (66) in (9), we obtain
\[ Y(\xi) = -\frac{A_2}{2}X^2 - A_1X - A_0. \] (67)
Combining (67) with (57), we obtain the exact solution to (58)
\[ X(\xi) = \frac{\sqrt{2\sqrt{\delta}}\tan(\sqrt{\alpha_0}\sqrt{\beta_0}(\xi - \xi_0))}{\sqrt{A_2}}, \] (68)
where \( \xi_0 \) is the integration constant. Therefore, the solutions of (52) are
\[ u_1(x, t) = e^{i\sqrt{f_0^2 + c_0^2k_1 - \gamma^2\sqrt{\lambda^2 - c_0^2k_2 - 2\lambda^2}}} \]
\[ \times \tan(\sqrt{f_0^2 + c_0^2k_1 - \gamma^2(\xi - 2\xi_0)}) \] (69)
\[ v_1(x, t) = \tan^2(\sqrt{f_0^2 + c_0^2k_1 - \gamma^2(\xi - 2\xi_0)}) \]
\[ \times \frac{f_0^2 + c_0^2k_1 - \gamma^2}{\delta}. \] (70)
\[ u_2(x,t) = e^{\eta}i \sqrt{\frac{-f_0^2 - c_0^2k_1 + \gamma^2}{k_2^2} \sqrt{k_2^2 - 2\lambda^2}} \times \tanh(\frac{\sqrt{-f_0^2 - c_0^2k_1 + \gamma^2} (\xi - 2\xi_0)}{k_2^2 - 2\lambda^2}) \]

\[ \times \tanh(\frac{\sqrt{-f_0^2 - c_0^2k_1 + \gamma^2} (\xi - 2\xi_0)}{k_2^2 - 2\lambda^2}) \] (71)

\[ v_2(x,t) = \tanh^2(\frac{\sqrt{-f_0^2 + c_0^2k_1 - \gamma^2} (\xi - 2\xi_0)}{k_2^2 - 2\lambda^2}) \times -\frac{f_0^2 + c_0^2k_1 - \gamma^2}{\delta} \] (72)

where \( k_1 \) and \( k_2 \) are

\[ k_1 = \alpha^2 + \beta^2 + \omega^2, \]
\[ k_2 = 1 + l^2 + n^2, \]

and \( \alpha = -l\beta + (-1 + \frac{1}{\xi_0})\gamma\lambda - n\omega. \)

6 Conclusions

In this work, we extend the application of first integral method to solve some nonlinear evolution systems. By means of this method new exact solutions to such evolution systems are obtained. The performance of this method is found to be reliable and effective. The Mathematica software was used to solve complicated and tedious algebraic calculations. The proposed method can be extended to other nonlinear problems in mathematical sciences.

Acknowledgement

The authors are grateful to the editor and anonymous referees for their helpful comments that improved this paper.

References


---

Marwan Alquran is Associate professor of Applied Mathematics at Jordan University of Science and Technology. His research field is on analytical and numerical solutions of nonlinear partial differential equations. Area of interest is developing algorithms to construct soliton solutions to nonlinear evolutionary equations. In May 2003, Alquran earned his Ph.D from Central Michigan University, USA.

Qutaibeh Katatbeh has used Mathematica extensively in his research areas of spectral theory, spectral bounds for Schrodinger operators, and eigenvalue problems, and in the areas of computational and mathematical physics. He has used Mathematica in his teaching projects and in the research for most of his publications. He established a Mathematica training center in the Middle East, supported by the International Bank, at Jordan University of Science and Technology. Katatbeh was appointed as a research professor in the department of mathematics and statistics at Concordia University in 2003, and promoted to Assistant professor in 2004. Currently he is an Associate professor at Jordan University of Science and Technology.

Banan Al-Shrida is Graduate student at Jordan University of Science and Technology. Al-Shrida earned her M.Sc. degree in Applied Mathematics in 2013 under the supervision of professor Marwan Alquran.