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Unsteady Drainage of Electrically Conducting Power Law Fluid

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Abstract: In this paper, the thin film flow of a power law MHD fluid on a vertical cylinder for a drainage problem has been studied. The nonlinear differential equation has been derived from the momentum equation by Jeffrey's approach. Series solutions have been obtained for velocity, flow rate and thickness of the fluid film by Perturbation method. The graphical results for velocity profile and thickness of the film are discussed and examined for different parameters of interest. Without MHD our problem reduces to well known Newtonian and power law problem.

Keywords: Thin film flow; power law MHD fluid; Jeffrey's approach ; Analytical solution.

1 Introduction

Especially in materials processing, chemical industry and biotechnology: In recent years, the flow of non-Newtonian fluid has gained attention due to its application in various fields of science, engineering and technology. It is a fact that the characteristics of non-Newtonian fluid to flow quite different from the linear viscous fluid. Thus, the well-known Navier-Stokes equations are not suitable for describing the behavior of non-Newtonian fluids. Just as linear viscous liquid, it is difficult to find a single model that has all the properties recommend the non-Newtonian fluids. Therefore, many models have been suggested to characterize the behavior of non-Newtonian fluid [1,4].

In the category of non-Newtonian fluids the power law model have been extensively studied because of mathematical simplicity and widespread industrial applications. During the last four decades significant progress has been made in the development of analytical solution and numerical algorithms of power law fluid flow problems [5,8].

In our previous work we discussed the theoretical study of steady flow for lift and drainage of Power law MHD fluid on a vertical cylinder. The derived governing nonlinear differential equation has then solved using Perturbation method [9]. Our main focus in this work is the study of thin film flow for a non-Newtonian fluid with MHD fluid properties. In a thin film flow, the fluid is partially bounded by a solid wall while the other surface is free to interact with another fluid, e.g., air. There are three main conditions which form basis for the formulation of thin films, namely, surface tension, centrifugal forces and gravitational forces. The analysis of thin film flow is important for designing chemical processing equipment. Probably the most striking daily life examples are rain water running down along a window and the flow of a paint down a wall. Study of thin film flows have established significant interest because of its realistic applications in physical and biological sciences [10,11]. There are many engineering applications where thin film flow shows the viscoelastic effects and MHD was originally applied to astrophysical and geophysical problems, where it is still very important, but more recently to the problem of fusion power, where the application is the creation and containment of hot plasmas by electromagnetic forces, since material walls would be destroyed. Astrophysical problems include solar structure, especially in the outer layers, the solar wind bathing the earth and other planets, and interstellar

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magnetic fields. The primary geophysical problem is planetary magnetism, produced by currents deep in the planet, a problem that has not been solved to any degree of satisfaction.
In this paper, we investigate the thin film flow down a vertical cylinder of a power law MHD fluid using Jeffrey's approach [12,16] for drainage problem, two cases are discussed, namely, Newtonian and non-Newtonian fluid respectively. In Newtonian case we find the exact solution while in power law series solution is obtained. To the best of our knowledge the analytical solution has not been reported elsewhere.

This letter is organized as follows. Section 2 contains the governing equation of power law fluid model. In section 3 the problem under consideration is formulated, section 4 is reserved for the solution of the problem and section 5 results and discussion. In Section 6 concluding remarks are given.

2 Basic Equations

The basic equations, governing the flow of incompressible power law MHD fluid neglecting the thermal effects, are:

$$\nabla . \mathbf{V} = 0. \tag{1}$$

$$\rho \frac{D\mathbf{V}}{Dt} = \rho \mathbf{f} - \nabla p + div\mathbf{S} + (\mathbf{J} \times \mathbf{B}), \tag{2}$$

where $\mathbf{f}$ is the body force, p is the dynamic pressure, $\mathbf{S}$ is the extra stress tensor. The term $\frac{D\mathbf{V}}{Dt}$ denotes the substantial acceleration consisting of the local derivative $\frac{\partial \mathbf{V}}{\partial t}$ and the convective derivative $\nabla.\mathbf{V}$ and $\mathbf{J}$ is the electric current density, $\mathbf{B}$ is the total magnetic field and $\mathbf{B} = \mathbf{B_0} + \mathbf{b}$ (where $\mathbf{B_0}$ represents the imposed magnetic field and $\mathbf{b}$ denotes the induced magnetic field). In the absence of displacement currents, the modified Ohm's law and Maxwell's equations [17,20] are,

$$\mathbf{J} = \sigma[\mathbf{E} + \mathbf{V} \times \mathbf{B}]. \tag{3}$$

$$div\mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \mu_m \mathbf{J}, \quad curl\mathbf{E} = -\frac{\partial B}{\partial t}. \tag{4}$$

Where σ is the electrical conductivity, the electric field is $\mathbf{E}$ and magnetic permeability is μ_m. By Ohm's law, and Maxwell's equations for the development of the magnetic flux $\mathbf{B}$ can be easily obtained. This is known as the magnetic induction equation to induce the movement of the electrically conductive fluid applied magnetic fields, a magnetic field is in a medium. We assume that the total magnetic field $\mathbf{B}$ is perpendicular to the velocity field $\mathbf{V}$ and the induced magnetic field $\mathbf{b}$ is negligible as compared to the applied magnetic field $\mathbf{B_0}$. In a small magnetic Reynolds number Since there is no external electrical field is used, and the polarization is negligible ionized fluid is the fluid flow to be independent of the electric field. Under this condition, the magneto-hydrodynamic force in Equation (2) can be put into the form,

$$\mathbf{J} \times \mathbf{B} = -\sigma \mathbf{B_0^2 V}. \tag{5}$$

As discussed in [5,8], the stress tensor defining a power law fluid is given by:

$$\mathbf{S} = \mu_{eff} \mathbf{A_1}, \tag{6}$$

$$\mu_{eff} = \eta \left| \sqrt{\frac{tr(\mathbf{A_1}^2)}{2}} \right|^{n-1}, \tag{7}$$

and where η is the coefficient of viscosity and n is the Power law index. The Rivilin-Ericksen tensor, $\mathbf{A_1}$ is defined by:

$$\mathbf{A}_1 = \nabla \mathbf{V} + (\nabla \mathbf{V})^T. \tag{8}$$

Remark: On behalf of consequent model for $n < 1$ the fluid is "pseudoplastic" for model or "shear thinning" for $n > 1$ the fluid is "dilatant" or "shear-thickening" and for $n = 1$ the Newtonian fluid is recovered.

3 Problem Formulation

Consider unsteady, laminar and parallel flow of an incompressible Power law MHD fluid slowly down an infinite vertical cylinder. As a result, a thin fluid film of thickness h which varies with time adheres to the cylinder and drains down under the action of gravity. The geometry of the problem down figure 6.1 shows that $rz-$ coordinate system has been chosen such that $r-axis$ is normal to the cylinder and $z-axis$ along the cylinder in downward direction. For simplicity, we assume that the fluid is non-conducting and the magnetic field is applied along the $r-axis$, there is no applied (force) pressure driving the flow and body force is only due to gravity. Here we shall pursue a velocity field and a stress field of the form:

$$\mathbf{V} = [0, 0, w(r,t)], \qquad \mathbf{S} = S(r,t). \tag{9}$$

Using equation (9), the continuity equation (1) is identically satisfied and by using equation (5) the momentum equation (2) reduces to,

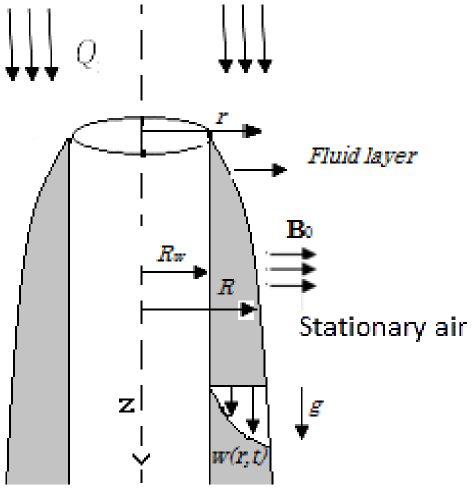

Fig.1. Geometry of the thin film flow downward to a vertical cylinder

r-component

$$0=-\frac{\partial p}{\partial r}, \tag{10}$$

θ-component

$$0=-\frac{\partial p}{\partial \theta}, \tag{11}$$

z-component

$$\rho\frac{\partial w}{\partial t}+\frac{\partial p}{\partial z}=\frac{\eta}{r}\frac{\partial}{\partial r}\left(r\left|\frac{\partial w}{\partial r}\right|^{n-1}\frac{\partial w}{\partial r}\right)+\rho g-\sigma B_0^2 w(r). \tag{12}$$

Equations (10) and (11) implies that $p=p(z)$ only. Assume that pressure p is atmospheric pressure i.e., p is zero (gauge pressure) everywhere. As we are discussing the drainage flow problem therefore we take $\frac{\partial w}{\partial r}$ positive. Thus equation (12) reduces to

$$\rho\frac{\partial w}{\partial t}=\frac{\eta}{r}\frac{\partial}{\partial r}\left(r\left(\frac{\partial w}{\partial r}\right)^{n}\right)+\rho g-\sigma B_0^2 w(r), \tag{13}$$

Neglecting acceleration term $\frac{\partial w}{\partial t}$ which is small compared gravity except in the initial emptying of the vessel, we get,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\left(\frac{\partial w}{\partial r}\right)^{n}\right)-\frac{\sigma B_0^2}{\eta}w(r)=-\frac{\rho g}{\eta}, \tag{14}$$

which is a nonlinear differential equation. The associated boundary conditions are:

$$\frac{\partial w}{dr}=0 \quad \text{at} \quad r=R, \tag{15}$$

$$w=0 \quad at \quad r=R_w. \tag{16}$$

Perturbation solution

We assume $\varepsilon=\frac{\sigma B_0^2}{\eta}$ to be a small parameter and velocity profile $w(r,\varepsilon)$ can be expressed as a power series given by,

$$w(r,\varepsilon)\approx w_0+\varepsilon w_1+\varepsilon^2 w_2+\ldots \tag{17}$$

By using equation (17) into equation (14) and (15) - (16) and equating coefficients of like powers of ε , we obtain the following set of problems along with their associated boundary conditions: **zeroth order problem**

$$\varepsilon^0:\frac{1}{r}\frac{d}{dr}\left(r\left(\frac{dw_0}{dr}\right)^{n}\right)=-\frac{\rho g}{\eta} \tag{18}$$

with boundary condition,

$$\frac{dw_0}{dr}=0 \quad \text{at} \quad r=R. \tag{19}$$

$$w_0=0 \quad \text{at} \quad r=R_w, \tag{20}$$

First order problem

$$\varepsilon^1:\frac{1}{r}\frac{d}{dr}\left(rn\left(\frac{dw_0}{dr}\right)^{n-1}\frac{dw_1}{dr}\right)-w_0=0 \tag{21}$$

with boundary conditions,

$$\frac{dw_1}{dr}=0 \quad \text{at} \quad r=R. \tag{22}$$

$$w_1=0 \quad \text{at} \quad r=R_w, \tag{23}$$

Here two cases arise:
Case-I: $n=1$ (Newtonian fluid)
Case-II: $n\neq 1$ (power law fluid)

3.1 Solution for Newtonian fluid

3.1.1 Velocity Profile

Zeroth order solution:
The solution of equation (18) by using boundary condition (19) and (20) is,

$$w_0=\frac{\rho g}{4\eta}\left[\left(R_w^2-r^2\right)+2R^2\ln\left(\frac{r}{R_w}\right)\right]. \tag{24}$$

First-order solution:
Substituting the zeroth order solution (24), into (21) and subject to the conditions (22) and (23) is given by

$$\begin{aligned} w_1 = &\frac{\rho g}{64\eta}\left[4\left(R_w^2-2R^2\right)r^2-r^4+8R^2R_w^2-3R_w^4\right.\\ &+4R^2\left(3R^2-2R_w^2-4R^2\ln\left(\frac{R}{R_w}\right)\right)\ln\left(\frac{r}{R_w}\right)\\ &\left.+8R^2r^2\ln\left(\frac{r}{R_w}\right)\right]. \end{aligned} \tag{25}$$

Thus the solution with perturbation method correct upto first order is,

$$\begin{aligned} w(r) = &\frac{\rho g}{4\eta}\left[\left(R_w^2-r^2\right)+2R^2\ln\left(\frac{r}{R_w}\right)\right]+\frac{\rho g\varepsilon}{64\eta}\\ &\left[4\left(R_w^2-2R^2\right)r^2-r^4+8R^2R_w^2-3R_w^4\right.\\ &+4R^2\left(3R^2-2R_w^2-4R^2\ln\left(\frac{R}{R_w}\right)\right) \end{aligned}$$

$$\ln\left(\frac{r}{R_w}\right)+8R^2r^2\ln\left(\frac{r}{R_w}\right)\Bigg]. \tag{26}$$

It is pointed out that if we set the perturbation parameter $\varepsilon=0$ in (26), we recover the solution for the same problem with a Newtonian fluid without MHD given in [21].

3.1.2 Volume Flow Rate

In dimensional form, the flow rate Q, is given by,

$$Q=\int_0^{2\pi}\int_{R_w}^{R} rw(r)drd\theta=2\pi\int_{R_w}^{R} rw(r)dr. \tag{27}$$

By making use of equation (26) in (27), we obtain,

$$\begin{aligned}Q=&-\frac{\rho g\pi}{8\eta}\Bigg[\left(R^2-R_w^2\right)^2-4R^4\ln\left(\frac{R}{R_w}\right)+2R^2\\&\left(R^2-R_w^2\right)-\frac{\varepsilon}{24}\left\{6\left(R_w^2-2R^2\right)\left(R^4-R_w^4\right)\right.\\&-\left(R^6-R_w^6\right)+3\left(8R_w^2R^2-3R_w^4\right)\left(R^2-R_w^2\right)\\&+6R^2\left(3R^2-2R_w^2-4R^2\ln\left(\frac{R}{R_w}\right)\right)\\&\left(2R^2\ln\left(\frac{R}{R_w}\right)-\left(R^2-R_w^2\right)\right)\\&\left.+3R^2\left(4R^4 ln\left(\frac{R}{R_w}\right)-\left(R^4-R_w^4\right)\right)\right\}\Bigg].\end{aligned} \tag{28}$$

3.1.3 Thickness of the fluid film

The volume flow rate in term of continuity equation is given by,

$$-\frac{\partial Q}{\partial z}=2\pi R\frac{\partial R}{\partial t}. \tag{29}$$

Substituting equation (28) in equation (29), after considerable simplification, we get,

$$\begin{aligned}-\frac{\partial z}{\partial t}=&-\frac{\rho g}{2\eta}\Bigg[\left(R^2-R_w^2\right)-2R^2\ln\left(\frac{R}{R_w}\right)-\frac{\varepsilon}{192}\\&\left\{192R^2R_w^2-132R^4-60R_w^4+24\ln\left(\frac{R}{R_w}\right)\right.\\&\left.\left(14R^4-8R_w^2R^2-12R^4\ln\left(\frac{R}{R_w}\right)\right)\right\}\Bigg].\end{aligned} \tag{30}$$

Now integrating equation (30) with respect to t and using the boundary condition $R(0,t)=R_w$, we get the relation between film thickness z and t as:

$$\begin{aligned}z=&-\frac{\rho gt}{2\eta}\Bigg[\left(R^2-R_w^2\right)-2R^2\ln\left(\frac{R}{R_w}\right)-\frac{\varepsilon}{192}\\&\left\{192R^2R_w^2-132R^4-60R_w^4+24\ln\left(\frac{R}{R_w}\right)\right.\\&\left.\left(14R^4-8R_w^2R^2-12R^4\ln\left(\frac{R}{R_w}\right)\right)\right\}\Bigg].\end{aligned} \tag{31}$$

For convex surface, we get,

$$\begin{aligned}z=&-\frac{\rho gtR_w^2}{2\eta}\Bigg[\left((1+\frac{h}{R_w})^2-1\right)-2\left(1+\frac{h}{R_w}\right)^2\ln(1\\&+\left.\frac{h}{R_w}\right)-\frac{\varepsilon R_w^2}{192}\left\{192\left(1+\frac{h}{R_w}\right)^2-132\left(1+\frac{h}{R_w}\right)^4\right.\\&-60+24\ln\left(1+\frac{h}{R_w}\right)\left(14\left(1+\frac{h}{R_w}\right)^4-8(1\right.\\&+\left.\left.\frac{h}{R_w}\right)^2-12\left(1+\frac{h}{R_w}\right)^4\ln\left(1+\frac{h}{R_w}\right)\right)\Bigg\}\Bigg].\end{aligned} \tag{32}$$

and for concave surface we arrive at,

$$\begin{aligned}z=&-\frac{\rho gtR_w^2}{2\eta}\Bigg[\left((1-\frac{h}{R_w})^2-1\right)-2\left(1-\frac{h}{R_w}\right)^2\ln(1\\&-\left.\frac{h}{R_w}\right)-\frac{\varepsilon R_w^2}{192}\left\{192\left(1-\frac{h}{R_w}\right)^2-132\left(1-\frac{h}{R_w}\right)^4\right.\\&-60+24\ln\left(1-\frac{h}{R_w}\right)\left(14\left(1-\frac{h}{R_w}\right)^4-8(1\right.\\&-\left.\left.\frac{h}{R_w}\right)^2-12\left(1-\frac{h}{R_w}\right)^4\ln\left(1-\frac{h}{R_w}\right)\right)\Bigg\}\Bigg].\end{aligned} \tag{33}$$

3.2 Solution for power law fluid fluid

3.2.1 Velocity Profile

Zeroth order solution:
By using binomial series and applying boundary condition (19) and (20), solution of equation (18) will be,

$$w_0=\left(\frac{\rho g}{2\eta}\right)^{\frac{1}{n}}\left(\sum_{i=0}^{\infty}\binom{\frac{1}{n}}{i}\frac{(-1)^iR^{-2i+\frac{2}{n}}}{2i-\frac{1}{n}+1}\left(r^{2i-\frac{1}{n}+1}-R^{2i-\frac{1}{n}+1}\right)\right). \tag{34}$$

First-order solution:
Making use of the zeroth-order solution (34) into (21), we acquire,

$$\begin{aligned}w_1=&\frac{1}{n}\left(\frac{\rho g}{2\eta}\right)^{\frac{2}{n}-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\frac{1}{n}}{i}\binom{\frac{1-n}{n}}{j}\frac{(-1)^{i+j}R^{-2i-2j+\frac{4}{n}-2}}{2i-\frac{1}{n}+1}\\&\left[\frac{1}{2i-\frac{1}{n}+3}\left\{\frac{\left(r^{2i+2j-\frac{2}{n}+4}-R_w^{2i+2j-\frac{2}{n}+4}\right)}{2i+2j-\frac{2}{n}+4}\right.\right.\\&-\left.\frac{R^{2i-\frac{1}{n}+3}}{2j-\frac{1}{n}+1}\left(r^{2j-\frac{1}{n}+1}-R_w^{2j-\frac{1}{n}+1}\right)\right\}\\&-\frac{R_w^{2i-\frac{1}{n}+1}}{2}\left\{\frac{\left(r^{2j-\frac{1}{n}+3}-R_w^{2j-\frac{1}{n}+3}\right)}{2j-\frac{1}{n}+3}\right.\end{aligned}$$

$$- \frac{R^2}{2j-\frac{1}{n}+1}\left(r^{2j-\frac{1}{n}+1}-R_w^{2j-\frac{1}{n}+1}\right)\Bigg\}\Bigg]. \quad (35)$$

Inserting equations (34) and (35) into series (17), one get the solution of equation (14) of the form:

$$w=\left(\frac{\rho g}{2\eta}\right)^{\frac{1}{n}}\left(\sum_{i=0}^{\infty}\binom{\frac{1}{n}}{i}\frac{(-1)^i R^{-2i+\frac{2}{n}}}{2i-\frac{1}{n}+1}\left(r^{2i-\frac{1}{n}+1}-R^{2i-\frac{1}{n}+1}\right)\right)$$
$$+\frac{\varepsilon}{n}\left(\frac{\rho g}{2\eta}\right)^{\frac{2}{n}-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\frac{1}{n}}{i}\binom{\frac{1-n}{n}}{j}\frac{(-1)^{i+j}R^{-2i-2j+\frac{4}{n}-2}}{2i-\frac{1}{n}+1}$$
$$\left[\frac{1}{2i-\frac{1}{n}+3}\left\{\frac{\left(r^{2i+2j-\frac{2}{n}+4}-R_w^{2i+2j-\frac{2}{n}+4}\right)}{2i+2j-\frac{2}{n}+4}\right.\right.$$
$$\left.-\frac{R^{2i-\frac{1}{n}+3}}{2j-\frac{1}{n}+1}\left(r^{2j-\frac{1}{n}+1}-R_w^{2j-\frac{1}{n}+1}\right)\right\}$$
$$-\frac{R_w^{2i-\frac{1}{n}+1}}{2}\left\{\frac{\left(r^{2j-\frac{1}{n}+3}-R_w^{2j-\frac{1}{n}+3}\right)}{2j-\frac{1}{n}+3}\right.$$
$$\left.\left.-\frac{R^2}{2j-\frac{1}{n}+1}\left(r^{2j-\frac{1}{n}+1}-R_w^{2j-\frac{1}{n}+1}\right)\right\}\right]. \quad (36)$$

Here if we set the perturbation parameter in (36), we recover the solution of same problem having power law fluid without MHD effects [22].

3.2.2 Volume Flow Rate

By making use of equation (36) in equation (27), we obtain,

$$Q=2\pi\left(\frac{\rho g}{2\eta}\right)^{\frac{1}{n}}\left[\sum_{i=0}^{\infty}\binom{\frac{1}{n}}{i}\frac{(-1)^i R^{-2i+\frac{2}{n}}}{2i-\frac{1}{n}+1}\right.$$
$$\left\{\frac{\left(R^{2i-\frac{1}{n}+3}-R_w^{2i-\frac{1}{n}+3}\right)}{2i-\frac{1}{n}+3}-\frac{R_w^{2i-\frac{1}{n}+1}(R^2-R_w^2)}{2}\right\}$$
$$+\frac{\varepsilon}{n}\left(\frac{\rho g}{2\eta}\right)^{\frac{1}{n}-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\frac{1}{n}}{i}\binom{\frac{1-n}{n}}{j}\frac{(-1)^{i+j}R^{-2i-2j+\frac{4}{n}-2}}{2i-\frac{1}{n}+1}$$
$$\left\{\frac{1}{\left(2i-\frac{1}{n}+3\right)\left(2i+2j-\frac{2}{n}+4\right)}\right.$$
$$\left(\frac{\left(R^{2i+2j-\frac{2}{n}+6}-R_w^{2i+2j-\frac{2}{n}+6}\right)}{2i+2j-\frac{2}{n}+6}\right.$$
$$\left.-\frac{R_w^{2i+2j-\frac{2}{n}+4}(R^2-R_w^2)}{2}\right)$$
$$-\frac{R^{2i-\frac{1}{n}+3}}{\left(2i-\frac{1}{n}+3\right)\left(2j-\frac{1}{n}+1\right)}\left(\frac{\left(R^{2j-\frac{1}{n}+3}-R_w^{2j-\frac{1}{n}+3}\right)}{2j-\frac{1}{n}+3}\right.$$
$$\left.-\frac{R_w^{2j-\frac{1}{n}+1}(R^2-R_w^2)}{2}\right)-\frac{R_w^{2i-\frac{1}{n}+1}}{2\left(2j-\frac{1}{n}+3\right)}$$
$$\left(\frac{\left(R^{2j-\frac{1}{n}+5}-R_w^{2j-\frac{1}{n}+5}\right)}{2j-\frac{1}{n}+5}-\frac{R_w^{2j-\frac{1}{n}+3}(R^2-R_w^2)}{2}\right)$$
$$+\frac{R_w^{2i-\frac{1}{n}+1}R^2}{2\left(2j-\frac{1}{n}+1\right)}\left(\frac{\left(R^{2j-\frac{1}{n}+3}-R_w^{2j-\frac{1}{n}+3}\right)}{2j-\frac{1}{n}+3}\right.$$
$$\left.\left.\left.-\frac{R_w^{2j-\frac{1}{n}+1}(R^2-R_w^2)}{2}\right)\right\}\right]. \quad (37)$$

3.2.3 Thickness of the fluid film

Simplifying equation (37) after making use of equation (29), one obtains,

$$\frac{\partial z}{\partial t}=\left(\frac{\rho g}{2\eta}\right)^{\frac{1}{n}}\sum_{i=0}^{\infty}\binom{\frac{1}{n}}{i}\frac{(-1)^i R^{-2i+\frac{2}{n}-2}}{2i-\frac{1}{n}+1}$$
$$\left[\frac{1}{2i-\frac{1}{n}+3}\left\{\left(\frac{1}{n}+3\right)R^{2i-\frac{1}{n}+3}\right.\right.$$
$$\left.-\left(-2i+\frac{2}{n}\right)R_w^{2i-\frac{1}{n}+3}\right\}-\frac{R_w^{2i-\frac{1}{n}+1}}{2}$$
$$\left.\left\{\left(2+\frac{2}{n}-2i\right)R^2-\left(-2i+\frac{2}{n}\right)R_w^2\right\}\right]$$
$$+\frac{\varepsilon}{n}\left(\frac{\rho g}{2\eta}\right)^{\frac{2}{n}-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\frac{1}{n}}{i}\binom{\frac{1-n}{n}}{j}$$
$$\frac{(-1)^{i+j}R^{-2i-2j+\frac{4}{n}-2}}{2i-\frac{1}{n}+1}$$
$$\left\{\frac{\left(R^{2i+2j-\frac{2}{n}+5}-RR_w^{2i+2j-\frac{2}{n}+4}\right)}{\left(2i-\frac{1}{n}+3\right)\left(2i+2j-\frac{2}{n}+4\right)}\right.$$
$$+\left(\frac{\left(R^{2j-\frac{1}{n}+3}-R_w^{2j-\frac{1}{n}+3}\right)}{2j-\frac{1}{n}+3}-\frac{R_w^{2j-\frac{1}{n}+1}(R^2-R_w^2)}{2}\right)$$
$$\left(\frac{RR_w^{2i-\frac{1}{n}+1}-R^{2i-\frac{1}{n}+2}}{2j-\frac{1}{n}+1}\right)-\left(R^{2j-\frac{1}{n}+2}-RR_w^{2j-\frac{1}{n}+1}\right)$$

$$\left(\frac{R^2R_w^{2i-\frac{1}{n}+1}}{2\left(2j-\frac{1}{n}+1\right)}-\frac{R^{2i-\frac{1}{n}+3}}{\left(2i-\frac{1}{n}+3\right)\left(2i-\frac{1}{n}+1\right)}\right)$$
$$+\frac{R_w^{2i-\frac{1}{n}+1}}{2\left(2j-\frac{1}{n}+3\right)}\left(R^{2j-\frac{1}{n}+4}-RR_w^{2j-\frac{1}{n}+3}\right)\Bigg\}. \qquad (38)$$

By integrating equation (38) with respect to t, and then using the boundary condition $R(0,t)=R$, we get the relation between film thickness z and t as:

$$z=\left(\frac{\rho g}{2\eta}\right)^{\frac{1}{n}}\sum_{i=0}^{\infty}\binom{\frac{1}{n}}{i}\frac{(-1)^iR^{-2i+\frac{2}{n}-2}}{2i-\frac{1}{n}+1}$$
$$\Bigg[\frac{1}{2i-\frac{1}{n}+3}\left\{\left(\frac{1}{n}+3\right)R^{2i-\frac{1}{n}+3}\right.$$
$$\left.-\left(-2i+\frac{2}{n}\right)R_w^{2i-\frac{1}{n}+3}\right\}-\frac{R_w^{2i-\frac{1}{n}+1}}{2}$$
$$\left\{\left(2+\frac{2}{n}-2i\right)R^2-\left(-2i+\frac{2}{n}\right)R_w^2\right\}\Bigg]t$$
$$+\frac{\varepsilon}{n}\left(\frac{\rho g}{2\eta}\right)^{\frac{2}{n}-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\frac{1}{n}}{i}\binom{\frac{1-n}{n}}{j}$$
$$\frac{(-1)^{i+j}R^{-2i-2j+\frac{4}{n}-2}}{2i-\frac{1}{n}+1}$$
$$\left\{\frac{\left(R^{2i+2j-\frac{2}{n}+5}-RR_w^{2i+2j-\frac{2}{n}+4}\right)}{\left(2i-\frac{1}{n}+3\right)\left(2i+2j-\frac{2}{n}+4\right)}\right.$$
$$+\left(\frac{\left(R^{2j-\frac{1}{n}+3}-R_w^{2j-\frac{1}{n}+3}\right)}{2j-\frac{1}{n}+3}-\frac{R_w^{2j-\frac{1}{n}+1}\left(R^2-R_w^2\right)}{2}\right)$$
$$\left(\frac{RR_w^{2i-\frac{1}{n}+1}-R^{2i-\frac{1}{n}+2}}{2j-\frac{1}{n}+1}\right)-\left(R^{2j-\frac{1}{n}+2}-RR_w^{2j-\frac{1}{n}+1}\right)$$
$$\left(\frac{R^2R_w^{2i-\frac{1}{n}+1}}{2\left(2j-\frac{1}{n}+1\right)}-\frac{R^{2i-\frac{1}{n}+3}}{\left(2i-\frac{1}{n}+3\right)\left(2i-\frac{1}{n}+1\right)}\right)$$
$$\left.+\frac{R_w^{2i-\frac{1}{n}+1}}{2\left(2j-\frac{1}{n}+3\right)}\left(R^{2j-\frac{1}{n}+4}-RR_w^{2j-\frac{1}{n}+3}\right)\right\}t. \qquad (39)$$

For a drainage on a convex surface, we get,

$$z=\left(\frac{\rho g}{2\eta}\right)^{\frac{1}{n}}\sum_{i=0}^{\infty}\binom{\frac{1}{n}}{i}\frac{(-1)^i(R_w+h)^{-2i+\frac{2}{n}-2}}{2i-\frac{1}{n}+1}$$
$$\Bigg[\frac{1}{2i-\frac{1}{n}+3}\left\{\left(\frac{1}{n}+3\right)(R_w+h)^{2i-\frac{1}{n}+3}\right.$$
$$\left.-\left(-2i+\frac{2}{n}\right)R_w^{2i-\frac{1}{n}+3}\right\}-\frac{R_w^{2i-\frac{1}{n}+1}}{2}$$
$$\left\{\left(2+\frac{2}{n}-2i\right)(R_w+h)^2-\left(-2i\right.\right.$$
$$\left.\left.+\frac{2}{n}\right)R_w^2\right\}\Bigg]t+\frac{\varepsilon}{n}\left(\frac{\rho g}{2\eta}\right)^{\frac{2}{n}-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}$$
$$\binom{\frac{1}{n}}{i}\binom{\frac{1-n}{n}}{j}\frac{(-1)^{i+j}(R_w+h)^{-2i-2j+\frac{4}{n}-2}}{2i-\frac{1}{n}+1}$$
$$\left\{\frac{\left((R_w+h)^{2i+2j-\frac{2}{n}+5}-(R_w+h)R_w^{2i+2j-\frac{2}{n}+4}\right)}{\left(2i-\frac{1}{n}+3\right)\left(2i+2j-\frac{2}{n}+4\right)}\right.$$
$$+\left(\frac{\left((R_w+h)^{2j-\frac{1}{n}+3}-R_w^{2j-\frac{1}{n}+3}\right)}{2j-\frac{1}{n}+3}\right.$$
$$\left.-\frac{R_w^{2j-\frac{1}{n}+1}\left((R_w+h)^2-R_w^2\right)}{2}\right)$$
$$\left(\frac{(R_w+h)R_w^{2i-\frac{1}{n}+1}-(R_w+h)^{2i-\frac{1}{n}+2}}{2j-\frac{1}{n}+1}\right)$$
$$-\left((R_w+h)^{2j-\frac{1}{n}+2}-(R_w+h)R_w^{2j-\frac{1}{n}+1}\right)$$
$$\left(\frac{(R_w+h)^2R_w^{2i-\frac{1}{n}+1}}{2\left(2j-\frac{1}{n}+1\right)}-\frac{(R_w+h)^{2i-\frac{1}{n}+3}}{\left(2i-\frac{1}{n}+3\right)\left(2i-\frac{1}{n}+1\right)}\right)$$
$$+\frac{R_w^{2i-\frac{1}{n}+1}}{2\left(2j-\frac{1}{n}+3\right)}\left((R_w+h)^{2j-\frac{1}{n}+4}\right.$$
$$\left.\left.-(R_w+h)R_w^{2j-\frac{1}{n}+3}\right)\right\}t. \qquad (40)$$

for drainage on concave surface, we arrive at,

$$z=\left(\frac{\rho g}{2\eta}\right)^{\frac{1}{n}}\sum_{i=0}^{\infty}\binom{\frac{1}{n}}{i}\frac{(-1)^i(R_w-h)^{-2i+\frac{2}{n}-2}}{2i-\frac{1}{n}+1}$$
$$\Bigg[\frac{1}{2i-\frac{1}{n}+3}\left\{\left(\frac{1}{n}+3\right)(R_w-h)^{2i-\frac{1}{n}+3}\right.$$
$$\left.-\left(-2i+\frac{2}{n}\right)R_w^{2i-\frac{1}{n}+3}\right\}-\frac{R_w^{2i-\frac{1}{n}+1}}{2}$$
$$\left\{\left(2+\frac{2}{n}-2i\right)(R_w-h)^2-\left(-2i\right.\right.$$
$$\left.\left.+\frac{2}{n}\right)R_w^2\right\}\Bigg]t+\frac{\varepsilon}{n}\left(\frac{\rho g}{2\eta}\right)^{\frac{2}{n}-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}$$
$$\binom{\frac{1}{n}}{i}\binom{\frac{1-n}{n}}{j}\frac{(-1)^{i+j}(R_w-h)^{-2i-2j+\frac{4}{n}-2}}{2i-\frac{1}{n}+1}$$

$$
\left\{\frac{\left((R_w-h)^{2i+2j-\frac{2}{n}+5}-(R_w-h)\,R_w^{2i+2j-\frac{2}{n}+4}\right)}{\left(2i-\frac{1}{n}+3\right)\left(2i+2j-\frac{2}{n}+4\right)}\right.
+\left(\frac{\left((R_w-h)^{2j-\frac{1}{n}+3}-R_w^{2j-\frac{1}{n}+3}\right)}{2j-\frac{1}{n}+3}
-\frac{R_w^{2j-\frac{1}{n}+1}\left((R_w-h)^2-R_w^2\right)}{2}\right)
\left(\frac{(R_w-h)\,R_w^{2i-\frac{1}{n}+1}-(R_w-h)^{2i-\frac{1}{n}+2}}{2j-\frac{1}{n}+1}\right)
-\left((R_w-h)^{2j-\frac{1}{n}+2}-(R_w-h)\,R_w^{2j-\frac{1}{n}+1}\right)
\left(\frac{(R_w-h)^2 R_w^{2i-\frac{1}{n}+1}}{2\left(2j-\frac{1}{n}+1\right)}-\frac{(R_w-h)^{2i-\frac{1}{n}+3}}{\left(2i-\frac{1}{n}+3\right)\left(2i-\frac{1}{n}+1\right)}\right)
+\frac{R_w^{2i-\frac{1}{n}+1}}{2\left(2j-\frac{1}{n}+3\right)}\left((R_w-h)^{2j-\frac{1}{n}+4}
-\left.(R_w-h)\,R_w^{2j-\frac{1}{n}+3}\right)\right\}t. \quad (41)
$$

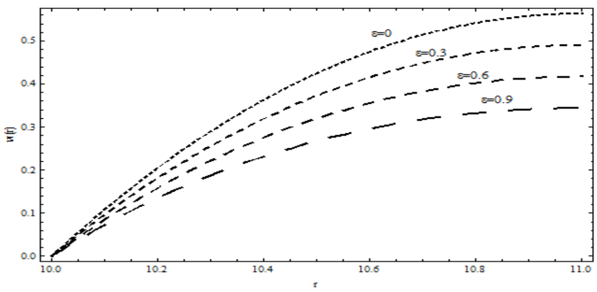

Fig.2.The effect of ε on velocity profile for Newtonian MHD fluid for drainage in thin film flow, when $\eta = 7poise$, $\rho = 0.78g/cm^3$, $R = 11cm$, $R_w = 10cm$.

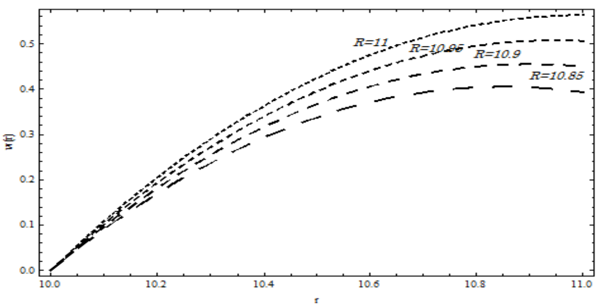

Fig.3. The effect of R on velocity profile for Newtonian MHD fluid for drainage in thin film flow, when $\eta = 7poise$, $\varepsilon = 0.001cm^{-2}$, $R_w = 10cm$, $\rho = 0.78g/cm^3$

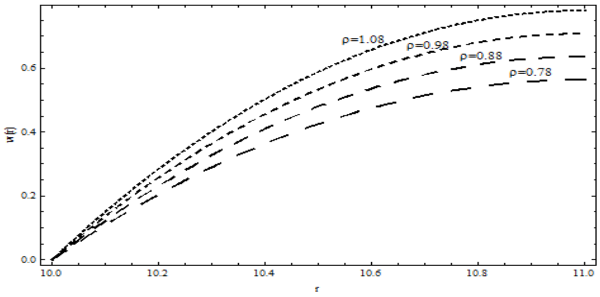

Fig.4. The effect of ρ on velocity profile for Newtonian MHD fluid for drainage in thin film flow, when $R = 11cm$, $\varepsilon = 0.001cm^{-2}$, $R_w = 10cm$, $\eta = 7poise$

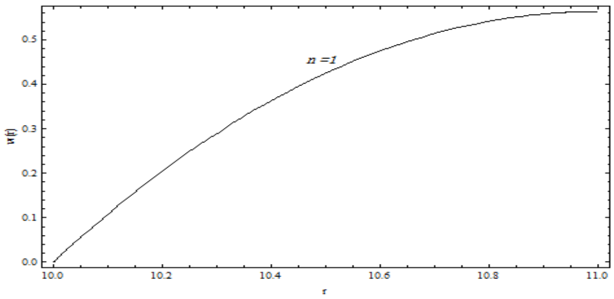

Fig.5. Velocity profile for Newtonian MHD fluid for drainage in thin film flow, when $R = 11cm$, $\varepsilon = 0.0001cm^{-2}$, $R_w = 10cm$, $\eta = 7poise$, $\rho = 0.78g/cm^3$

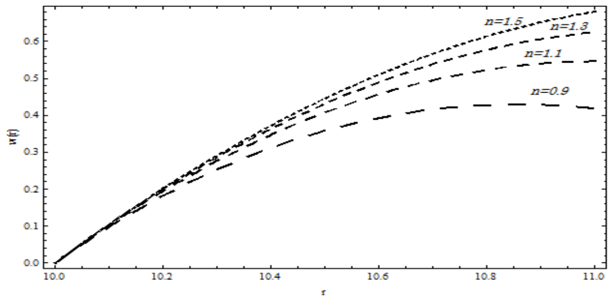

Fig.6. The effect of n on velocity profile for power law MHD fluid, when $R = 11cm$, $\varepsilon = 0.0001cm^{-2}$, $R_w = 10cm$, $\eta = 7poise$, $\rho = 0.78g/cm^3$

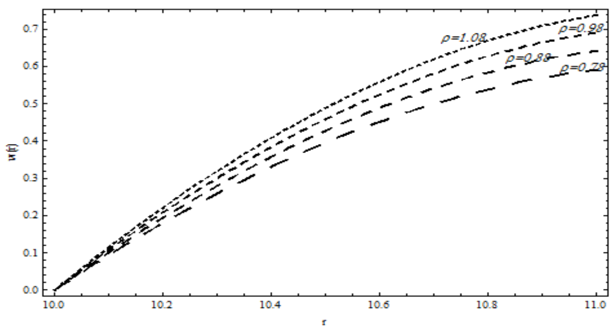

Fig.7. The effect of ρ on velocity profile for power law MHD fluid when $n = 1.7$, $R = 11cm$, $\varepsilon = 0.001cm^{-2}$, $R_w = 10cm$, $\eta = 7poise$, $\rho = 0.78g/cm^3$

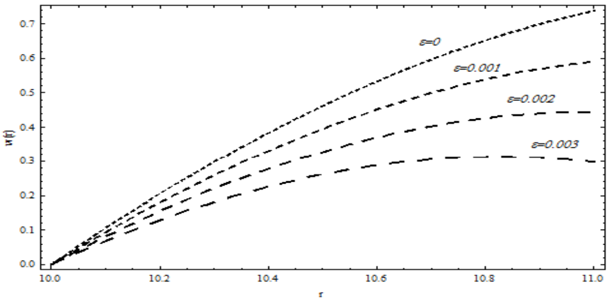

Fig.8. The effect of ε on velocity profile for power law MHD fluid, when $n = 1.7$, $R = 11cm$, $R_w = 10cm$, $\eta = 7poise$, $\rho = 0.78g/cm^3$

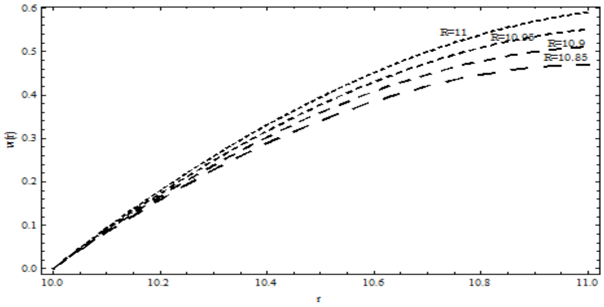

Fig.9. The effect of R on velocity profile for power law MHD fluid, when $n = 1.7$, $\varepsilon = 0.001cm^{-2}$, $R_w = 10cm$, $\eta = 7poise$, $\rho = 0.78g/cm^3$

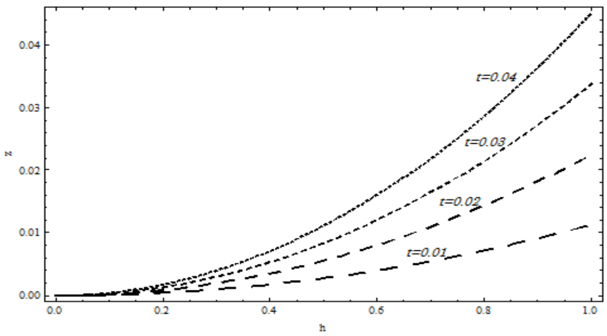

Fig.10. Growth of film thickness for convex surface for different value of t, for Newtonian MHD fluid, when $\eta = 7poise$, $\varepsilon = 0.001cm^{-2}$ $R_w = 10cm$, $R = 11cm$,$\rho = 0.78g/cm^3$

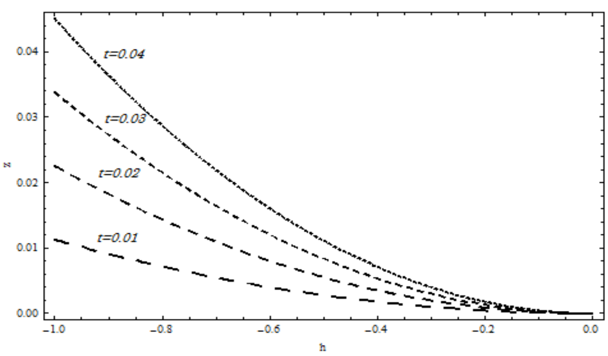

Fig.11. Growth of film thickness for concave surface for different value of t, for Newtonian MHD fluid, when $\eta = 7poise$, $\varepsilon = 0.001cm^{-2}$, $R_w = 10cm$, $R = 11cm$, $\rho = 0.78g/cm^3$.

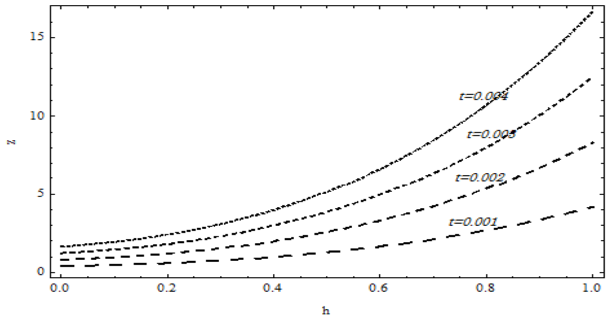

Fig.12. Growth of film thickness for convex surface for different value of t, for power law MHD fluid, when $\eta = 7poise$, $\varepsilon = 0.059cm^{-2}$, $R_w = 10cm$, $n = 0.36$ and $\rho = 0.138g/cm^3$.

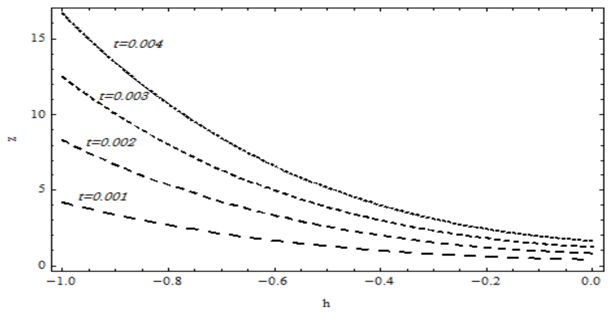

Fig.13. Growth of film thickness for concave surface for different value of t, for power law MHD fluid, when $\eta = 7poise$, $\varepsilon = 0.059cm^{-2}$, $R_w = 10cm$, $n = 0.36$ and $\rho = 0.138g/cm^3$.

4 RESULTS AND DISCUSSION

The analysis determine the effects of power law index n, magnetic parameter ε, density ρ and R over velocity profile and growth of film thickness for convex and concave surfaces of fluid film. The results are given in figures (2) - (13). The variation of axial velocity for n, ε, ρ and R for both Newtonian and power law MHD fluid in case of drainage is displayed in figures (2) - (9). From figures (2) - (9), we observed, that with an increase in n, ρ and R, results in increase of velocity profile while the same decreases with increase in ε. The difference of time t for growth of film thickness of fluid film in figures (10) - (13) have been plotted where it is observed that the growth of thickness of fluid film increases for increasing time t for all Newtonian and power law MHD fluid.

5 CONCLUDING REMARKS

We have presented results in the thin film flow field of a fluid, which is called the Power law MHD fluid, on a vertical cylinder for drainage problem. The resulting nonlinear differential equation has been solved by perturbation method, which is effective and reliable method for the proposed problem. The velocity profile, flow rate and thickness of the fluid film have been derived for the title problem.

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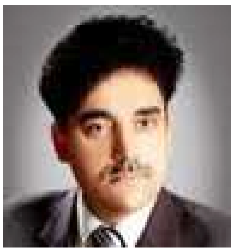

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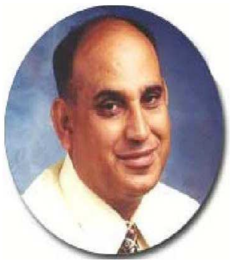

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