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# A New Approximate Analytical Solution of Kuramoto – Sivashinsky Equation Using Homotopy Analysis Method

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**Abstract:** In this paper, Homotopy Analysis Method (HAM) is applied to obtain approximate analytical solution of modified Kuramoto–Sivashinsky (KS) equation. HAM provides a simple way to adjust and control the convergence region of the series solution by introducing several parameters, namely, the auxiliary parameter, $h$, the auxiliary function, $H(x,t)$, the initial guess, $u_0(x,t)$ and the auxiliary linear operator, $L$, as stated in [1]. The obtained results show that HAM yields approximate analytical solutions which are quite close to the exact solution of KS equation, which proves the strength of HAM.

**Keywords:** Kuramoto−Sivashinsky equation, homotopy analysis method, approximate analytical solutions, Maple.

## 1. Introduction

HAM [1,2], devised by Shi-Jun Liao in 1992, is a general analytical approach to obtain approximate analytical solutions of various types of nonlinear equations, including ordinary and partial differential equations, algebraic equations, differential-integral equations and differential-difference equations. The method distinguishes itself from other analytical methods in the following four aspects: (i.) It is a series expansion method, but is independent of small physical parameters. Thus, it is applicable for not only weakly but also strongly nonlinear problems. (ii.) HAM unifies three methods: Lyapunov artificial small parameter method, delta expansion method and Adomian decomposition method. (iii.) HAM provides a simple way to ensure the convergence of the solution. Furthermore, it provides some degree of freedom to choose the base function of the desired solution. (iv.) HAM can easily be associated with some other mathematical methods including series expansion methods, integral transform methods and some other numerical methods.

In this paper, HAM is used to obtain approximate analytical solution of modified KS equation. The Generalized Kuramoto-Sivashinsky (GKS) equation is defined [3] as

$$u_t + \beta u^\alpha u_x + \gamma u^\tau u_{xx} + \delta u_{xxxx} = 0, \tag{1}$$

where $\alpha, \beta, \gamma, \tau, \delta \in R$ and $\alpha, \beta, \gamma, \delta \neq 0$. When $\alpha = \beta = 1$ and $\tau = 0$, (1) reduces to original KS equation [4, 5]. This equation was derived by Kuramoto in the study of phase turbulence in the Belousov-Zhabotinsky reaction [3]. An extension of this equation to two or more spatial dimensions was given by Sivashinsky in the study of propagation of a frame front for the case of mild combustion. The KS equation has some application areas including the representation of one class of pattern formation [6,7] and in model of bifurcation and chaos [8,9].

In the past several decades, various methods have been proposed to solve the KS equation. Some of these methods are Chebyshev spectral collocation scheme [10], lattice Boltzmann technique [11], method of radial basis functions [12], local discontinuous Galerkin method [13], tanh function method [14], variational iteration method [15], Perturbation methods [16]. Recently, various other methods [17]-[20] have also been proposed to construct exact solutions of the KS equation. Some other methods [21] have been developed for finding exact solutions of some other similar nonlinear evolution equations. KS equation

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is also known as KdV–Burgers–Kuramoto (KBK) equation [22]. Numerical techniques [23,24] based on the finite difference and collocation methods are proposed for the solution of GKS equation. Using B-spline functions, a method to solve GKS equation is proposed in [25].

The organization of this paper is as follows: In Section 2, a short description of HAM is presented. Section 3 provides some experimental results as an application of HAM to obtain approximate analytical solution of the KS equation. Conclusions and discussions are given in the final section.

## 2. Homotopy Analysis Method

In this section we describe the HAM shortly. Let us consider the following nonlinear partial differential equation

$$ND[u(x,t)] = 0 \tag{2}$$

where $ND$ is a nonlinear operator, $x$ and $t$ denote independent variables and $u(x,t)$ is an unknown function. Using (2) and by means of generalizing the traditional concept of homotopy, Liao [2] constructs the so called zero-order deformation equation:

$$(1-q)L(U(x,t;q)-u_0(x,t)) = qhH(x,t)ND(U(x,t;q)) \tag{3}$$

where $q \in [0,1]$ is embedding parameter, $h \neq 0$ is an auxiliary parameter, $H(x,t) \neq 0$ is an auxiliary function, $L$ is an auxiliary linear operator, $u_0(x,t)$ is an initial guess of $u(x,t)$ and $U(x,t;q)$ is an unknown function on the independent variables $x$, $t$ and $q$.

If we write $q = 0$ and $q = 1$, then, we get following two equations, respectively:

$$U(x,t;0) = u_0(x,t), \;\; U(x,t;1) = u(x,t). \tag{4}$$

Using the parameter $q$, we can expand $U(x,t;q)$ into Taylor series as follows:

$$U(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)q^m \tag{5}$$

where

$$u_m = \frac{1}{m!}\frac{\partial^m u(t,q)}{\partial q^m}\bigg|_{q=0}.$$

Assume that the auxiliary linear operator, the initial guess, the auxiliary parameter, $h$, and the auxiliary function, $H(x,t)$, are selected such that the series (5) is convergent at $q = 1$, then due to (4), we have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t). \tag{6}$$

Let us define the vector

$$\mathbf{u}_m(x,t) = (u_0(x,t), u_1(x,t), u_2(x,t), ..., u_n(x,t)). \tag{7}$$

Differentiating (3) $m$ times with respect to $q$, then setting $q = 0$, and dividing the resulting equation by $m!$, we have the so called $m^{th}$-order deformation equation as follows:

$$L\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = hH(x,t)R_m(\mathbf{u}_{m-1}) \tag{8}$$

where

$$R_m\left(\mathbf{u}_{m-1}\right) = \frac{1}{(m-1)!}\frac{\partial^{m-1} ND(u(t;q))}{\partial q^{m-1}}\bigg|_{q=0}. \tag{9}$$

Now, for $m \geq 1$, the solution of (8) becomes

$$u_m\left(x,t\right) = \chi_m u_{m-1}\left(x,t\right) + hL^{-1}NR\left[\overrightarrow{u}_{m-1}\left(x,t\right)\right]$$

where

$$\chi_m = \begin{cases} 0 & m \leq 1 \\ 1 & m > 1. \end{cases}$$

The so-called $m^{th}$-order deformation equation (8) is a linear equation which can be easily solved with a computer algebra system, for example, Maple as we do in this paper.

## 3. Experimental Results: Application of HAM to KS Equation

In this section we apply the HAM to the modified KS equation to obtain approximate analytical solutions of this equation. Consider the modified Kuramoto-Sivashinsky equation

$$u_t + uu_x + u_{xx} + u_{xxxx} = 0 \tag{10}$$

with initial condition

$$u(x,0) = c + \tfrac{5}{19}\sqrt{\tfrac{11}{19}} \left[11\tanh^3(k(x-x_0)) - 9\tanh(k(x-x_0))\right].$$

In [26], the exact solution of (10) is given as

$$u(x,t) = c + \frac{5}{19}\sqrt{\frac{11}{19}}[11\tanh^3(k(x-ct-x_0)) - 9\tanh(k(x-ct-x_0))]. \tag{11}$$

Next, based on (10) we are motivated to define the following nonlinear fractional partial differential operator:

$$ND\left(u\left(x,t;q\right)\right) = u_t\left(x,t;q\right) + u\left(x,t;q\right)u_x\left(x,t;q\right) + pu_{xx}\left(x,t;q\right) + qu_{xxxx}\left(x,t;q\right). \tag{12}$$

Using (12), we construct a $0^{th}$-order deformation equation as

$$\left(1-q\right)L\left(u\left(x,t;q\right) - u_0\left(x,t\right)\right) = hqNDu\left(x,t;q\right). \tag{13}$$

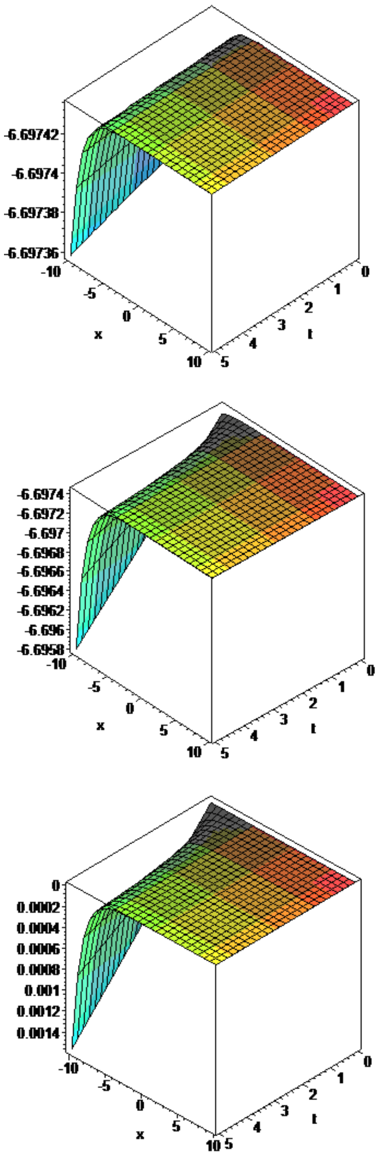


**Figure 1** Figures from top to bottom: exact solution, approximate solution and error function.

For $q = 0$ and $q = 1$, respectively, we can write

$$u(x,t;0) = u_0(x,t) = u(x,0), \quad u(x,t;1) = u(x,t). \tag{14}$$

According to (8) and (9), we obtain the $m^{th}$-order deformation equation

$$L\left(u_m\left(x,t\right) - \chi_m u_{m-1}\left(x,t\right)\right) = hNR\left(\mathbf{u}_{m-1}\left(x,t\right)\right) \tag{15}$$

where

$$NR\left(u_m(x,t)\right) = D_t u_m\left(x,t\right) + \sum_{i=0}^{m-1} u_i(x,t)(u_{m-1-i})_x(x,t) + p(u_{m-1})_{xx}(x,t) + q(u_{m-1})_{xxxx}(x,t).$$

Thus, for $m \geq 1$, the solution of (15) becomes

$$u_m\left(x,t\right) = \chi_m u_{m-1}\left(x,t\right) + hL^{-1}NR\left[\overrightarrow{u}_{m-1}\left(x,t\right)\right]. \tag{16}$$

For $p = q = 1$ and $h = -1$, from (10), (14) and (16), the initial condition of $u(x,t)$ is given by

$$u_0(x,t) = c + \frac{55}{361}\sqrt{209}\left(\tanh\left(k\left(x - x0\right)\right)\right)^3 - 9\tanh\left(k\left(x - x0\right)\right).$$

Using the recurrence equation (16), we obtain $u_1(x,t) =$

By using the first two terms of the Taylor expansion

$$u(x,t) = u_0(x,t) + u_1(x,t) + ...$$

we can write:

$$u(x,t) \approx u_0(x,t) + u_1(x,t)$$

Table 1 shows the application of HAM to the approximate analytical solution of KS equation and exact solution for different values of $x$, $t$ and $c = 0.1, k = \frac{1}{2}\sqrt{11/19}, x_0 = -25$. Figure 1 illustrates the exact and approximate solutions and error functions for the same values of $x, t, c, k$ and $x_0$.

**Table 1** Solutions obtained by HAM and exact solution of KS eq.

| x | t | HAM solution | Exact solution |
|---|---|---|---|
| -1 | 1 | -6.697435131 | -6.697435464 |
| -0.1 | 0.1 | -6.697435480 | -6.697435496 |
| -0.01 | 0.01 | -6.697435496 | -6.697435497 |
| -0.001 | 0.001 | -6.697435498 | -6.697435498 |
| 0 | 0 | -6.697435498 | -6.697435498 |
| 0.001 | 0.001 | -6.697435498 | -6.697435498 |
| 0.01 | 0.01 | -6.697435496 | -6.697435498 |
| 0.1 | 0.1 | -6.697435485 | -6.697435499 |
| 1 | 1 | -6.697435439 | -6.697435511 |

## 4. Conclusion

In this paper, we first described HAM shortly, and then used it to obtain approximate analytical solutions of modified GKS equation. We wrote these analytical solutions explicitly and compared them with the exact solutions of KS equation. We used the Maple Computer Algebra System

$$\left(\begin{array}{l}\left(c+\frac{55}{361}\sqrt{209}\left(\tanh\left(k\left(x-x_0\right)\right)\right)^3-9\tanh\left(k\left(x-x_0\right)\right)\right)\\ \cdot\left(\frac{165}{361}\sqrt{209}\left(\tanh\left(k\left(x-x_0\right)\right)\right)^2\left(1-\left(\tanh\left(k\left(x-x_0\right)\right)\right)^2\right)k-\right)\\ +\frac{330}{361}\sqrt{209}\tanh\left(k\left(x-x_0\right)\right)\left(1-\left(\tanh\left(k\left(x-x_0\right)\right)\right)^2\right)^2k^2\\ -\frac{330}{361}\sqrt{209}\left(\tanh\left(k\left(x-x0\right)\right)\right)^3\left(1-\left(\tanh\left(k\left(x-x_0\right)\right)\right)^2\right)k^2\\ +18\tanh\left(k\left(x-x_0\right)\right)\left(1-\left(\tanh\left(k\left(x-x_0\right)\right)\right)^2\right)k^2\\ -\frac{6600}{361}\sqrt{209}\left(1-\left(\tanh\left(k\left(x-x_0\right)\right)\right)^2\right)^3k^4\tanh\left(k\left(x-x_0\right)\right)\\ +\frac{11880}{361}\sqrt{209}\left(\tanh\left(k\left(x-x_0\right)\right)\right)^3\left(1-\left(\tanh\left(k\left(x-x_0\right)\right)\right)^2\right)^2k^4\\ -\frac{1320}{361}\sqrt{209}\left(\tanh\left(k\left(x-x_0\right)\right)\right)^5\left(1-\left(\tanh\left(k\left(x-x_0\right)\right)\right)^2\right)k^4\\ -144\left(1-\left(\tanh\left(k\left(x-x_0\right)\right)\right)^2\right)^2k^4\tanh\left(k\left(x-x_0\right)\right)\\ +72\left(\tanh\left(k\left(x-x_0\right)\right)\right)^3\left(1-\left(\tanh\left(k\left(x-x_0\right)\right)\right)^2\right)k^4\end{array}\right)\cdot t$$

in the numerical computations and observed that the algorithm works quite fast and only a small number of iterations are needed to obtain satisfactory results which proves the strength of HAM and Maple. In a future work we plan to exploit this method to approximate analytical solutions of some other nonlinear equations.

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