Mathematical Sciences Letters
An International Journal

http://dx.doi.org/10.18576/msl/110202

Modulus Functions Sequence-Based Operator Ideal

Nashat Faried , *Mohamed Ali and Hanan Sakr**

Mathematics Department, Faculty of Science, Ain Shams University, Cairo, Egypt

Received: 2 Feb. 2022, Revised: 2 Mar. 2022, Accepted: 22 Mar. 2022
Published online: 1 May 2022

Abstract: The purpose of this article is to establish the ideal of bounded linear operators between arbitrary Banach spaces, for which the approximation numbers sequence is a member of the space of sequences given by a sequence of modulus functions. In addition, we developed an operator ideal utilizing certain well-known spaces such as Ces*á*ro sequence space and Orlicz sequence space as particular examples of our results. Furthermore, we showed that the operators of the finite rank are dense in the operator ideal produced by these spaces. Moreover, we verified that the operator ideal components formed by them are complete. Our results generalize those in [1] by Faried and Bakery.

Keywords: Approximation numbers, Modulus function, Operator ideal

1 Introduction

Because of its vast applications in the fixed point hypothesis, the geometry of Banach spaces, the eigenvalues hypothesis, and the spectral hypothesis, the hypothesis of operator ideal aims has a unique importance in the helpful investigation. Diverse scalar grouping spaces determine major participation of the administrator aims in the family of normed spaces or Banach spaces in direct practical evaluation.

In the literature, various operator ideals were defined by using sequences of different $s-$numbers of bounded linear operators. For example, the operator ideals were, initially, created by Pietsch [2] using ℓ^p, where $0 < p < \infty$ (the space of all $p-$absolutely summable sequences of real numbers), and the approximation numbers. Later, Faried and Bakery [1] formed operator ideals with their approximation numbers by taking the space ℓ_ϕ, and the space $ces((p_n))$, where ϕ is an Orlicz function, and (p_n) is a sequence of positive reals. After that, Bakery ([3] and [4]) formed an operator ideal, as well as, its approximation numbers using a Ces*á*ro type spaces distinct mean including Lacunary sequence. In addition, using weighted Ces*á*ro sequence space, Maji and Srivastava [5] constructed a family of $s-$type $ces(p,q)$ operators and proved that it creates a quasi-Banach operator ideal. Moreover, Bakery [6] formed an operator ideal using Norlund sequence space.

After that, Bakery [7] studied mappings of type special space of sequences (sss), as well as, Faried and Bakery [8] investigated small operator ideals formed by $s-$numbers on generalized Ces*á*ro and Orlicz sequence spaces. Furthermore, Bakery and Mohammed ([9], [10], and [11]) introduced small pre-quasi Banach operator ideals of type Orlicz-Ces*á*ro mean sequence spaces. Also, they established some properties of pre-quasi operator ideal of type generalized Ces*á*ro sequence space defined by weighted means and of pre-quasi normed generalized de La Vall*é*e Poussin's mean sequence space. Later, Bakery and Abou Elmatty [12] examined some properties of pre-quasi norm on Orlicz sequence space, as well as, Bakery and Mohamed [13] studied the nonlinearity of extended $s-$type weighted Nakano sequence spaces of fuzzy functions with some applications. For more details and discussions about these efforts and more related results and topics, one can refer to these papers [14], [15], [16], [17], [18], [19], [20], [21], [22], and [23].

The goal of this paper is to illustrate that the ideal of bounded linear operators between arbitrary Banach spaces exists when the approximation numbers sequence belongs to the space of sequences defined by a sequence of modulus functions. Furthermore, we created an operator ideal by using well-known spaces like Ces*á*ro sequence space and Orlicz sequence space as examples of our findings. We also demonstrated that, in the operator ideal formed by these spaces, the operators of finite rank

* Corresponding author e-mail: hananhassan@edu.asu.edu.eg

are dense. We integrated that the created operator's ideal components are complete. Our results are not only compatible with those of Faried and Bakery [1], but also generalize them.

The rest of this work is organized as the following. The aim of section (2) is to investigate the preliminaries and definitions needed in the sequel results. Section (3) is devoted to introduce the linear issue, the topological issue, and the ideal components completeness related to the new-constructed spaces. In section (4), conclusion remarks about our work and future related works are introduced.

2 Definitions and Preliminaries

This section displays a number of notations and concepts that are used in the following (most of them can be accessed in [24], [25], [26], [27], [2], and [28]): Suppose that ℓ^0 denotes the space containing every real or complex sequence. A sequence space is defined as any subspace of ℓ^0. Let us use the symbols $\mathbb{N} = \{1,2,3,...\}$, and $\mathbb{R}$ to denote the set of natural numbers, and the set of real numbers, respectively. In addition, the space containing every bounded linear operator from a Banach space X to a Banach space Y could own the symbol of $\mathscr{B}(X,Y)$. That is to say that

$$\mathscr{B}(X,Y) = \{T : X \to Y;\ T \ linear\ and\ bounded\ X,Y\ Banach\ spaces\}.$$

Furthermore, e_n denotes $(0,0,...,1,0,0,...)$, in which one exists at n^{th} position for all $n \in \mathbb{N}$.

Now, we give some definitions and results about operator ideals and $s-$number sequences.

Definition 2.1.[24] A finite rank operator is a bounded linear operator whose dimension of the range space is finite. It should be noted that the space of all finite rank operators is symbolized for $\mathfrak{F}(X,Y)$.

Definition 2.2.[24] A metric injection is a one-to-one operator with a closed range and with a norm equals one.

Definition 2.3.[28] A map $s : T \to (s_n(T))$ which allocate to any operator $T \in \mathscr{B}(X,Y)$ a unique non-negative scalar sequence $(s_n(T))_{n=0}^{\infty}$ is said to be an $s-$function, as well as, the number $s_n(T)$ is said to be the n^{th} $s-$number of T, provided that the seven required conditions are verified, for any arbitrary Banach spaces X, Y, X_0, Y_0, as follows:

1. For any $T \in \mathscr{B}(X,Y)$, we have $\|T\| = s_0(T) \geq s_1(T) \geq s_2(T) \geq ... \geq 0$.
2. For any $T,V \in \mathscr{B}(X,Y)$, we have $s_{n+m}(T+V) \leq s_n(T) + s_m(V)$.
3. For any $T,V \in \mathscr{B}(X,Y)$, we have $s_n(T+V) \leq s_n(T) + \|V\|$.
4. Ideal property: For any $V \in \mathscr{B}(X_0,X)$, $T \in \mathscr{B}(X,Y)$ and $U \in \mathscr{B}(Y,Y_0)$, we have $s_n(UTV) \leq \|U\| s_n(T)\|V\|$.
5. For $T \in \mathscr{B}(X,Y)$ and $\lambda \in \mathbb{R}$, we obtain that $s_n(\lambda T) = |\lambda| s_n(T)$.
6. For any $T \in \mathscr{B}(X,Y)$, we have $s_n(T) = 0$, where $n \geq rank(T)$.
7. $s_{q\geq n}(I_n) = 0$ or $s_{q<n}(I_n) = 1$, taking into account that I_n is the unit operator on ℓ_2^n (Hilbert space of dimension n).

Example 2.1.[28] One can list some $s-$numbers important examples:

1. Approximation numbers $\alpha_n(T) = \inf\{\|T - S\| : S \in \mathscr{B}(X,Y)\ and\ rank(S) \leq n\}$.
2. Gel'fand numbers $c_n(T) = \alpha_n(J_Y T)$, taking into account that J_Y is a metric injection from the space Y into a higher space $\ell^\infty(\Lambda)$ for suitable index Λ.
3. Kolmogorov numbers $d_n(T) = \inf_{dimY\leq n} \sup_{\|x\|\leq 1} \inf_{y\in Y} \|Tx - y\|$.
4. Tichomirov numbers $d_n^*(T) = d_n(J_Y T)$.

For more details about these examples, one can refer to [1], and [28]. We concentrate only on approximation numbers $\alpha_n(T)$ throughout the rest of the paper in constructing operator ideals and establishing more related results.

Definition 2.4.[28] The operator ideal is a subclass $\mathscr{U}$ of $\mathscr{B} := \{\mathscr{B}(X,Y) : X,Y\ Banach\ spaces\}$, such that $\{\mathscr{U}(X,Y) : X,Y\ Banach\ spaces\}$ fulfil the three required components conditions as follows:

1. $I_K \in \mathscr{U}$, taking into account that K represents the one-dimensional Banach space.
2. $\mathscr{U}(X,Y)$ is a vector space on $\mathbb{R}$.
3. Assuming that $V \in \mathscr{B}(X_0,X)$, $T \in \mathscr{U}(X,Y)$, $U \in \mathscr{B}(Y,Y_0)$, therefore we have $UTV \in \mathscr{U}(X_0,Y_0)$.

Lemma 2.1.[1] For any Banach space Y, the space $\mathscr{B}(X,Y)$ is again a Banach space.

Lemma 2.2.[1] If $T,V \in \mathscr{B}(X,Y)$, then $|s_n(T) - s_n(V)| \leq \|T - V\|$ for $n = 1,2,3,....$

Definition 2.5.[1] The special space of sequences (sss) is defined as a class of linear sequence spaces F satisfying the three required conditions as follows:

1. F is a vector space over $\mathbb{R}$, as well as, for all $n \in \mathbb{N}$, we have $e_n \in F$.
2. For $n \in \mathbb{N}$, we obtain that: If $x = (x_n) \in \ell^0, y = (y_n) \in F$ and $|x_n| \leq |y_n|$, then $x \in F$ (i.e., F is solid).
3. Provided that $(x_n)_{n=0}^{\infty} \in F$, hence we get $(x_{[\frac{n}{2}]})_{n=0}^{\infty} = (x_0,x_0,x_1,x_1,x_2,x_2,...) \in F$, taking into account that $[\frac{n}{2}]$ stands for the integral part of $\frac{n}{2}$.

Definition 2.6.[1] $\mathscr{U}_F^{app} := \{\mathscr{U}_F^{app}(X,Y) : X,Y\ Banach\ spaces\}$, taking into account that $\mathscr{U}_F^{app}(X,Y) := \{T \in \mathscr{B}(X,Y) : (\alpha_n(T))_{n=0}^{\infty} \in F\}$ are its components.

Theorem 2.1.[1] If F is a special space of sequences (sss), then $\mathscr{U}_F^{app}$ is an operator ideal.

Proof. For the proof of this theorem, one can refer to [1].

Definition 2.7.[25] $f : [0,\infty) \to [0,\infty)$ is known as a modulus function if it satisfies the following:

1. $x = 0$ if and only if $f(x) = 0$.
2. For any $x, y \geq 0$, we have $f(x+y) \leq f(x) + f(y)$.
3. f is continuous from right direction of zero.
4. f is increasing.

Definition 2.8.[25] $f : [0,\infty) \to [0,\infty)$ is said to be a convex function, if for a scalar λ, one can obtain that $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$, for all $x, y \geq 0$.

Definition 2.9.[25] Assume that any function $\phi : [0,\infty) \to [0,\infty)$ is satisfying non-decreasing property, continuity property and convexity property along with $\phi(0) = 0$, $\phi(x) > 0$ when $x > 0$ and $\phi(x) \to \infty$ when $x \to \infty$, then one can call it an Orlicz function.

Remark 2.1.[25] In the event that the subadditivity of f is replaced by convexity, it returns to be an Orlicz function ϕ.

Kolk ([29], and [30]) used a sequence of modulus functions (f_k) to construct the spaces $Z((f_k))$ for $Z \in \{\ell, \ell^\infty, c, c_0\}$. On $\ell((f_k)) = \{x = (x_k) \in \ell^0 : \sum_{k=0}^{\infty} f_k(|x_k|) < \infty\}$, define the functional $\rho(x) = \sum_{k=0}^{\infty} f_k(|x_k|)$.

3 Main Results

In this section, the ideal including bounded linear operators defined between arbitrary Banach spaces, for which approximation numbers sequence is determined by a sequence of modulus functions, is investigated.

3.1 Linear issue

One can start investigating the operator ideals produced by approximation numbers, and the sequence spaces characterized by a sequence of modulus functions. In such spaces, the class of all bounded linear operators defined between any arbitrary Banach spaces along with the n^{th} approximation numbers generated by the bounded linear operators in those spaces constructs an operator ideal.

The following notes are very important and will be used in the sequel results.

Remark 3.1. For a modulus function f, one can obtain that $f(\lambda x) \leq 2\lambda f(x)$, where $\lambda \in \mathbb{R}$ and $\lambda > 1$.

We obtain from condition (2) in Definition (2.7) that, $f(2x) \leq 2f(x)$, $f(2^2x) \leq 2^2 f(x)$, ..., $f(2^{n+1}x) \leq 2^{n+1} f(x)$ for all $n \in \mathbb{N}$.
Since $\lambda \in \mathbb{R}$ and $\lambda > 1$ we have $2^n \leq \lambda \leq 2^{n+1}$ for some $n \in \mathbb{N}$,
then from condition (3) in Definition (2.7), we have
$f(\lambda x) \leq f(2^{n+1}x) \leq 2^{n+1} f(x) = 2 \cdot 2^n f(x) \leq 2\lambda f(x)$.

Remark 3.2. For a modulus function f, one can get $\frac{1}{2}\lambda f(x) \leq f(\lambda x)$, where $\lambda \in \mathbb{R}$ and $\lambda < 1$.

In fact, from condition (2) in Definition (2.7),
$f(2x) \leq 2f(x)$, then $\frac{1}{2} f(2x) \leq f(x)$,
also $f(2^2x) \leq 2^2 f(x)$, then $\frac{1}{2^2} f(2^2x) \leq f(x)$,
and so on until...
$f(2^{n+1}x) \leq 2^{n+1} f(x)$, i.e. $\frac{1}{2^{n+1}} f(2^{n+1}x) \leq f(x)$ for all $n \in \mathbb{N}$.
Since $\lambda \in \mathbb{R}$ and $\lambda < 1$ we have $\frac{1}{2^{n+1}} \leq \lambda \leq \frac{1}{2^n}$ for some $n \in \mathbb{N}$,
then from condition (3) in Definition (2.7), we have
$\frac{1}{2}\lambda f(x) \leq \frac{1}{2}\frac{1}{2^n} f(x) = \frac{1}{2^{n+1}} f(x) \leq f(\frac{1}{2^{n+1}}x) \leq f(\lambda x)$.

Remark 3.3. If we take:

1. $f_k(x) = x^p$, for all k, then $\ell((f_k))$ is reduced to ℓ^p $(0 < p < \infty)$ studied by Pietsch [2].
2. $f_k(|x_k|) = (\frac{\sum_{k=0}^{\infty}|x_k|}{n+1})^{p_k}$, for all k, $p_k \geq 1$, then $\ell((f_k))$ coincides with the space $ces((p_k))$ studied by Faried and Bakery [1].
3. $f_k(|x_k|) = (\frac{\sum_{k=0}^{\infty}|x_k|}{n+1})^{p}$, for all k, $1 < p < \infty$, then $\ell((f_k))$ coincides with the space $ces(p)$ studied by Lee [31].
4. $f_k(x) = \phi_k(x) = \phi(x)$, Orlicz function, for all k, then $\ell((f_k))$ coincides with the space ℓ_ϕ studied by Faried and Bakery [1].

Theorem 3.1. $\ell((f_k))$ is a special space of sequences (sss), provided that the sequence $(f_k(h))$ is monotonic decreasing for all h.

Proof.

1. (a)i Suppose that $x, y \in \ell((f_k))$, then we obtain that $\sum_{k=0}^{\infty} f_k(|x_k|) < \infty$ and $\sum_{k=0}^{\infty} f_k(|y_k|) < \infty$. Therefore,

$$\sum_{k=0}^{\infty} f_k(|x_k + y_k|) \leq \sum_{k=0}^{\infty} f_k(|x_k| + |y_k|).$$

Since f_k are modulus functions for all k, using condition (2) from Definition (2.7), we have:

$$\sum_{k=0}^{\infty} f_k(|x_k + y_k|) \leq \sum_{k=0}^{\infty} f_k(|x_k|) + \sum_{k=0}^{\infty} f_k(|y_k|) < \infty.$$

Then $x + y \in \ell((f_k))$.

ii. Let $x \in \ell((f_k))$ and $\lambda \in \mathbb{R}$, then we have $\sum_{k=0}^{\infty} f_k(|x_k|) < \infty$. By using Remark (3.1), we get:

$$\sum_{k=0}^{\infty} f_k(|\lambda x_k|) \leq \max\{1, 2|\lambda|\} \sum_{k=0}^{\infty} f_k(|x_k|) < \infty.$$

So $\lambda x \in \ell((f_k))$.
From (i) and (ii), one can obtain that $\ell((f_k))$ is a vector space over $\mathbb{R}$.

(b) Let us show that $\{e_n\} \subseteq \ell((f_k))$. We have:

$$\sum_{k=0}^{\infty} f_k(|e_n(k)|) = f_n(1) < \infty.$$

Hence $e_n \in \ell((f_k))$, for all $n \in \mathbb{N}$.

2. Assume that $x = (x_k) \in \ell^0$, $y = (y_k) \in \ell((f_k))$ and $|x_k| \leq |y_k|$, for all k. Since f_k are increasing for any k, then we have

$$\sum_{k=0}^{\infty} f_k(|x_k|) \leq \sum_{k=0}^{\infty} f_k(|y_k|) < \infty.$$

Therefore $\ell((f_k))$ is a solid space.

3. Let $(x_k) \in \ell((f_k))$, then we have $\sum_{k=0}^{\infty} f_k(|x_k|) < \infty$. Thus

$$\sum_{k=0}^{\infty} f_k(|x_{[\frac{n}{2}]}|) = \sum_{k=0}^{\infty} f_{2k}(|x_k|) + \sum_{k=0}^{\infty} f_{2k+1}(|x_k|).$$

Using the given that the sequence $(f_k(h))$ is monotonic decreasing for all h,
i.e. $f_{2k}(h) \leq f_k(h)$ and $f_{2k+1}(h) \leq f_k(h)$, for all h, then

$$\sum_{k=0}^{\infty} f_k(|x_{[\frac{n}{2}]}|) \leq \sum_{k=0}^{\infty} f_k(|x_k|) + \sum_{k=0}^{\infty} f_k(|x_k|) < \infty.$$

And hence $(x_{[\frac{n}{2}]}) \in \ell((f_k))$.

Hence, by (1), (2) and (3), $\ell((f_k))$ satisfies the three required conditions of the special space of sequences. That is to say that $\ell((f_k))$ is an sss.

We can deduce the following results from Theorem (3.1). Furthermore, the proof for any following corollary is direct by using Theorem (2.1), and Theorem (3.1).

Corollary 3.1. $\mathscr{U}^{app}_{\ell((f_k))}$ is an operator ideal, provided that we have the sequence $(f_k(h))$ is monotonic decreasing for all h.

Corollary 3.2. $\mathscr{U}^{app}_{\ell_p}$ is an operator ideal, where $0 < p < \infty$.

Corollary 3.3. $\mathscr{U}^{app}_{ces((p_k))}$ is an operator ideal (also under the condition that $\liminf_{n\to\infty} p_n > 1$ and $\limsup_{n\to\infty} p_n < \infty$, considered by Faried and Bakery [1], the same result can be found).

Corollary 3.4. Assuming that $1 < p < \infty$, therefore we have $\mathscr{U}^{app}_{ces_p}$ is an operator ideal.

Corollary 3.5. Given ϕ is an Orlicz function, we get that $\mathscr{U}^{app}_{\ell_\phi}$ is an operator ideal.

3.2 Topological issue

We prove here that the finite rank operators ideal in the class of Banach spaces is dense in $\mathscr{U}^{app}_{\ell((f_k))}$.

Theorem 3.2. If the sequence of modulus functions $(f_k(h))$ is monotonic decreasing for all h and $kf_k(1) \geq d$, $d > 0$, then all finite rank operators space in the class of Banach spaces is dense in $\mathscr{U}^{app}_{\ell((f_k))}(X,Y)$, i.e. $\overline{\mathfrak{F}(X,Y)} = \mathscr{U}^{app}_{\ell((f_k))}(X,Y)$.

Proof. Define $\rho(T) = \sum_{n=0}^{\infty} f_n(\alpha_n(T))$ on $\mathscr{U}^{app}_{\ell((f_k))}(X,Y)$.
Firstly, we prove that each finite operator $T \in \mathfrak{F}(X,Y)$ belongs to the space $\mathscr{U}^{app}_{\ell((f_k))}(X,Y)$. Assume that $T \in \mathfrak{F}(X,Y)$, then $\alpha_n(T) = 0$, for all $n \geq rank(T)$. So we have:

$$\sum_{n=0}^{\infty} f_n(\alpha_n(T)) = \sum_{n=0}^{(rank(T))-1} f_n(\alpha_n(T)) < \infty.$$

Therefore $T \in \mathscr{U}^{app}_{\ell((f_k))}(X,Y)$.
To show that $\mathscr{U}^{app}_{\ell((f_k))}(X,Y) \subseteq \overline{\mathfrak{F}(X,Y)}$.
By taking $T \in \mathscr{U}^{app}_{\ell((f_k))}(X,Y)$, one can obtain $(\alpha_n(T))_{n=0}^{\infty} \in \ell((f_k))$, i.e. $\sum_{n=0}^{\infty} f_n(\alpha_n(T)) < \infty$.
Let $\eta > 0$, hence there is $s \in \mathbb{N}$ where

(1)

$$and\ \alpha_{2s} < 1.$$

By using that $\alpha_n(T)$ is decreasing for all $n \in \mathbb{N}$ and with the help of the fact that the sequence of modulus functions $(f_k(h))$ is monotonic decreasing for all h, we get:

$$s^2 f_{2s}(\alpha_{2s}(T)) \leq \sum_{n=s+1}^{s+s^2} f_n(\alpha_n(T)) \leq \sum_{n=s}^{\infty} f_n(\alpha_n(T)).$$

By using (1), we have:

(2)

Since f_{2s} is a modulus function and we have $\alpha_{2s} < 1$, then by using Remark (3.2), we obtain $\frac{s^2}{2} f_{2s}(1)\alpha_{2s}(T) < \frac{d\eta}{4f_0(1)}$. By the condition that $kf_k(1) \geq d$, $d > 0$, then $\frac{s}{4} d\alpha_{2s}(T) < \frac{d\eta}{4f_0(1)}$. Hence

(3)

Since $\alpha_{2s} = \inf\{||T - A|| : rank(A) \leq 2s\}$ and from (3), then there is $A \in \mathfrak{F}_{2s}(X,Y)$, for which $rank(A) \leq 2s$,

(4)

Then, we have

$$\begin{aligned}
\mathbf{d}(T,A) &= \rho(\alpha_n(T-A))_{n=0}^{\infty} \\
&= \sum_{n=0}^{\infty} f_n(\alpha_n(T-A)) \\
&= \sum_{n=0}^{3s-1} f_n(\alpha_n(T-A)) + \sum_{n=3s}^{\infty} f_n(\alpha_n(T-A)) \\
&\leq 3sf_0(||T-A||) + \sum_{n=s}^{\infty} f_{n+2s}(\alpha_{n+2s}(T-A)).
\end{aligned}$$

Obviously, one has $\alpha_{n+2s}(T-A) \leq \alpha_n(T) - \alpha_{2s}(A) = \alpha_n(T)$. Then

$$\mathbf{d}(T,A) \leq 3sf_0(||T-A||) + \sum_{n=s}^{\infty} f_{n+2s}(\alpha_n(T)).$$

By using Remark (3.1), we get:

$$\mathbf{d}(T,A) \le 6sf_0(1)||T-A|| + \sum_{n=s}^{\infty} f_{n+2s}(\alpha_n(T)).$$

Since the sequence of modulus functions $(f_k(h))$ is monotonic decreasing for all h,
so $f_{n+2s} \le f_n$, for $n \in \mathbb{N}$. Hence, we obtain:

$$\mathbf{d}(T,A) \le 6sf_0(1)||T-A|| + \sum_{n=s}^{\infty} f_n(\alpha_n(T)).$$

From (2) and (4), we have:

$$\begin{aligned}\mathbf{d}(T,A) &\le 6f_0(1)\frac{\eta}{f_0(1)} + \frac{d\eta}{4f_0(1)} \\ &= 6\eta + \frac{d\eta}{4f_0(1)}.\end{aligned}$$

Therefore for $\varepsilon > 0$, there is $s \in \mathbb{N}$, where $\mathbf{d}(T,A) \le \varepsilon$, and $\varepsilon = 6\eta + \frac{d\eta}{4f_0(1)}$. Hence, we obtain $\overline{\mathfrak{F}(X,Y)} = \mathscr{U}^{app}_{\ell((f_k))}(X,Y)$.

One can infer the sequel corollaries from Theorem (3.2). Moreover, the proof for any following result is clear from Theorem (3.2).

Corollary 3.6. $\overline{\mathfrak{F}(X,Y)} = \mathscr{U}^{app}_{\ell_p}(X,Y)$, where $0 < p < \infty$.

Corollary 3.7. $\overline{\mathfrak{F}(X,Y)} = \mathscr{U}^{app}_{ces((p_k))}(X,Y)$, where $p_k \ge 1$.

Corollary 3.8. $\overline{\mathfrak{F}(X,Y)} = \mathscr{U}^{app}_{ces_p}(X,Y)$, where $1 < p < \infty$.

Corollary 3.9. $\overline{\mathfrak{F}(X,Y)} = \mathscr{U}^{app}_{\ell_\phi}(X,Y)$, where ϕ is an Orlicz function.

3.3 Ideal components completeness

For the sequence space defined by a sequence of modulus functions, one can establish the completeness of the components of the ideal $\mathscr{U}^{app}_{\ell((f_k))}(X,Y)$.

Theorem 3.3. $\mathscr{U}^{app}_{\ell((f_k))}(X,Y)$ is complete, provided that Y is complete.

Proof. Let $(T_n)_{n=0}^{\infty}$ be a Cauchy sequence in $\mathscr{U}^{app}_{\ell((f_k))}(X,Y)$. Then we have:

(5)

But $\rho(T_n - T_m) := \sum_{n=0}^{\infty} f_n(\alpha_n(T_n - T_m)) \ge f_0(\alpha_0(T_n - T_m))$. Suppose that $||T_n - T_m|| \nrightarrow 0$, then there exists $\beta > 0$, such that $||T_{n_k} - T_{m_k}|| \ge \beta > 0$. Since f_0 is nondecreasing function, then $f_0(||T_{n_k} - T_{m_k}||) \ge f_0(\beta) \ge f_0(0) = 0$. So we have $\rho(T_{n_k} - T_{m_k}) \nrightarrow 0$ which is a contradiction with the given. And hence $||T_n - T_m||_{\overrightarrow{n,m\to\infty}} 0$ must hold. Since Y is complete, hence $\mathscr{B}(X,Y)$ is also complete. As we get $||T_n - T_m|| \to 0$ as $n,m \to \infty$, i.e. $(T_n)_{n=0}^{\infty}$ is a Cauchy sequence in $\mathscr{B}(X,Y)$, hence we have that it is convergent, i.e. there exists $T \in \mathscr{B}(X,Y)$ such that $||T_n - T||_{\overrightarrow{n\to\infty}} 0$. From (5), we have:

(6)

It is obvious that $\alpha_r(T_n - T) = \alpha_r(T_n - T_m + T_m - T) \le \alpha_0(T_m - T) + \alpha_r(T_n - T_m)$. Since f_r are nondecreasing functions for every r, then we get:

$$f_r(\alpha_r(T_n - T)) \le f_r(\alpha_0(T_m - T) + \alpha_r(T_n - T_m)).$$

And by subadditivity of f_r for every r, we have:

$$f_r(\alpha_r(T_n - T)) \le f_r(\alpha_0(T_m - T)) + f_r(\alpha_r(T_n - T_m)).$$

So we obtain:

$$\begin{aligned}|f_r(\alpha_r(T_n - T)) - f_r(\alpha_r(T_n - T_m))| &\le f_r(\alpha_0(T_m - T)) \\ &= f_r(||T_m - T||).\end{aligned}$$

By using the continuity of f_r at zero for every r and since $||T_m - T||_{\overrightarrow{m\to\infty}} 0$,
then $f_r(||T_m - T||)_{\overrightarrow{m\to\infty}} 0$. And hence

(7)

From (6), we have:

$$\sum_{r=0}^{k} f_r(\alpha_r(T_n - T_m)) < \varepsilon,\ for\ every\ k.$$

Therefore, from (7), we have:

$$\sum_{r=0}^{k} f_r(\alpha_r(T_n - T)) < \varepsilon,\ for\ every\ k.$$

By taking $k \to \infty$, we obtain:

$$\sum_{r=0}^{\infty} f_r(\alpha_r(T_n - T)) < \varepsilon.$$

Then for $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$, satisfying $\rho(T_n - T) < \varepsilon$, for $n \ge n_0$. Then $(T_n)_{n=0}^{\infty}$ is a convergent sequence in $\mathscr{U}^{app}_{\ell((f_k))}(X,Y)$. We have $T - T_n \in \mathscr{U}^{app}_{\ell((f_k))}(X,Y)$ and $T_n \in \mathscr{U}^{app}_{\ell((f_k))}(X,Y)$. By linearity of $\mathscr{U}^{app}_{\ell((f_k))}(X,Y)$, then we obtain $T = T - T_n + T_n \in \mathscr{U}^{app}_{\ell((f_k))}(X,Y)$. So $\rho(T_n - T)_{\overrightarrow{n\to\infty}} 0$ and $T = \rho - \lim_n T_n$. Therefore $\mathscr{U}^{app}_{\ell((f_k))}(X,Y)$ is complete, which is required.

The following results can be derived as special cases from Theorem (3.3). In addition, the proof for any of them is obvious from Theorem (3.3).

Corollary 3.10. $\mathscr{U}^{app}_{\ell_p}(X,Y)$ is complete, provided that Y is complete, where $0 < p < \infty$.

Corollary 3.11. $\mathscr{U}^{app}_{ces((p_k))}(X,Y)$ is complete, provided that Y is complete (also under the condition that $\liminf_{n\to\infty} p_n > 1$ and $\limsup_{n\to\infty} p_n < \infty$, considered by Faried and Bakery [1], the same result can be found).

Corollary 3.12. $\mathscr{U}^{app}_{ces_p}(X,Y)$ is complete, provided that Y is complete, if $1 < p < \infty$.

Corollary 3.13. $\mathscr{U}^{app}_{\ell_\phi}(X,Y)$ is complete, provided that Y is complete, where ϕ is an Orlicz function.

Example 3.1. We have the following examples by taking the following special cases of f_k:

1. $f_k(x) = \frac{x^{p_k}}{k+1}$, for all k, then $\ell((f_k)) = \{x = (x_k) \in \ell^0 : \sum_{k=0}^{\infty} \frac{x^{p_k}}{k+1} < \infty\}$. It satisfies the condition $kf_k(1) \geq d$, $d > 0$ by taking, for example, $d = \frac{1}{2}$, then we have $\frac{k|1|^{p_k}}{k+1} \geq d$ since $\frac{1}{k+1} \geq \frac{1}{2k}$, for all k.
2. $f_k(x) = \frac{x^{p_k}}{\sqrt{k+1}}$, for all k, then $\ell((f_k)) = \{x = (x_k) \in \ell^0 : \sum_{k=0}^{\infty} \frac{x^{p_k}}{\sqrt{k+1}} < \infty\}$. It satisfies the condition $kf_k(1) \geq d$, $d > 0$ by taking, for example, $d = \frac{1}{\sqrt{2}}$, then we have $\frac{k|1|^{p_k}}{\sqrt{k+1}} \geq d$ since $\frac{1}{\sqrt{k+1}} \geq \frac{1}{\sqrt{2k}}$, for all k.
3. $f_k(|x_k|) = (\frac{\sum_{k=0}^{n}|x_k|}{n+1})^{p_k}$, for all k, then $\ell((f_k))$ coincides with the space $ces((p_k)) = \{x = (x_k) \in \ell^0 : \sum_{n=0}^{\infty}(\frac{\sum_{k=0}^{n}|x_k|}{n+1})^{p_k} < \infty\}$ studied by Faried and Bakery [1]. It satisfies the condition $kf_k(1) \geq d$, $d > 0$ by taking, for example, $d = 1$, then we have $(\frac{n}{n+1})^{p_k} \geq \frac{1}{n}$, $p_k \geq 1$ since $\frac{n}{n+1} \geq \frac{1}{n}$, for all n.
4. $f_k(|x_k|) = (\frac{\sum_{k=0}^{n}|x_k|}{n+1})^{p}$, for all k, then $\ell((f_k))$ coincides with the space $ces(p) = \{x = (x_k) \in \ell^0 : \sum_{n=0}^{\infty}(\frac{\sum_{k=0}^{n}|x_k|}{n+1})^{p} < \infty\}$ studied by Lee [31]. It satisfies the condition $kf_k(1) \geq d$, $d > 0$ by taking, for example, $d = 1$, then we have $(\frac{n}{n+1})^{p} \geq \frac{1}{n}$, $1 < p < \infty$ since $\frac{n}{n+1} \geq \frac{1}{n}$, for all n.
5. $f_k(x) = \phi_k(x) = \phi(x)$, Orlicz function, for all k, then $\ell((f_k))$ is reduced to ℓ_ϕ studied by Faried and Bakery [1].
6. $f_k(x) = x^p$, for all k, then $\ell((f_k))$ is reduced to ℓ^p, $0 < p < \infty$ studied by Pietsch [2].

4 Conclusion and future works

Forming an operator ideal whose sequence of approximation numbers belongs to some sequence spaces including powers or weights has been studied by many mathematicians. But in our study, we formed an operator ideal using a sequence of modulus functions which is a new problem to solve. Luckily, it appears to be a wider space that includes some well-known spaces as a special case of it. In addition, these results facilitate the previous long methods used to obtain the operator ideal of some well-known spaces and there is an agreement in the final results between them. Furthermore, the modulus functions sequence space, which we formed an operator ideal with, can be seen as a generalization for many spaces, as well as, it is not a special case of any other spaces which we can form an operator ideal with. This investigation fills some gaps in the literature. By employing similar techniques in this paper, the authors may be able to present novel results. The authors can introduce some new-constructed ideals of bounded linear operators between arbitrary Banach spaces whose approximation numbers sequence belongs to the space of sequences defined by a sequence of generalized modulus functions. Moreover, they can use Gel'fand numbers, Kolmogorov numbers, or Tichomirov numbers instead of approximation numbers to produce new results.

Conflict of interest

The authors declare that they have no conflict of interest.

Funding

The authors don't receive any financial support for their research.

Authors' contributions

The authors contributed equally to the paper. All of the authors read and approved the final version of the paper.

Data availability statement

Not applicable.

Acknowledgements

The authors would like to thank the editor and the referees for their valuable suggestions and comments that improved the paper.

References

[1] N. Faried and A.A. Bakery, Mappings of type Orlicz and generalized Cesáro sequence space, J. Inequal. app. **2013:186** (2013).
[2] A. Pietsch, Operator Ideals, North-Holland, Amsterdam, 1980.
[3] A.A. Bakery, Mappings of type generalized de La Vallée Poussin's mean, J. Inequal. app. **2013:518** (2013).
[4] A.A. Bakery, Operator ideal of Cesáro type sequence spaces involving lacunary sequence, Abstr. Appl. Anal. **2014:419560** (2014).
[5] A. Maji and P.D. Srivastava, On operator ideals using weighted Cesáro sequence space, J. Egyp. Math. Soc. **22**, 446-452 (2014).
[6] A.A. Bakery, Operator ideal of Norlund-type sequence spaces, J. Inequal. app. **2015:255** (2015).

[7] A.A. Bakery, Mappings of type special space of sequences, J. Function Spaces, **2016**, Article ID 4372471 (2016).
[8] N. Faried and A.A. Bakery, Small operator ideals formed by $s-$numbers on generalized Cesáro and Orlicz sequence spaces, J. Inequal. app. **2018:357** (2018).
[9] A.A. Bakery and M.M. Mohammed, Small pre-quasi Banach operator ideals of type Orlicz-Cesáro mean sequence spaces, J. Function Spaces, **2019**, Article ID 7265010 (2019).
[10] A.A. Bakery and M.M. Mohammed, Some properties of pre-quasi operator ideal of type generalized Cesáro sequence space defined by weighted means, Open Math., **17**, 1703-1715 (2019).
[11] A.A. Bakery and M.M. Mohammed, Some properties of pre-quasi normed generalized de La Vallée Poussin's mean sequence space, J. Function Spaces, **2020**, Article ID 7265010 (2020).
[12] A.A. Bakery and A.R. Abou Elmatty, Some properties of pre-quasi norm on Orlicz sequence space, J. Inequal. app. **2020:55** (2020).
[13] A.A. Bakery and E.A.E. Mohamed, On the nonlinearity of extended $s-$type weighted Nakano sequence spaces of fuzzy functions with some applications, J. Function Spaces, **2022**, Article ID 2746942 (2022).
[14] N. Faried, M.S.S. Ali and H.H. Sakr, On generalized fractional order difference sequence spaces defined by a sequence of modulus functions, Math. Sci. Lett. **6(2)**, 163-168 (2017).
[15] N. Faried, M. Ali and H. Sakr, Generalized difference sequence spaces of fractional-order via orlicz-functions sequence, Math. Sci. Lett. **10(3)**, 101-107 (2021).
[16] N. Faried, M.S.S. Ali and H.H. Sakr, Vague soft matrix-based decision-making, Glob. J. Pure Appl. Math. **15 (5)**, 755-780 (2019).
[17] N. Faried, M.S.S. Ali and H.H. Sakr, Fuzzy soft Hilbert spaces, J. Math. Comp. Sci. **22 (2021)**, 142-157 (2020).
[18] N. Faried, M.S.S. Ali and H.H. Sakr, On fuzzy soft linear operators in fuzzy soft Hilbert spaces, Abst. Appl. Anal. **2020**, Article ID 5804957 (2020).
[19] N. Faried, M.S.S. Ali and H.H. Sakr, Fuzzy soft symmetric operators, Ann. Fuzzy Math. Inform. **19 (3)**, 275-280 (2020).
[20] N. Faried, M.S.S. Ali and H.H. Sakr, Fuzzy soft hermitian operators, Adv. Math. Sci. J. **9 (1)**, 73-82 (2020).
[21] N. Faried, M. Ali and H. Sakr, On FS normal operators, Math. Sci. Lett. **10(2)**, 41-46 (2021).
[22] N. Faried, M. Ali and H. Sakr, A note on FS isometry operators, Math. Sci. Lett. **10(1)**, 1-3 (2021).
[23] N. Faried, M. Ali and H. Sakr, A theoretical approach on unitary operators in fuzzy soft settings, Math. Sci. Lett. **11(1)**, 45-49 (2022).
[24] A. Lima and E. Oja, Ideals of finite rank operators, intersection properties of balls and the approximation property, Stud. Math. **133**, 175-186 (1999).
[25] H. Nakano, Concave modulars, J. Math. Soc. Japan **5**, 29-49 (1953).
[26] A. Pietsch, Nuclear Locally Convex Spaces, Springer-Verlag Berlin Heidelberg, New York, 1972.
[27] A. Pietsch, s-numbers of operators in Banach spaces, Stud. Math. **51**, 201-223 (1974).
[28] A. Pietsch, Eigenvalues and s-numbers, Cambridge University Press, New York, 1986.
[29] E. Kolk, On strong boundedness and summmability with respect to a sequence of moduli, Acta. Comment. Univ. Tartu. **960**, 41-50 (1993).
[30] E. Kolk, Inclusion theorems for some sequence spaces defined by a sequence of moduli, Acta. Comment. Univ. Tartu. **970**, 65-72 (1994).
[31] P.Y. Lee, Cesáro sequence spaces, Math. Chron. **13**, 29-45 (1984).