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On Joint Density for Concomitants of Generalized Order Statistics Based on Morgenstern Family

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Abstract: Concomitants of ordered random variables have been studied for one and two variables about different subjects such as moments, recurrence relation, uncertainty and so on. In this paper, the joint density of the concomitants of generalized order statistics (GOS's) for Morgenstern family is proposed, study on moments of such model is considered. Statistical inferences such as maximum likelihood (ML) estimation, Bayesian estimation under different types of loss function and Bayesian prediction are obtained for the association parameter of Morgenstern family. Applications of these inferences are presented.

Keywords: Bayesian Estimation, Bayesian Prediction, Concomitants, Generalized order statistics, Maximum likelihood estimation, Morgenstern family.

1 Introduction

The Morgenstern family discussed by Johnson and Kotz [13] provides a flexible family that can be used in such contexts, which is specified by the distribution function (df) and the probability density function (pdf), respectively, as follows:

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)[1+\alpha(1-F_X(x))(1-F_Y(y))], \tag{1}$$

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)[1+\alpha(2F_X(x)-1)(2F_Y(y)-1)], \tag{2}$$

where $-1 \leq \alpha \leq 1$, and $f_X(x)$, $f_Y(y)$, and $F_X(y)$, $F_Y(y)$ are the marginal pdf's and df's of X and Y respectively. The association parameter α is known as the dependence parameter of the random variables X and Y. If α is zero, then X and Y are independent. The conditional pdf of Y given X is given by:

$$f_{Y|X}(y \mid x) = f_Y(y)[1+\alpha(2F_X(x)-1)(2F_Y(y)-1)], \quad -1 \leq \alpha \leq 1. \tag{3}$$

The general theory of concomitants of order statistics has originally studied by David et al. [6]. Let (X_i,Y_i), $i=1,2,\cdots,n$, be n pairs of independent random variables from some bivariate population with df $F(x,y)$. Let $X_{(r:n)}$ be the $r-th$ order statistics, then the Y value associated with $X_{(r:n)}$ is called the concomitant of the $r-th$ order statistics and is denoted by $Y_{[r:n]}$. Sometimes exact information are available only on the concomitants variable since the other variable is only ranked and not measured exactly, consider for example a group of patients ranked according to the value of their response to a treatment and subsequently the values of their blood test are observed only on those patients whose initial value exceeds a threshold, in this situation we have information only on the concomitants variable. The pdf of $Y_{[r:n]}$ is given by:

$$g_{[r:n]}(y) = g_{Y_{[r:n]}}(y) = \int_{-\infty}^{\infty} f_{Y|X}(y \mid x)f_{(r:n)}(x)\mathrm{d}x, \tag{4}$$

where $f_{(r:n)}(x)$ is the pdf of $X_{(r:n)}$. For $1 \leq r_1 < \cdots < r_k \leq n$, the joint density for $Y_{[r_1:n]},\cdots,Y_{[r_k:n]}$ is given by:

$$\begin{aligned} g_{[r_1,\cdots,r_k:n]}(y_1,\cdots,y_k) &= g_{Y_{[r_1:n]},\cdots,Y_{[r_k:n]}}(y_1,\cdots,y_k) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{x_k}\cdots\int_{-\infty}^{x_2}\prod_{i=1}^{k} f_{Y|X}(y_i|x_i) \\ &\quad \times f_{(r_1,\cdots,r_k:n)}(x_1,\cdots,x_k)\mathrm{d}x_1\cdots\mathrm{d}x_k, \end{aligned} \tag{5}$$

where $f_{(r_1,\cdots,r_k:n)}(x_1,\cdots,x_k) = f_{X_{r_1:n},\cdots,X_{r_k:n}}(x_1,\cdots,x_k)$. That is, in general, the joint concomitants of order

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statistics $Y_{[r_1:n]}, \cdots, Y_{[r_k:n]}$ is dependent, where $f_{(r_1,\cdots,r_k:n)}(x_1,\cdots,x_k)$ is the the joint density of $X_{r_1:n},\cdots,X_{r_k:n}$.

Kamps [14] has introduced the GOS's, the joint density function of the first r GOS's $X_{(1,n,m,k)}, X_{(2,n,m,k)}, \cdots, X_{(r,n,m,k)}$, $1 \le r \le n$ is given by:

$$f_{(1,2,\cdots,r,n,m,k)}(x_1,\cdots,x_r) = f_{X_{(1,n,m,k)},\cdots,X_{(r,n,m,k)}}(x_1,\cdots,x_r)$$
$$= c_{r-1}(\prod_{i=1}^{r} f_X(x_i))(\prod_{i=1}^{r-1}(1-F_X(x_i))^m)(1-F_X(x_r))^{\gamma_r-1},$$
$$x_1 \le x_2 \le \ldots \le x_r, \tag{6}$$

with parameters $n \in \mathbb{N}$, $k > 0$, $m \in \mathbb{R}$, such that $\gamma_r = k+(n-r)(m+1) > 0$, $c_{r-1} = \prod_{j=1}^{r} \gamma_j$ for all $1 \le r \le n$.

In Bayesian approach, the performance depends on the prior information about the unknown parameters and the loss function. The prior information can be expressed by the experimenter, who has some beliefs about the unknown parameters and their statistical distributions. Traditionally, most authors have been used squared error (SE) loss function which is symmetric although the use of symmetric loss functions may be inappropriate. One disadvantage when using SE loss is that it penalizes overestimation or underestimation. Overestimation of a parameter can lead to more severe or less severe consequences than underestimation, or vice versa. Subsequently, the use of an asymmetrical loss function, which associates greater importance to overestimation or underestimation, can be considered for the estimation of the parameter, many authors study the SE loss in Bayesian inferences see for example, Calabria and Pulcini [5], Singh et al. [20] and Jaheen ([11] and [12]).

A very useful asymmetric loss function known as the linear exponential (LINEX) loss function, was first introduced by Varian [23] and was widely used by several authers. This function rises approximately exponentially on one side of zero and approximately linearly on the other side.

The general entropy (GE) loss is also asymmetric loss function which is used in several papers, for example, Dey et al. [7], Dey and Liu [8] and Soliman ([21] and [22]).

In many practical problems, one would wish to use previous data to predict a future observation from the same population. One way to do this is to construct an interval which will contain the future observation with a specified probability. This interval is called a prediction interval. Bayesian prediction bounds for future observations have been discussed by several authors, including AL-Hussaini and Jaheen [3], AL-Hussaini and Ahmed [2], Raqab and Madi [17] and Abdel-Aty et al. [1].

In this paper, we consider the Bayesian and non-Bayesian inferences of the association parameter α for concomitants of GOS's based on Morgenstern family. The Bayes estimators are obtained under symmetric SE, asymmetric LINEX and GE loss functions using a non-informative prior. Also, we consider this Bayesian approach to predict the future ordered observations based on the observed ordered data. The organization of the article is as follows: In Section 2, joint density for concomitants of GOS's based on Morgenstern family is considered, also, the moments and the moment generating function of the considered model is studied, and give an example of this moments for subfamilies of Morgenstern distributions. In Section 3, we obtain Bayesian and non-Bayesian estimation, and study the Bayesian prediction of one sample. Application of this results based on order statistics as special cases from generalized order statistics is applied in Section 4. Finally, some conclusions and comments are given in Section 5.

2 Joint Density for Concomitants of Generalized Order Statistics

Based on Morgenstern family, the marginal and joint density of one and two concomitants of GOS's was obtained and studied. In this section, we drive the form of the joint density of the first r concomitants of GOS's $Y_{[1,n,m,k]}, Y_{[2,n,m,k]}, \cdots, Y_{[r,n,m,k]}$, $1 \le r \le n$ for Morgenstern family.

For the Morgenstern family with pdf given by (2), from (4), the density function of the concomitant of the $r-th$ GOS's $Y_{[r,n,m,k]}$, $1 \le r \le n$, is given by:

$$g_{[r,n,m,k]}(y) = f_Y(y)\left[1+\alpha C^*(r,n,m,k)(2F_Y(y)-1)\right], \tag{7}$$

where $C^*(r,n,m,k) = 1-2C(r,n,m,k) = 1-2\frac{\prod_{i=1}^{r}\gamma_i}{\prod_{i=1}^{r}(\gamma_i+1)}$. From (5), the joint density of the concomitants $Y_{[r,n,m,k]}$ and $Y_{[s,n,m,k]}$ of the $r-th$ and $s-th$ GOS's, $1 \le r < s \le n$, has the form:

$$\begin{aligned} g_{[r,s,n,m,k]}(y_1,y_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{x_2} f_{Y|X}(y_1 \mid x_1) f_{Y|X}(y_2 \mid x_2) \\ &\quad \times f_{(r,s,n,m,k)}(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2 \\ &= f_Y(y_1)f_Y(y_2)[1+\alpha C_r(2F_Y(y_1)-1) \\ &\quad +\alpha C_s(2F_Y(y_2)-1) \\ &\quad +\alpha^2 C_{rs}(2F_Y(y_1)-1)(2F_Y(y_2)-1)], \end{aligned} \tag{8}$$

where

$$C_r = C^*(r,n,m,k) = 1-2C(r,n,m,k) = 1-2\frac{\prod_{i=1}^{r}\gamma_i}{\prod_{i=1}^{r}(\gamma_i+1)},$$

$$C_s = C^*(s,n,m,k) = 1 - 2C(s,n,m,k) = 1 - 2\frac{\prod_{i=1}^{s}\gamma_i}{\prod_{i=1}^{s}(\gamma_i+1)},$$

$$C_{rs} = 1 - 2C(r,n,m,k) - 2C(s,n,m,k) + 4\frac{(\prod_{i=1}^{r}\gamma_i)(\prod_{j=r+1}^{s}\gamma_j)}{(\prod_{i=1}^{r}(\gamma_i+2))(\prod_{j=r+1}^{s}(\gamma_j+1))},$$

see Beg and Ahsanullah [4].

Now, to construct the joint density, we can note that it's can be as the form of the product of $f_{Y|X}(y_i \mid x_i)$, $\forall i = 1,2,\cdots,r$, and for any integration it will be a constants of γ's. So, from (5), (6) and (3), we can say that the joint density of the first r concomitants of GOS's $Y_{[1,n,m,k]}, Y_{[2,n,m,k]}, \cdots, Y_{[r,n,m,k]}$, $1 \le r \le n$ for Morgenstern family is given by:

$$\begin{aligned} g_{[1,2,\cdots,r,n,m,k]}(y_1,\cdots,y_r) &= \int_{-\infty}^{\infty}\int_{-\infty}^{x_k}\cdots\int_{-\infty}^{x_2}\prod_{i=1}^{r} f_{Y|X}(y_i|x_i) \\ &\times f_{(1,2,\cdots,r,n,m,k)}(x_1,\cdots,x_r)\mathrm{d}x_1\cdots\mathrm{d}x_r \\ &= \left(\prod_{i=1}^{r} f_Y(y_i)\right)\Bigg[1+\alpha\sum_{i=1}^{r}C_i(2F_Y(y_i)-1) \\ &+\alpha^2\sum_{i,j=1,i\neq j}^{r}C_{ij}(2F_Y(y_i)-1)(2F_Y(y_j)-1) \\ &+\alpha^3\sum_{i,j,k=1,i\neq j\neq k}^{r}C_{ijk}(2F_Y(y_i)-1) \\ &\times(2F_Y(y_j)-1)(2F_Y(y_k)-1) \\ &+\cdots+\alpha^{r-1} \\ &\times\sum_{i_1,i_2,\cdots,i_r=1,i_1\neq i_2\neq\cdots\neq i_r}^{r}C_{i_1i_2\cdots i_r}(2F_Y(y_{i_1})-1) \\ &\times(2F_Y(y_{i_2})-1)\cdots(2F_Y(y_{i_r})-1) \\ &+\alpha^rC^*\prod_{i=1}^{r}(2F_Y(y_i)-1)\Bigg], \end{aligned} \tag{9}$$

where $C_i = C^*(i,n,m,k)$, $i = 1,2,\cdots,r$, all the constants C's are constant functions of γ's.

Remark 2.1. Equation (2) can't be written as a product of $[1+\alpha A_i(2F_Y(y_i)-1)]$,
$\forall i = 1,2,\cdots,r$, in other ward:

$$g_{[1,2,\cdots,r,n,m,k]}(y_1,\cdots,y_r) \neq \prod_{i=1}^{r} f_Y(y_i)[1+\alpha A_i(2F_Y(y_i)-1)],$$

where all the constants A's are constant functions of γ's.

2.1 Moments and moment generating function of concomitants

Equation (2) can be written as:

$$\begin{aligned} &g_{[1,2,\cdots,r,n,m,k]}(y_1,\cdots,y_r) = \\ &\prod_{i=1}^{r} f_{Y_{1:1}}(y_i)+\alpha\sum_{i=1}^{r}C_i(f_{Y_{2:2}}(y_i)-f_{Y_{1:1}}(y_i))\prod_{t=1,i\neq t}^{r} f_{Y_{1:1}}(y_t) \\ &+\alpha^2\sum_{i,j=1,i\neq j}^{r}C_{ij}(f_{Y_{2:2}}(y_i)-f_{Y_{1:1}}(y_i)) \\ &\times(f_{Y_{2:2}}(y_j)-f_{Y_{1:1}}(y_j))\prod_{t=1,i\neq j\neq t}^{r} f_{Y_{1:1}}(y_t)+\cdots+\alpha^{r-1} \\ &\times\sum_{i_1,i_2,\cdots,i_r=1,i_1\neq i_2\neq\cdots\neq i_r}^{r}C_{i_1i_2\cdots i_r} \\ &\times(f_{Y_{2:2}}(y_{i_1})-f_{Y_{1:1}}(y_{i_1}))\cdots(f_{Y_{2:2}}(y_{i_r})-f_{Y_{1:1}}(y_{i_r})) \\ &+\alpha^rC^*\prod_{i=1}^{r}(f_{Y_{2:2}}(y_i)-f_{Y_{1:1}}(y_i)), \end{aligned} \tag{10}$$

where $f_{Y_{1:1}}(y) = f_Y(y)$ is the pdf of $Y_{1:1}$, the first ordinary order statistic of a random sample of size one of Y variate, and $f_{Y_{2:2}}(y) = 2F_Y(y)f_Y(y)$ is the pdf of $Y_{2:2}$, the second ordinary order statistic of a random sample of size two of Y variate. Using this result, we derive the product moments, denoted by $\mu_{[1,2,\cdots,r,n,m,k]}^{(l_1,l_2,\cdots,l_r)}$, $l_1,l_2,\cdots,l_r > 0$, as follow:

$$\begin{aligned} \mu_{[1,2,\cdots,r,n,m,k]}^{(l_1,l_2,\cdots,l_r)} &= E[Y_{[1,n,m,k]}^{l_1}Y_{[2,n,m,k]}^{l_2}\cdots Y_{[r,n,m,k]}^{l_r}] \\ &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}y_1^{l_1}\cdots y_r^{l_r}g_{[1,2,\cdots,r,n,m,k]}(y_1,\cdots,y_r)\mathrm{d}y_1\cdots\mathrm{d}y_r \\ &= \prod_{i=1}^{r}\mu_{1:1}^{l_i}+\alpha\sum_{i=1}^{r}C_i(\mu_{2:2}^{l_i}-\mu_{1:1}^{l_i})\prod_{t=1,i\neq t}^{r}\mu_{1:1}^{l_t} \\ &+\alpha^2\sum_{i,j=1,i\neq j}^{r}C_{ij}(\mu_{2:2}^{l_i}-\mu_{1:1}^{l_i})(\mu_{2:2}^{l_j}-\mu_{1:1}^{l_j}) \\ &\times\prod_{t=1,i\neq j\neq t}^{r}\mu_{1:1}^{l_t}+\cdots+\alpha^{r-1} \\ &+\sum_{i_1,i_2,\cdots,i_r=1,i_1\neq i_2\neq\cdots\neq i_r}^{r}C_{i_1i_2\cdots i_r}(\mu_{2:2}^{l_{i_1}}-\mu_{1:1}^{l_{i_1}})\cdots \\ &\times(\mu_{2:2}^{l_{i_r}}-\mu_{1:1}^{l_{i_r}})+\alpha^rC^*\prod_{i=1}^{r}(\mu_{2:2}^{l_i}-\mu_{1:1}^{l_i}), \end{aligned} \tag{11}$$

where $\mu_{1:1}^{l_i} = \int_{-\infty}^{\infty}y_i^{l_i}f_{Y_{1:1}}(y_i)\mathrm{d}y_i$, $\mu_{2:2}^{l_i} = \int_{-\infty}^{\infty}y_i^{l_i}f_{Y_{2:2}}(y_i)\mathrm{d}y_i$.

The joint moment generating function of

$Y_{[1,n,m,k]}, Y_{[2,n,m,k]}, \cdots, Y_{[r,n,m,k]}$, $1 \le r \le n$ is given by:

$$M_{[1,2,\cdots,r,n,m,k]}(\theta_1,\cdots,\theta_r) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\{\theta_1 y_1+\cdots+\theta_r y_r\} \times g_{[1,2,\cdots,r,n,m,k]}(y_1,\cdots,y_r)dy_1\cdots dy_r$$
$$=\prod_{i=1}^{r}M_{1:1}(\theta_i)+\alpha\sum_{i=1}^{r}C_i(M_{2:2}(\theta_i)-M_{1:1}(\theta_i)) \times\prod_{t=1,i\neq t}^{r}M_{1:1}(\theta_t)+\alpha^2\sum_{i,j=1,i\neq j}^{r}C_{ij}(M_{2:2}(\theta_i)-M_{1:1}(\theta_i)) \times(M_{2:2}(\theta_j)-M_{1:1}(\theta_j))\prod_{t=1,i\neq j\neq t}^{r}M_{1:1}(\theta_t)+\cdots+\alpha^{r-1} \times\sum_{i_1,i_2,\cdots,i_r=1,i_1\neq i_2\neq\cdots\neq i_r}^{r}C_{i_1i_2\cdots i_r} \times(M_{2:2}(\theta_{i_1})-M_{1:1}(\theta_{i_1}))\cdots(M_{2:2}(\theta_{i_r})-M_{1:1}(\theta_{i_r})) +\alpha^rC^*\prod_{i=1}^{r}(M_{2:2}(\theta_i)-M_{1:1}(\theta_i)). \tag{12}$$

Differentiating (12) with respect to θ_1, θ_2, $\cdots$, θ_r, l_1 times, l_2 times, $\cdots$, l_r times, respectively, and putting $\theta_1=\theta_2=\cdots=\theta_r=0$, we get (11).

Remark 2.2. From (11) and (12) one can deduce product moments and joint moment generating function for the different models of *GOS*'s such as, order statistics (with $m=0$ and $k=1$) and record values (with $m=-1$ and $k=1$). Furthermore, we can obtain results for concomitants of *GOS*'s, order statistics and record values corresponding to different bivariate distributions of the Morgenstern family by specifying the respective marginal distributions from the general results that we obtained.

2.1.1 Weibull Distribution

Here we will give an application of the results that have been obtained for subfamilies of Morgenstern distributions such as Weibull distribution. The pdf and cdf for Weibull distribution are given by, respectively:

$$f(y)=\frac{a}{\lambda^a}y^{a-1}e^{-\frac{y^a}{\lambda^a}}, \tag{13}$$

$$F(y)=1-e^{-\frac{y^a}{\lambda^a}},\ 0\le y<\infty,\ a,\ \lambda>0, \tag{14}$$

then, from (2) and (11), the joint moments of $Y_{[1,n,m,k]}, Y_{[2,n,m,k]}, \cdots, Y_{[r,n,m,k]}$, $1 \le r \le n$ is given by:

$$\mu_{[1,2,\cdots,r,n,m,k]}^{(l_1,l_2,\cdots,l_r)}=\prod_{i=1}^{r}\lambda^{l_i}\Gamma(\frac{l_i}{a}+1) +\alpha\sum_{i=1}^{r}C_i\lambda^{l_i}\Gamma(\frac{l_i}{a}+1)\left[1-2^{-\frac{l_i}{a}}\right] \times\prod_{t=1,i\neq t}^{r}\lambda^{l_t}\Gamma(\frac{l_t}{a}+1) +\cdots+\alpha^{r-1} \times\sum_{i_1,i_2,\cdots,i_r=1,i_1\neq i_2\neq\cdots\neq i_r}^{r}C_{i_1i_2\cdots i_r} \times\lambda^{l_{i_1}}\Gamma(\frac{l_{i_1}}{a}+1)\left[1-2^{-\frac{l_{i_1}}{a}}\right]\times\cdots \times\lambda^{l_{i_r}}\Gamma(\frac{l_{i_r}}{a}+1)\left[1-2^{-\frac{l_{i_r}}{a}}\right] +\alpha^rC^*\prod_{i=1}^{r}\lambda^{l_i}\Gamma(\frac{l_i}{a}+1)\left[1-2^{-\frac{l_i}{a}}\right], \tag{15}$$

3 Classical Estimation and Bayesian Estimation and Prediction for Concomitants of *GOS*'s

In this section, we study and obtain classical estimation such as ML estimation and Bayesian estimation using non-informative prior under SE, LINEX and GE loss functions for the association parameter α.

3.1 Maximum Likelihood Estimation

Suppose that $\mathbf{y}=(y_1,y_2,\cdots,y_r)$ is a concomitants of *GOS*'s sample. From (2), the log-likelihood function is given by:

$$\log L(\alpha|\mathbf{y}) = \sum_{i=1}^{r}\log(f_Y(y_i)+\log\Big[1+\alpha\sum_{i=1}^{r}C_i(2F_Y(y_i)-1) +\alpha^2\sum_{i,j=1,i\neq j}^{r}C_{ij}(2F_Y(y_i)-1)(2F_Y(y_j)-1) +\alpha^3\sum_{i,j,k=1,i\neq j\neq k}^{r}C_{ijk}(2F_Y(y_i)-1)(2F_Y(y_j)-1) \times(2F_Y(y_k)-1) +\cdots+\alpha^{r-1}\sum_{i_1,i_2,\cdots,i_r=1,i_1\neq i_2\neq\cdots\neq i_r}^{r}C_{i_1i_2\cdots i_r}(2F_Y(y_{i_1})-1) \times(2F_Y(y_{i_2})-1)\cdots(2F_Y(y_{i_r})-1) +\alpha^rC^*\prod_{i=1}^{r}(2F_Y(y_i)-1)\Big]. \tag{16}$$

By differentiating (16) for the association parameter α, we can obtain the ML estimation $\hat{\alpha}_{ML}$ of α by solving the following equation:

$$\sum_{i=1}^{r} C_i(2F_Y(y_i)-1)+2\alpha\sum_{i,j=1,i\neq j}^{r} C_{ij}(2F_Y(y_i)-1) \times(2F_Y(y_j)-1)+\cdots +(r-1)\alpha^{r-2}\sum_{i_1,i_2,\cdots,i_r=1,i_1\neq i_2\neq\cdots\neq i_r}^{r} C_{i_1i_2\cdots i_r}(2F_Y(y_{i_1})-1) \times(2F_Y(y_{i_2})-1)\cdots(2F_Y(y_{i_r})-1) +r\alpha^{r-1}C^*\prod_{i=1}^{r}(2F_Y(y_i)-1)=0. \quad (17)$$

3.2 Bayesian estimation

If we have enough information about the parameter we should use informative prior, otherwise it is better to consider non-informative prior. Since the range of α is definite in $\mathbb{R}$, $-1\leq\alpha\leq 1$, then we will use a suitable prior distribution. In this paper we consider the following non-informative prior which is the uniform distribution as follow:

$$\pi(\alpha)=\frac{1}{2}, -1\leq\alpha\leq 1. \quad (18)$$

Combining the likelihood function, (2), and prior function, (18), then the posterior density of α given $\mathbf{y}$ is given by:

$$\pi^*(\alpha|\mathbf{y})=\frac{L(\alpha|\mathbf{y})\pi(\alpha)}{\int_{-1}^{1}L(\alpha|\mathbf{y})\pi(\alpha)d\alpha} =\frac{1}{D}\Bigg[1+\alpha\sum_{i=1}^{r}C_i(2F_Y(y_i)-1) +\alpha^2\sum_{i,j=1,i\neq j}^{r}C_{ij}(2F_Y(y_i)-1)(2F_Y(y_j)-1)+\cdots +\alpha^{r-1}\sum_{i_1,i_2,\cdots,i_r=1,i_1\neq i_2\neq\cdots\neq i_r}^{r}C_{i_1i_2\cdots i_r}(2F_Y(y_{i_1})-1) \times(2F_Y(y_{i_2})-1)\cdots(2F_Y(y_{i_r})-1) +\alpha^rC^*\prod_{i=1}^{r}(2F_Y(y_i)-1)\Bigg], \quad (19)$$

where

$$D=\begin{cases} 2+\frac{2}{3}\sum_{i,j=1,i\neq j}^{r}C_{ij}(2F_Y(y_i)-1)(2F_Y(y_j)-1)+\cdots \\ +\frac{2}{r}\sum_{i_1,i_2,\cdots,i_r=1,i_1\neq i_2\neq\cdots\neq i_r}^{r}C_{i_1i_2\cdots i_r} \\ \times(2F_Y(y_{i_1})-1)(2F_Y(y_{i_2})-1)\cdots(2F_Y(y_{i_r})-1), \\ r\ odd \\ \\ 2+\frac{2}{3}\sum_{i,j=1,i\neq j}^{r}C_{ij}(2F_Y(y_i)-1)(2F_Y(y_j)-1)+\cdots \\ +\frac{2}{r+1}C^*\prod_{i=1}^{r}(2F_Y(y_i)-1), r\ even. \end{cases} \quad (20)$$

3.2.1 Bayes estimator of α based on SE loss function

The the posterior $p-th$ moment of α is given by:

$$\hat{\alpha}_{BS}^p=E(\alpha^p|\mathbf{y})=\int_{-1}^{1}\alpha^p\pi^*(\alpha|\mathbf{y})d\alpha =\frac{1}{D}\Bigg[\frac{\alpha^{p+1}}{p+1}+\frac{\alpha^{p+2}}{p+2}\sum_{i=1}^{r}C_i(2F_Y(y_i)-1)+\cdots +\frac{\alpha^{p+r}}{p+r}\sum_{i_1,i_2,\cdots,i_r=1,i_1\neq i_2\neq\cdots\neq i_r}^{r}C_{i_1i_2\cdots i_r}(2F_Y(y_{i_1})-1) \times(2F_Y(y_{i_2})-1)\cdots(2F_Y(y_{i_r})-1) +\frac{\alpha^{p+r+1}}{p+r+1}C^*\prod_{i=1}^{r}(2F_Y(y_i)-1)\Bigg]_{-1}^{1}.$$

According to the value of r and p we can get a closed form of $\hat{\alpha}_{BS}^p$. At $p=1$, the Bayes estimator $\hat{\alpha}_{BS}$ under SE loss function is given by:

$$\hat{\alpha}_{BS}=E_\alpha(\alpha|\mathbf{y}) =\begin{cases} \frac{1}{D}l\big[\frac{2}{3}\sum_{i=1}^{r}C_i(2F_Y(y_i)-1)+\cdots \\ +\frac{2}{r+2}C^*\prod_{i=1}^{r}(2F_Y(y_i)-1)\big],\ r\ odd \\ \\ \frac{1}{D}\big[\frac{2}{3}\sum_{i=1}^{r}C_i(2F_Y(y_i)-1)+\cdots \\ +\frac{2}{r+1}\sum_{i_1,i_2,\cdots,i_r=1,i_1\neq i_2\neq\cdots\neq i_r}^{r}C_{i_1i_2\cdots i_r}. \\ \times.(2F_Y(y_{i_1})-1)(2F_Y(y_{i_2})-1)\cdots(2F_Y(y_{i_r})-1)\big], \\ r\ even. \end{cases} \quad (21)$$

3.2.2 Bayes estimator of α based on LINEX loss function

The Bayes estimator $\hat{\alpha}_{BL}$ under LINEX loss function is given by:

$$\hat{\alpha}_{BL}=\frac{-1}{\lambda}\log\left[\int_{-1}^{1}e^{-\lambda\alpha}\pi^*(\alpha|\mathbf{y})d\alpha\right] =\frac{-1}{\lambda}\log\Bigg[\int_{-1}^{1}\frac{1}{D}\Bigg(e^{-\lambda\alpha}+\alpha e^{-\lambda\alpha}\sum_{i=1}^{r}C_i(2F_Y(y_i)-1) +\cdots +\alpha^re^{-\lambda\alpha}C^*\prod_{i=1}^{r}(2F_Y(y_i)-1)\Bigg)d\alpha\Bigg] =\frac{-1}{\lambda}\log\Bigg[\frac{1}{D}\Bigg(\beta_0(\lambda)+\beta_1(\lambda)\sum_{i=1}^{r}C_i(2F_Y(y_i)-1)+\cdots +\beta_r(\lambda)C^*\prod_{i=1}^{r}(2F_Y(y_i)-1)\Bigg)\Bigg], \quad (22)$$

where

$$\beta_n(\lambda)=\int_{-1}^{1}\alpha^ne^{-\lambda\alpha}d\alpha =\lambda^{n-1}[\Gamma(n+1,-\lambda)-\Gamma(n+1,\lambda)], n=0,1,\cdots,$$

$\Gamma(a,b)$ is the incomplete gamma function, $\lambda \neq 0$.

3.2.3 Bayes estimator of α based on GE loss function

The Bayes estimator $\hat{\alpha}_{BG}$ under GE loss function is given by:

$$\begin{aligned}\hat{\alpha}_{BG} &= \left[E_{\alpha}(\alpha^{-\nu}|\mathbf{y})\right]^{\frac{-1}{\nu}} \\ &= \left[\int_{-1}^{1} \alpha^{-\nu}\pi^{*}(\alpha|\mathbf{y})\mathrm{d}\alpha\right]^{\frac{-1}{\nu}} \\ &= \left[\frac{1}{D}\left(\frac{\alpha^{-\nu+1}}{-\nu+1} + \frac{\alpha^{-\nu+2}}{-\nu+2}\sum_{i=1}^{r} C_i(2F_Y(y_i)-1) + \cdots \right.\right. \\ &\quad \left.\left. + \frac{\alpha^{-\nu+r+1}}{-\nu+r+1}C^{*}\prod_{i=1}^{r}(2F_Y(y_i)-1)\right)\Bigg|_{-1}^{1}\right]^{\frac{-1}{\nu}}, \quad (23)\end{aligned}$$

$\nu < 0$, and we will get the closed form according to the value of ν and r.

Remark 3.1. It can be shown that, when $\nu = 1$, the Bayesian estimate in (23) coincides with the Bayesian estimate under the weighted SE loss function. Similarly, when $\nu = -1$, the Bayesian estimate in (23) coincides with the Bayesian estimate under the SE loss function.

3.3 Bayesian prediction

For one sample prediction, the problem is to predict the future observation in a sample based on early failures from the sample. Consider the first r concomitants of GOS's $Y_{[1,n,m,k]}, Y_{[2,n,m,k]}, \cdots, Y_{[r,n,m,k]}$ from a random sample of size n , $1 \leq r \leq n$. For the remaining $(n-r)$ components, let $W_{[s,n,m,k]} = Y_{[r+s,n,m,k]}$, $s = 1,\ldots,n-r$, denotes the concomitants of the future lifetime of the $s-th$ component to fail, $1 \leq s \leq n-r$. If we want to find the prediction bounds for $W_{[s,n,m,k]}$, from (7) and (8), the conditional density function of the $s-th$ future concomitants of GOS's given that the first r concomitants of GOS's is given by:

$$\begin{aligned}g(w_s|\alpha,y_r) &= \frac{g_{[r,s,n,m,k]}(y_r,w_s)}{g_{[r,n,m,k]}(y_r)} \\ &= f_Y(w_s)\left[1 + \frac{1}{1+\alpha C_r(2F_Y(y_r)-1)}\left(\alpha C_s(2F_Y(w_s)-1)\right.\right. \\ &\quad \left.\left. + \alpha^2 C_{rs}(2F_Y(y_r)-1)(2F_Y(w_s)-1)\right)\right]. \quad (24)\end{aligned}$$

The prediction will be the rest of the given observation $Y_{[r+1,n,m,k]}, Y_{[r+2,n,m,k]}, \ldots, Y_{[n,n,m,k]}$ in case of concomitants of GOS's. The predictive density function of $W_{[s,n,m,k]}$ given the first r concomitants of GOS's can be obtained by multiplying the conditional density function (24) by the posterior density function (19), so it can be written as:

$$g(w_s|y_r) = \int_{-1}^{1} g(w_s|\alpha,y_r)\pi^{*}(\alpha|\mathbf{y})\mathrm{d}\alpha. \quad (25)$$

Hence, the predictive survival function for the $s-th$ future concomitants of GOS's is given by:

$$P[W_{[s,n,m,k]} > \xi|y_r] = \int_{\xi}^{\infty} g(w_s|y_r)\mathrm{d}w_s. \quad (26)$$

So the lower and upper $100\tau\%$ prediction bounds $[L(x), U(x)]$ for Y_s is obtained by equating (26) to $(1+\tau)/2$ and $(1-\tau)/2$, respectively. For the long computations of the integrations in (25) and (26) we prefer to leave it and solve it numerically.

4 Numerical Results

In this section, for type-II censored order statistics samples (with $m = 0$ and $k = 1$), we select a Subfamily of Morgenstern family to apply it in our results, we obtain some numerical results for the exponential distribution (from Equation (14) at $\lambda = 1$, $a = 1$ we obtain exponential distribution) according to the following steps:

1. Using the inverse transformation method, $Y = -\log(1-U)$ where U is the uniformly distributed random variate, we generate a bivariate sample (X_i, Y_i), $i = 1,2,\ldots,n$, from Morgensterns standard bivariate exponential distribution (i.e., the marginal is the standard exponential distribution).
2. For Morgenstern family, Pearson's product-moment correlation coefficient is $\rho = \frac{\alpha}{3}$, which clearly ranges from $\frac{-1}{3}$ to $\frac{1}{3}$. For exponential distribution we have $\rho = \frac{\alpha}{4}$, see Schucany et al. [19].

5 Conclusion and Comments

We consider a form of joint density of the first r concomitants of GOS's based on Morgenstern family, but for the constants it can be solved numerically. From the relation between this joint density and the order statistics we can get the moments and the moment generating function of such model. For estimation of association/correlation parameter α we use the ML estimation and Bayesian estimation with non-informative prior under SE, LINEX and GE loss functions, and based on exponential distribution subfamily we note the following of the simulation study, see Table (1):

1. We choose the generated data that not contains outlier, as it effects on the results and make the estimation of α exceeds its range specially for ML estimation.

Table 1: Different types of estimation of α. The lower, the upper and the width of the 95% prediction intervals at $r = 7$.

r	$\hat{\alpha}_{ML}$	$\hat{\alpha}_{BS}$	$\hat{\alpha}_{BL}$						$\hat{\alpha}_{BG}$		
			$\lambda=-1$	$\lambda=-2$	$\lambda=-3$	$\lambda=1$	$\lambda=2$	$\lambda=3$	$\nu=-1$	$\nu=-2$	$\nu=-4$
1	–	-0.0788	0.0845	0.2296	0.3442	-0.2328	-0.3574	-0.45106	-0.0788	0.57735	0.6687
2	-0.61306	-0.01865	0.1418	0.2792	0.3856	-0.1769	-0.3095	-0.4109	-0.01865	0.5738	0.6130
3	-2.0589	-0.10456	0.0565	0.2028	0.3201	-0.2537	-0.3734	-0.46337	-0.10456	0.5746	0.6665
4	-0.5098	-0.04093	0.1172	0.2553	0.3638	-0.1945	-0.3223	-0.4202	-0.04093	0.5682	0.6611
5	-0.3949	-0.0304	0.1276	0.2646	0.3718	-0.1852	-0.3146	-0.4141	-0.0304	0.5686	0.6615
6	-0.2295	-0.0209	0.1362	0.2717	0.3775	-0.1758	-0.30617	-0.4066	-0.0209	0.5673	0.6604
7	-1.3682	-0.0985	0.06009	0.2038	0.3196	-0.2463	-0.3658	-0.4561	-0.0985	0.5699	0.6625
8	-0.8066	-0.07312	0.0845	0.2253	0.3375	-0.2228	-0.3455	-0.4391	-0.07312	0.5671	0.6602
9	-0.400493	-0.0434	0.1131	0.2502	0.3585	-0.1952	-0.3218	-0.4191	-0.0434	0.56508	0.6585
10	0.7052	0.1443	0.2793	0.3881	0.4714	-0.0043	-0.1450	-0.26305	0.1443	0.5578	0.6523
Bayesian prediction											
	LY_8	UY_8	Width	LY_9	UY_9	Width	LY_{10}	UY_{10}	Width	–	–
7	0.026374	3.72774	3.7013	0.0264935	3.73186	3.7053	0.026614	3.73596	3.7093	–	–

2. All the estimation of α does not exceed the range $[-1,1]$, except for some values of ML estimation, and $\hat{\alpha}_{BG} \in]0,1]$.
3. At $r = 1$ we can't obtain the ML estimation.
4. For $\hat{\alpha}_{BG}$, we note that $\nu \in 2\mathbb{Z}^- \cup \{-1\}$.
5. At $\nu = -1$, we find that $\hat{\alpha}_{BG} = \hat{\alpha}_{BS}$.
6. For fixed r, $\hat{\alpha}_{BL}$ decreases as λ increases, and $\hat{\alpha}_{BG}$ increases and tends to 1 as ν decreases.
7. We can estimate ρ by $\hat{\rho} = \frac{\hat{\alpha}}{4}$ for the suggested distribution.
8. The Bayesian prediction provides lower and upper limits of 95% prediction intervals, it is observed that the prediction intervals tend be wider when s increase. This is a natural, since the prediction of the future order statistic that is far a way from the last observed value has less accuracy than that of other future order statistics.

Conflict of Interest

The authors declare that they have no conflict of interest.

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References

[1] Y. Abdel-Aty, J. Franz and M.A.W. Mahmoud, Bayesian prediction based on generalized order statistics using multiply type II censored, Statistics **41**, 495-504 (2007).
[2] E.K. Al-Hussaini and A.A. Ahmed, On Bayesian predictive distributions of generalized order statistics, Metrika **57**, 165-176 (2003).
[3] E.K. AL-Hussaini and Z.F. Jaheen, Bayesian prediction bounds for the Burr type XII model, Communications in Statistics-Theory and Methods **24**, 1829-1842 (1995).
[4] M.I. Beg and M. Ahsanullah, Concomitants of generalized order statistics from Farlie-Gumbel-Morgensterm distributions, Statistical Methodology **5**, 1-20 (2008).
[5] R. Calabria and G. Pulcini, Point estimation under asymmetric loss functions for left truncated exponential samples, Communications in Statistics-Theory and Methods **25**, 285-600 (1996).
[6] H.A. David, M.J. O'Connell and S.S. Yang, Distribution and expected value of the rank of a concomitant of an order statistic, Annals of Statistics **5**, 216-223 (1977).
[7] D.K. Dey, M. Ghosh and C. Srinivasan, Simultaneous estimation of parameters under entropy loss, Journal of Statistical Planning and Inference **25**, 347-363 (1987).
[8] D.K. Dey and P.L. Liu, On comparison of estimators in a generalized life model, Microelectronics Reliability **33**, 207-221 (1992).
[9] Dinesh Kumar, Umesh Singh, Sanjay Kumar Singh and Souradip Mukherjee, The New Probability Distribution: An Aspect to a LifeTime Distribution, Mathematical Science Letters **6**, 35-42 (2017).
[10] H. Fathy Riad and E. H. Hafez, Point and Interval Estimation for Frechet Distribution Based on Progressive First Failure Censored Data, Mathematical Science Letters **9**, 181-191 (2020).
[11] Z.F. Jaheen, Empirical Bayes inference for generalized exponential distribution based on records, Communications in Statistics-Theory and Methods **33**, 851-1861 (2004a).
[12] Z.F. Jaheen, Empirical Bayes analysis of record statistics based on linex and quadratic loss functions, Computers and Mathematics with Applications **47**, 947-954 (2004b).
[13] N.L. Johnson and S. Kotz, On some generalized Farlie-Gumbel-Morgenstern distributions, Communications in Statistics-Theory and Methods **4**, 415-427 (1975).
[14] U. Kamps, A concept of generalized order statistics, Journal of Statistical Planning and Inference **48**, 1-23 (1995).
[15] M. M. Mohie El-Din, M. M. Amein, H. E. El-Attar and E. H. Hafez, Symmetric and Asymmetric Bayesian Estimation For Lindley Distribution Based on Progressive First Failure

Censored Data, Mathematical Science Letters **6**, 255-261 (2017).
[16] M. M. Mohie El-Din, Nahed S. A. Ali, M. M. Amein and M. S. Mohamed, Statistical Inference of Concomitants Based on Morgenstern Family under Generalized Order Statistics, Mathematical Science Letters **5**, 243-254 (2016).
[17] M.Z. Raqab and M.T. Madi, Bayesian inference for the generalized exponential distribution, Journal of Statistical Computation and Simulation **69**, 109-124 (2005).
[18] Rashad M. EL-Sagheer, Assessing the Lifetime Performance Index of Extreme Value Model Based on Progressive Type-II Censored Samples, Mathematical Science Letters **6**, 279-286 (2017).
[19] W.R. Schucany, W.C. Parr and J.E. Boyer, Correlation structure in Farlie-Gumbel-Morgenstern distributions, Biometrika **65**, 650-653 (1978).
[20] U. Singh, P.K. Gupta and S. Upadhyay, Estimation of exponentiated Weibull shape parameter under linex loss function, Communications in Statistics-Simulation and Computation **31**, 523-537 (2002).
[21] A.A. Soliman, Estimation of parameters of life from progressively censored data using Burr-XII Model, IEEE Transactions on Reliability **54**, 34-42 (2005).
[22] A.A. Soliman, Estimators for the finite mixture of Rayleigh model based on progressively censored data, Communications in Statistics-Theory and Methods **35**, 803-820 (2006).
[23] H.R. Varian, A Bayesian approach to real state assessment In: Stephen, E. F., Zellner, A. (Eds.), Studies in Bayesian Econometrics and Statistics in Honor of Leonard J. Savage. North-Holland, Amsterdam, 195-208 (1975).