Journal of Statistics Applications & Probability Letters
An International Journal

http://dx.doi.org/10.18576/jsapl/080103

Cubic Transmuted Burr-XII Distribution with Properties and Applications

Sumaia Akter, Md. Awwal Islam Khan, Md. Shohel Rana and Md. Mahabubur Rahman[*]

Department of Statistics, Islamic University, Kushtia-7003, Bangladesh

Received: 16 Apr. 2020, Revised: 10 Aug. 2020, Accepted: 16 Aug. 2020
Published online: 1 Jan. 2021

Abstract: In this paper, a cubic transmutation approach, proposed by Rahman et al. [1], is used to introduce the cubic transmuted Burr-XII distribution. The distributional properties like moments, moment generating function, characteristic function, quantile function, reliability function and hazard rate function are discussed for the proposed distribution along with the distributions of different order statistics. The maximum likelihood estimation of the model parameters has been discussed for the proposed distribution. In order to investigate the applicability of the proposed distribution two real-life applications have been considered.

Keywords: Burr-XII distribution, cubic transmutation, maximum likelihood estimation, order statistics, reliability analysis

1 Introduction

Probability distributions are being used to model real-world phenomena. Nowadays, generalizing the probability distribution is a very common practice due to capture more complex real-world problems. Still, there are so many real-wold problems that do not follow any of the standard probability models. So, generalizing the probability distributions is an ever-existing process.

The Pearson distribution is a class of continuous probability distributions, in which the density function $y = f(x)$ is defined to be any valid solution to the differential equation, see [2], can be expressed as

$$\frac{1}{y}\frac{dy}{dx} = \frac{(x-a)}{c_0 + c_1 x + c_2 x^2}, \tag{1}$$

where c_0, c_1 and c_2 are constants. The Burr [3], cumulative distribution function $y = F(x)$ is obtained from the following simple differential equation

$$\frac{1}{y}\frac{dy}{dx} = (1-y)g(x,y). \tag{2}$$

It is to be noted that (2) comprises 12 distributions, is very similar to (1) for $g(x,y) = (c_0 + c_1 x + c_2 x^2)^{-1}$. The Burr-XII distribution or simply the Burr distribution is a member of a system of continuous distributions (2) for a positive random variable. It is also well known as the Singh-Maddala distribution [4], and sometimes called the "generalized log-logistic distribution". It is most commonly used to model household income in the U.S. and it has limited applicability. The Burr-XII distribution has the density function, see [5], as

$$g(x) = \alpha\beta \frac{x^{\alpha-1}}{(1+x^{\alpha})^{\beta+1}},\ x \in \mathbb{R}^+,$$

where $\alpha \in \mathbb{R}^+$ and $\beta \in \mathbb{R}^+$ are the shape and scale parameters respectively. Burr distribution becomes the Pareto Type II (Lomax) distribution for $\alpha = 1$ and it is a special case of the Champernowne distribution, see [6], often called the Fisk distribution, see [7], for $\beta = 1$.

[*] Corresponding author e-mail: mmriu.stat@gmail.com

The main target of this article is to increase the applicability of the Burr-XII distribution specially in the area of household income, environmental, biology, engineering, reliability, insurance and other areas of life. For doing so, a second-order Burr-XII distribution is introduced that can capture the complex behavior in the real-life datasets.

The layout plan of the article: In Section 2, the cubic transmuted Burr-XII distribution is introduced. Some of the distributional properties are described in Section 3 along with the distributions of the different order statistics in Section 4. The maximum likelihood estimation of the model parameters is described in Section 5. Section 6 describes two real-life applications of the proposed distribution. Some concluding remarks are listed at the end.

2 Cubic Transmuted Burr-XII Distribution

The Burr-XII distribution was first introduced in the literature by Burr [3], which has the distribution function as

$$G(x) = 1-(1+x^{\alpha})^{-\beta}, \; x \in \mathbb{R}^{+}, \tag{3}$$

where $\alpha \in \mathbb{R}^{+}$ and $\beta \in \mathbb{R}^{+}$. Further detail studying the distribution, see [8,9]. Shaw and Buckley [10], introduced the transmuted family of distribution to solve the problems related to the financial mathematics. This family has the following cumulative distribution function

$$F(x) = (1+\lambda)\, G(x) - \lambda G^{2}(x), \; x \in \mathbb{R}, \tag{4}$$

where $\lambda \in [-1,1]$ and $G(x)$ is the base distribution function of any standard probability distribution. For $\lambda = 0$, (4) has the base distribution function. Maurya et al. [11], use (3) in (4) and developed transmuted Burr-XII distribution which has the following cumulative distribution function

$$F(x) = 1+\left[(\lambda-1)(x^{\alpha}+1)^{-\beta} - \lambda\,(x^{\alpha}+1)^{-2\beta}\right], \; x \in \mathbb{R}^{+},$$

where $\alpha, \beta \in \mathbb{R}^{+}$ and $\lambda \in [-1,1]$. Consider the distribution function of a cubic transmuted family proposed by Rahman et al. [1], which is written as

$$F(x) = (1-\lambda)G(x) + 3\lambda G^{2}(x) - 2\lambda G^{3}(x), \; x \in \mathbb{R}, \tag{5}$$

where $\lambda \in [-1,1]$ and $G(x)$ is the distribution function of any baseline probability distribution. In order to introduce the distribution function of the proposed CTBurr-XII distribution, use (3) in (5) which can be further expressed as

$$F(x) = (x^{\alpha}+1)^{-3\beta}\left\{(x^{\alpha}+1)^{\beta}-1\right\}\left[\lambda\left\{(x^{\alpha}+1)^{\beta}-2\right\}+(x^{\alpha}+1)^{2\beta}\right], \; x \in \mathbb{R}^{+}, \tag{6}$$

where $\alpha, \beta \in \mathbb{R}^{+}$ and $\lambda \in [-1,1]$. The corresponding probability density function of the proposed CTBurr-XII distribution is obtained by differentiating (6) with respect to x, which can be stated in the following definition.

Definition: *A continuous random variable X is said to have a proposed CTBurr-XII distribution if its probability density function can be written as follow*

$$f(x) = \alpha\, \beta\, x^{\alpha-1}\, (x^{\alpha}+1)^{-3\beta-1}\left[-6\lambda + 6\lambda\,(x^{\alpha}+1)^{\beta} - (\lambda-1)(x^{\alpha}+1)^{2\beta}\right], \; x \in \mathbb{R}^{+}, \tag{7}$$

where $\alpha, \beta \in \mathbb{R}^{+}$ *and* $\lambda \in [-1,1]$.

Some of the possible shapes for the density and distribution functions of the proposed CTBurr-XII distribution are presented in Fig. 1. It has been observed from the figure that the proposed model has the capability to capture the complex behavior of the real-life datasets.

3 Distributional Properties

Some of the important distributional properties of the proposed CTBurr-XII distribution, presented in (7), are discussed in the following subsections.

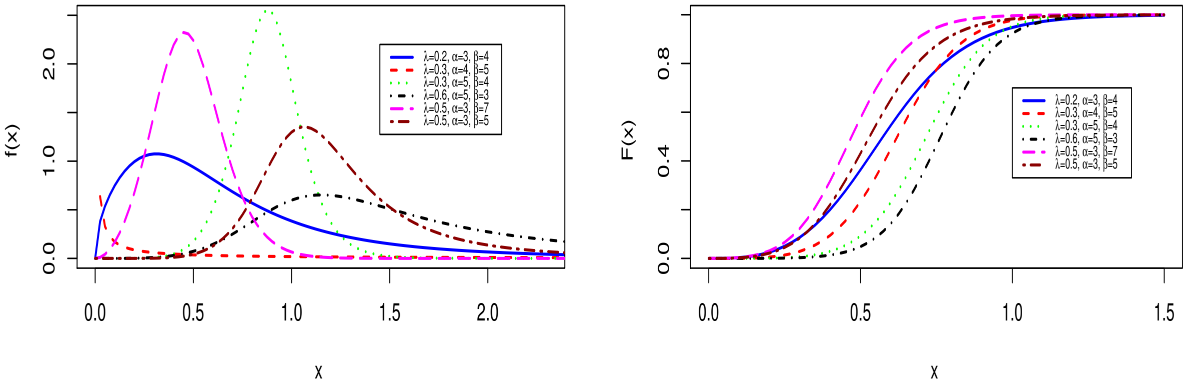

Fig. 1: The density and distribution functions of the proposed CTBurr-XII distribution

3.1 Moments

The moment is a specific quantitative measure of the shape of a probability function. For a specific probability distribution, the first moment is the expected value, the second central moment is the variance, the third standardized moment is the skewness, and the fourth standardized moment is the kurtosis. The *rth* raw moments of the proposed CTBurr-XII distribution is obtained as

$$\mu_r' = \Gamma\left(\frac{r+\alpha}{\alpha}\right)\left[\frac{(1-\lambda)\Gamma\left(\beta-\frac{r}{\alpha}\right)}{\Gamma(\beta)}+\frac{3\lambda\Gamma\left(2\beta-\frac{r}{\alpha}\right)}{\Gamma(2\beta)}-\frac{2\lambda\Gamma\left(3\beta-\frac{r}{\alpha}\right)}{\Gamma(3\beta)}\right], \alpha\beta > r. \tag{8}$$

The mean of the proposed distribution is obtained by setting $r=1$ in equation (8) and expressed as follow

$$\mu = \mu_1' = \Gamma\left(\frac{1+\alpha}{\alpha}\right)\left[\frac{(1-\lambda)\Gamma\left(\beta-\frac{1}{\alpha}\right)}{\Gamma(\beta)}+\frac{3\lambda\Gamma\left(2\beta-\frac{1}{\alpha}\right)}{\Gamma(2\beta)}-\frac{2\lambda\Gamma\left(3\beta-\frac{1}{\alpha}\right)}{\Gamma(3\beta)}\right].$$

The variance of the proposed CTBurr-XII distribution is obtained as

$$\begin{aligned}\sigma^2 = \mu_2' - (\mu_1')^2 = \Gamma\left(\frac{2+\alpha}{\alpha}\right)&\left[\frac{(1-\lambda)\Gamma\left(\beta-\frac{2}{\alpha}\right)}{\Gamma(\beta)}+\frac{3\lambda\Gamma\left(2\beta-\frac{2}{\alpha}\right)}{\Gamma(2\beta)}-\frac{2\lambda\Gamma\left(3\beta-\frac{2}{\alpha}\right)}{\Gamma(3\beta)}\right]\\ &-\left[\Gamma\left(\frac{1+\alpha}{\alpha}\right)\left\{\frac{(1-\lambda)\Gamma\left(\beta-\frac{1}{\alpha}\right)}{\Gamma(\beta)}+\frac{3\lambda\Gamma\left(2\beta-\frac{1}{\alpha}\right)}{\Gamma(2\beta)}-\frac{2\lambda\Gamma\left(3\beta-\frac{1}{\alpha}\right)}{\Gamma(3\beta)}\right\}\right]^2.\end{aligned}$$

One can obtain all the higher moments by using $r>2$ in equation (8).

3.2 Moment Generating Function

The moment generating function for the proposed CTBurr-XII distribution is stated by the following theorem.

Theorem 1. *Let a continuous random variable X follows the CTBurr-XII distribution, then the moment generating function $M_X(t)$ of X is*

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\cdot\Gamma\left(\frac{r+\alpha}{\alpha}\right)\left[\frac{(1-\lambda)\Gamma\left(\beta-\frac{r}{\alpha}\right)}{\Gamma(\beta)}+\frac{3\lambda\Gamma\left(2\beta-\frac{r}{\alpha}\right)}{\Gamma(2\beta)}-\frac{2\lambda\Gamma\left(3\beta-\frac{r}{\alpha}\right)}{\Gamma(3\beta)}\right].$$

Proof. The moment generating function is defined as

$$M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} f(x) \mathrm{d}x,$$

where $f(x)$ is given in (7). Using the series representation of e^{tx} given by Gradshteyn and Ryzhik [12], we have

$$M_x(t) = \int_0^\infty \sum_{r=0}^\infty \frac{t^r}{r!} x^r f(x) \mathrm{d}x = \sum_{r=0}^\infty \frac{t^r}{r!} E(X^r). \tag{9}$$

Using $E(X^r)$ from (8) in (9), we have the moment generating function $M_X(t)$.

3.3 Characteristic Function

The characteristic function for the proposed CTBurr-XII distribution is stated by the following theorem.

Theorem 2. *Let a continuous random X follows the CTBurr-XII distribution, then the characteristic function $\phi_X(t)$ is*

$$\phi_X(t) = \sum_{r=0}^\infty \frac{(it)^r}{r!} \cdot \Gamma\left(\frac{r+\alpha}{\alpha}\right) \left[\frac{(1-\lambda)\Gamma\left(\beta - \frac{r}{\alpha}\right)}{\Gamma(\beta)} + \frac{3\lambda\Gamma\left(2\beta - \frac{r}{\alpha}\right)}{\Gamma(2\beta)} - \frac{2\lambda\Gamma\left(3\beta - \frac{r}{\alpha}\right)}{\Gamma(3\beta)}\right].$$

where, $i = \sqrt{-1}$ is the imaginary unit and $t \in \mathbb{R}$.

Proof. The proof is simple like moment generating function.

3.4 Quantile Function and Median

The quantile function for the proposed CTBurr-XII distribution is obtained by solving $F(x) = q$, see for example [13], and further proceed as follow

$$(x^\alpha+1)^{-3\beta}\left\{(x^\alpha+1)^\beta - 1\right\}\left[\lambda\left\{(x^\alpha+1)^\beta - 2\right\} + (x^\alpha+1)^{2\beta}\right] = q,$$

which can be further obtained as

$$x_q = \left[\left\{\frac{\sqrt[3]{\delta_1 + \sqrt{\delta_2^2 + 4\delta_3^3} + \theta}}{3\sqrt[3]{2}(q-1)} - \frac{\sqrt[3]{2}\delta_3}{3(q-1)\sqrt[3]{\delta_1 + \sqrt{\delta_2^2 + 4\delta_3^3} + \theta}} - \frac{1-\lambda}{3(q-1)}\right\}^{1/\beta} - 1\right]^{1/\alpha}, \tag{10}$$

where $\theta = 54\lambda q^2 - 27\lambda^2 q - 81\lambda q - 2$, $\delta_1 = 2\lambda^3 + 21\lambda^2 + 33\lambda$, $\delta_2 = 2\lambda^3 + 21\lambda^2 + 33\lambda + \theta$, and $\delta_3 = -\lambda^2 - 7\lambda + 9\lambda q - 1$. By using (10), one can easily obtain the first quartile (Q_1), second quartile (Q_2) or median and third quartile (Q_3) by setting $q = 0.25,\ 0.50,$ and 0.75 respectively.

3.5 Generating Random Sample

The random number from the proposed CTBurr-XII distribution is generated by setting $F(x) = u$, see for example [14], and further obtained as

$$(x^\alpha+1)^{-3\beta}\left\{(x^\alpha+1)^\beta - 1\right\}\left[\lambda\left\{(x^\alpha+1)^\beta - 2\right\} + (x^\alpha+1)^{2\beta}\right] = u,$$

where $u \sim U(0,1)$, and the above can be further obtained as

$$X = \left[\left\{\frac{\sqrt[3]{\delta_1 + \sqrt{\delta_2^2 + 4\delta_3^3} + \theta}}{3\sqrt[3]{2}(u-1)} - \frac{\sqrt[3]{2}\delta_3}{3(u-1)\sqrt[3]{\delta_1 + \sqrt{\delta_2^2 + 4\delta_3^3} + \theta}} - \frac{1-\lambda}{3(u-1)}\right\}^{1/\beta} - 1\right]^{1/\alpha}, \tag{11}$$

where $\theta = 54\lambda u^2 - 27\lambda^2 u - 81\lambda u - 2$, $\delta_1 = 2\lambda^3 + 21\lambda^2 + 33\lambda$, $\delta_2 = 2\lambda^3 + 21\lambda^2 + 33\lambda + \theta$, and $\delta_3 = -\lambda^2 - 7\lambda + 9\lambda u - 1$. Hence, one can use (11) to generate random samples from the proposed CTBurr-XII distribution.

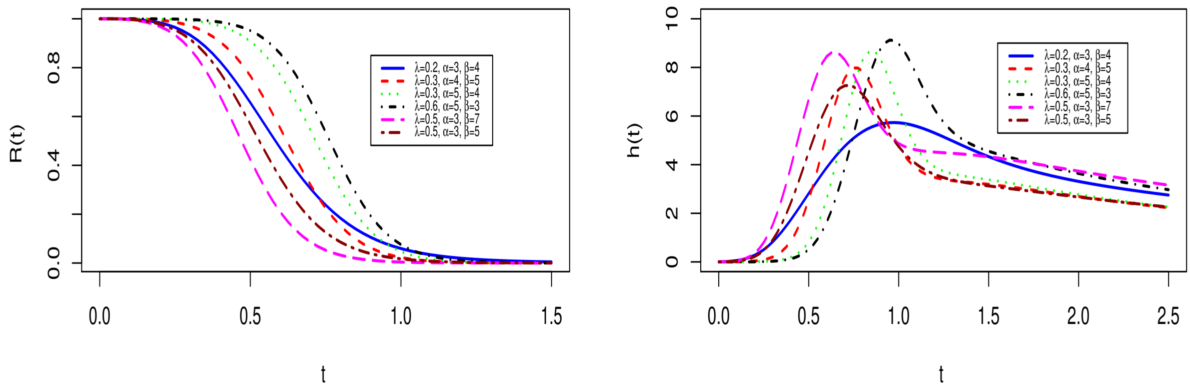

Fig. 2: The reliability and hazard rate functions of the proposed CTBurr-XII distribution

3.6 Reliability Analysis

The reliability function is simply the complement of a distribution function and is defined for the proposed CTBurr-XII distribution as

$$R(t) = 1 - F(t) = 1 - (t^{\alpha}+1)^{-3\beta}\left\{(t^{\alpha}+1)^{\beta}-1\right\}\left[\lambda\left\{(t^{\alpha}+1)^{\beta}-2\right\}+(t^{\alpha}+1)^{2\beta}\right].$$

The hazard function is the ratio of the probability distribution function to the reliability function and is given by

$$h(t) = \frac{f(t)}{R(t)} = \frac{f(t)}{1-F(t)} = \frac{\alpha\beta t^{\alpha-1}(t^{\alpha}+1)^{-3\beta-1}\left[-6\lambda+6\lambda(t^{\alpha}+1)^{\beta}-(\lambda-1)(t^{\alpha}+1)^{2\beta}\right]}{1-(t^{\alpha}+1)^{-3\beta}\left\{(t^{\alpha}+1)^{\beta}-1\right\}\left[\lambda\left\{(t^{\alpha}+1)^{\beta}-2\right\}+(t^{\alpha}+1)^{2\beta}\right]}.$$

Fig. 2 describes several plots of the reliability and hazard rate functions for the proposed CTBurr-XII distribution. It has been observed from the figure that the proposed distribution has the capability to capture several increasing then decreasing hazard rate functions.

4 Order Statistics

The probability density function of the rth order statistic for the proposed CTBurr-XII distribution is given as follow

$$\begin{aligned} f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!}&\left[\alpha\beta x^{\alpha-1}(x^{\alpha}+1)^{-3\beta-1}\times\left\{-6\lambda+6\lambda(x^{\alpha}+1)^{\beta}-(\lambda-1)(x^{\alpha}+1)^{2\beta}\right\}\right] \\ &\times\left[(x^{\alpha}+1)^{-3\beta}\left\{(x^{\alpha}+1)^{\beta}-1\right\}\left\{\lambda\left\{(x^{\alpha}+1)^{\beta}-2\right\}+(x^{\alpha}+1)^{2\beta}\right\}\right]^{r-1} \\ &\times\left[1-(x^{\alpha}+1)^{-3\beta}\left\{(x^{\alpha}+1)^{\beta}-1\right\}\left\{\lambda\left\{(x^{\alpha}+1)^{\beta}-2\right\}+(x^{\alpha}+1)^{2\beta}\right\}\right]^{n-r}, \end{aligned}$$

where $r = 1,2,\cdots,n$. Using $r = 1$, obtain the density function of lowest order statistic $X_{1:n}$, and is given as

$$\begin{aligned} f_{X_{1:n}}(x) = n&\left[\alpha\beta x^{\alpha-1}(x^{\alpha}+1)^{-3\beta-1}\left\{-6\lambda+6\lambda(x^{\alpha}+1)^{\beta}-(\lambda-1)(x^{\alpha}+1)^{2\beta}\right\}\right] \\ &\times\left[1-(x^{\alpha}+1)^{-3\beta}\left\{(x^{\alpha}+1)^{\beta}-1\right\}\left\{\lambda\left\{(x^{\alpha}+1)^{\beta}-2\right\}+(x^{\alpha}+1)^{2\beta}\right\}\right]^{n-1}, \end{aligned}$$

also for using $r=n$, the density function of highest order statistic $X_{n:n}$, is obtain by

$$f_{X_{n:n}}(x)=n\left[\alpha\beta x^{\alpha-1}(x^{\alpha}+1)^{-3\beta-1}\left\{-6\lambda+6\lambda(x^{\alpha}+1)^{\beta}-(\lambda-1)(x^{\alpha}+1)^{2\beta}\right\}\right]$$
$$\times\left[(x^{\alpha}+1)^{-3\beta}\left\{(x^{\alpha}+1)^{\beta}-1\right\}\left\{\lambda\left\{(x^{\alpha}+1)^{\beta}-2\right\}+(x^{\alpha}+1)^{2\beta}\right\}\right]^{n-1}.$$

Note that for $\lambda=0$, it has the density function of the rth order statistic for Burr-XII distribution, as follow

$$g_{X_{r:n}}(x)=\frac{n!}{(r-1)!(n-r)!}\frac{\alpha\beta x^{\alpha-1}}{(x^{\alpha}+1)^{\beta+1}}\left[1-(x^{\alpha}+1)^{-\beta}\right]^{r-1}\times\left[1-1-(x^{\alpha}+1)^{-\beta}\right]^{n-r},r=1,2,\cdots,n.$$

The kth order moment of $X_{r:n}$ for the proposed CTBurr-XII distribution is obtained by using the equation

$$E(X_{r:n}^{k})=\int_{0}^{\infty}x_r^{k}\cdot f_{X_{r:n}}(x)\cdot \mathrm{d}x.$$

5 Estimation

In this section, the estimation of the model parameters for the proposed CTBurr-XII distribution has been conducted. This was down by using maximum likelihood estimation technique. For doing this, consider a random sample $x_1,x_2,\cdots,x_n$ of size n from the proposed CTBurr-XII distribution, which has the likelihood function as

$$L=\prod_{i=1}^{n}\left[\alpha\beta x^{\alpha-1}(x^{\alpha}+1)^{-\beta-1}(-6\lambda+6\lambda(x^{\alpha}+1)^{\beta}-(\lambda-1)(x^{\alpha}+1)^{2\beta})\right],$$

and the equivalent log-likelihood function $l=ln(L)$ is

$$l=n\log[\alpha\beta]+(\alpha-1)\sum_{i=1}^{n}\log[x_i]+(-3\beta-1)\sum_{i=1}^{n}\log[1+x_i^{\alpha}]+\sum_{i=1}^{n}\log\left[6\lambda(x_i^{\alpha}+1)^{\beta}-(\lambda-1)(x_i^{\alpha}+1)^{2\beta}-6\lambda\right]. \quad (12)$$

The maximum likelihood estimates of α, β and λ are obtained by maximizing the log-likelihood function given in (12). For doing so, taking the derivatives with respect to unknown parameters and further proceed as

$$\frac{\partial l}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\log(x_i)-(3\beta+1)\sum_{i=1}^{n}\frac{x_i^{\alpha}\log(x_i)}{x_i^{\alpha}+1}+\sum_{i=1}^{n}\frac{6\beta\lambda x_i^{\alpha}\log(x_i)(x_i^{\alpha}+1)^{\beta-1}-2(\beta(\lambda-1))x_i^{\alpha}\log(x_i)(x_i^{\alpha}+1)^{2\beta-1}}{6\lambda(x_i^{\alpha}+1)^{\beta}-(\lambda-1)(x_i^{\alpha}+1)^{2\beta}-6\lambda},$$

$$\frac{\partial l}{\partial\beta}=\frac{n}{\beta}-3\sum_{i=1}^{n}\log[x_i^{\alpha}+1]+\sum_{i=1}^{n}\frac{6\lambda(x_i^{\alpha}+1)^{\beta}\log(x_i^{\alpha}+1)-2(\lambda-1)(x_i^{\alpha}+1)^{2\beta}\log(x_i^{\alpha}+1)}{6\lambda(x_i^{\alpha}+1)^{\beta}-(\lambda-1)(x_i^{\alpha}+1)^{2\beta}-6\lambda},$$

and

$$\frac{\partial l}{\partial\lambda}=\sum_{i=1}^{n}\frac{6(x_i^{\alpha}+1)^{\beta}-(x_i^{\alpha}+1)^{2\beta}-6}{6\lambda(x_i^{\alpha}+1)^{\beta}-(\lambda-1)(x_i^{\alpha}+1)^{2\beta}-6\lambda}.$$

Now setting $\frac{\partial l}{\partial\alpha}=0$, $\frac{\partial l}{\partial\beta}=0$ and $\frac{\partial l}{\partial\lambda}=0$ and solving the resulting non-linear system of equations gives the maximum likelihood estimate $\hat{\Theta}=\left(\hat{\alpha},\hat{\beta},\hat{\lambda}\right)'$ of $\Theta=(\alpha,\beta,\lambda)'$. Hence, theoretical solution is very much complex even sometimes impossible for this nonlinear set of equations. In order to get the numerical solution, apply R-package "bbmle", for more details see [15]. Also as $n\to\infty$, the asymptotic distribution of the MLE's are, see for example [16,17], given as

$$\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\\ \hat{\lambda}\end{pmatrix}\sim N\left[\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix},\begin{pmatrix}\hat{V}_{11}&\hat{V}_{12}&\hat{V}_{13}\\ \hat{V}_{21}&\hat{V}_{22}&\hat{V}_{23}\\ \hat{V}_{31}&\hat{V}_{32}&\hat{V}_{33}\end{pmatrix}\right].$$

The asymptotic variance–covariance matrix V, of the estimates $\hat{\alpha},\hat{\beta},\hat{\lambda}$ is obtained by inverting Hessian matrix; see "Appendix". An approximate 100(1 - α) two-sided confidence intervals for α, β and λ are given by:

$$\hat{\alpha}\pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{22}},\ \hat{\beta}\pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{33}},\ \text{and}\ \hat{\lambda}\pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{11}},$$

where Z_{α} is the αth percentile of the standard normal distribution.

6 Applications

In order to check the applicability, two real-life applications have been conducted for the proposed CTBurr-XII distribution, which are described by the following two subsections.

6.1 Fatigue Fracture Data

This dataset represents the life of fatigue fracture of Kevlar 373/ epoxy subjected to constant pressure at 90% stress level until all had failed. The data was extracted from [18] and it has previously been used by Barlow et al. [19]. The summary statistics of the dataset are given in Table 1, and observed that it has positively skewed distribution.

Table 1: The summary of the fatigue fracture and the bladder cancer datasets

Dataset	Min.	Q1	Median	Mean	Q3	Max
Fatigue Fracture	0.0251	0.9048	1.7362	1.9592	2.2959	9.0960
Bladder Cancer	0.080	3.348	6.395	9.366	11.838	79.05

In order to asses the practicality of the proposed CTBurr-XII distribution, several other distributions like TBurr-XII and Burr-XII are selected. As first step, estimate the model parameters with corresponding standard errors of the proposed model along with selected models, and estimated values are presented in Table 2.

Table 2: MLE's of the parameters and respective *SE*'s for selected distributions along with proposed CTBurr-XII distribution

Distribution	Parameters	Estimate	SE
CTBurr-XII	λ	0.7225	0.2050
	α	1.7379	0.2656
	β	0.6609	0.0791
TBurr-XII	λ	5.72e-08	0.2238
	α	1.0221	0.1871
	β	1.000	0.1063
Burr-XII	α	2.2306	0.2670
	β	0.6656	0.0946

The estimated plots for the selected models along with the proposed CTBurr-XII distribution are plotted over the empirical cdf plot, and presented in the left of Fig. 3. Hence, from the figures, observed that the dataset fitted well with the proposed distribution as compared with other selected models. Critically observed that three parameters proposed CTBurr-XII distribution is suitable than three parameters TBurr-XII distribution for modeling this dataset.

Again, some model selection criteria like Log-likelihood, Akaike's information criterion (AIC), corrected Akaike information criterion (AICc), Bayesian information criterion (BIC) are selected to asses the practicality of the proposed model. The calculated model selection criteria values are presented in Table 3. According to the obtained model selection criteria values, it has been seen that the proposed CTBurr-XII model fitted well as compared with other competing models.

Table 3: Selection criteria values obtained for selected models

Distribution	Log-likelihood	AIC	AICc	BIC
CTBurr-XII	- 125.9509	257.018	258.2351	265.894
TBurr-XII	- 148.0415	302.083	302.4164	309.0752
Burr-XII	- 128.5534	261.1068	261.2712	265.7682

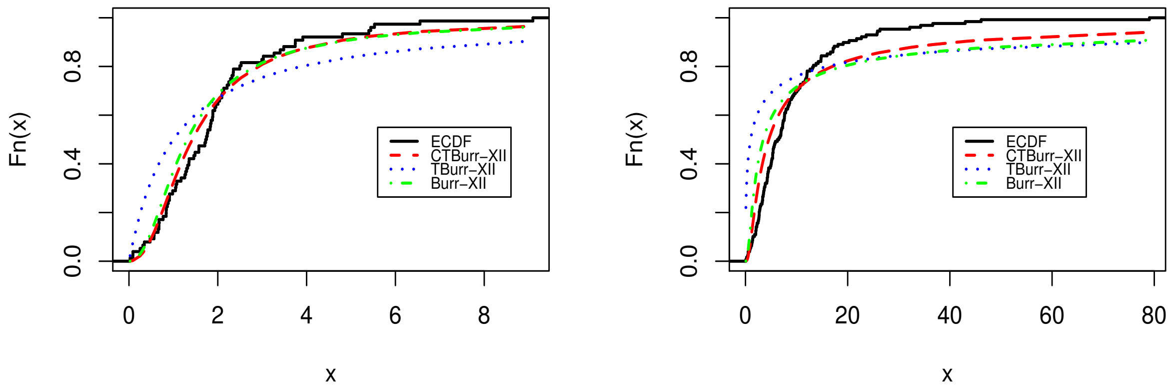

Fig. 3: The estimated distribution functions for the selected models along with the proposed CTBurr-XII distribution are plotted over the empirical distribution functions for the fatigue fracture (left) and bladder cancer (right) datasets

6.2 Bladder Cancer Data

The second dataset represents the remission times (in months) of a random sample of 128 bladder cancer patients as reported in [20]. Summary statistics of the dataset are described in Table 1, and observed that it has the right-skewed distribution.

The estimated values of the model parameters with corresponding standard errors for the selected models are given in Table 4. The estimated plots for the selected models along with the proposed CTBurr-XII distribution are plotted over the empirical cdf plot, and presented in the right of Fig. 3. Hence, from the figures, observed that the dataset fitted well with the proposed distribution as compared with other selected models. Hence again critically observed that three parameters proposed CTBurr-XII distribution is suitable than three parameters TBurr-XII distribution for modeling this dataset.

Table 4: MLE's of the parameters and respective SE's for selected distributions along with proposed CTBurr-XII distribution

Distribution	Parameters	Estimate	SE
CTBurr-XII	λ	0.9731	0.4603
	α	1.6102	0.3479
	β	0.2742	0.0274
TBurr-XII	λ	5.09e-07	0.5072
	α	0.4999	0.0111
	β	0.9991	0.0951
Burr-XII	α	2.3346	0.3539
	β	0.2337	0.3994

The obtained model selection criteria like Log-likelihood, AIC, AICc and BIC values are presented in Table 5. According to the obtained model selection criteria values, it has also been seen that the proposed CTBurr-XII model fitted well as compared with other competing models.

Table 5: Selection criteria values obtained for selected models

Distribution	Log-likelihood	AIC	AICc	BIC
CTBurr-XII	- 432.7963	871.5926	871.7861	880.1487
TBurr-XII	- 472.0219	950.0438	950.2374	958.5999
Burr-XII	- 453.5166	911.0332	911.1292	916.7372

The proposed CTBurr-XII distribution shows a quit better fit than any other selected model for this study. One important point should be noted here that, the proposed CTBurr-XII distribution and TBurr-XII distribution contains three parameters, but the proposed model was flexible enough to handle both of the above two datasets.

7 Concluding Remarks

This article introduced a new CTBurr-XII distribution. The important distributional properties along with the distributions of different order statistics are discussed. The proposed distribution is applied on the life-time datasets and observed the quite better fit than any other selected models used in this study. Hopefully, this distribution will be flexible enough to handle more complex real-life datasets arising in different areas of life.

Acknowledgement

The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

References

[1] Rahman, M. M., Al-Zahrani, B., Shahbaz, S. H., and Shahbaz, M. Q., Cubic Transmuted Uniform Distribution: An Alternative to Beta and Kumaraswamy Distributions, European Journal of Pure and Applied Mathematics **12**, 1106–1121 (2019).
[2] Pearson, K., Contributions to the mathematical theory of evolution, II: Skew variation in homogeneous material, Philosophical Transactions of the Royal Society **186**, 343–414 (1895).
[3] Burr, I. W., Cumulative frequency functions. The Annals of mathematical statistics **13**(2), 215-232 (1942).
[4] Singh, S., Maddala, G., A function for the size distribution of incomes, In 29. Econometrica **44**(5), 963-970 (1976).
[5] Tadikamalla, Pandu R., A Look at the Burr and Related Distributions, International Statistical Review **48**(3), 337-344 (1980).
[6] Kleiber, C., Kotz, S., Statistical Size Distributions in Economics and Actuarial Sciences, Vol. 417, John Wiley & Sons. (2003).
[7] Champernowne, D. G., The graduation of income distributions, Econometrica **20**(4), 591-614 (1952).
[8] Rodriguez, R. N., A guide to Burr Type XII distributions, Biometrika **64**(1), 129–134 (1977).
[9] Johnson, N. L., Kotz, S., Balakrishnan, N., Continuous Univariate Distributions, Vol. 1, 2nd ed., John Wiley, New York (1994).
[10] Shaw, W. T. and Buckley, I. R., The alchemy of probability distributions: beyond gram-charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map, Research report, (2009).
[11] Maurya, R. K., Tripathi, Y. M., and Rastogi, M. K., Transmuted Burr-XII distribution, Journal of the Indian Society for Probability and Statistics **18**(2), 177-193 (2017).
[12] Gradshteyn, I. S., Ryzhik, I. M., Table of integrals, series, and products, Elsevier, New York (2007).
[13] Rahman, M. M., Al-Zahrani, B., and Shahbaz, M. Q., Cubic Transmuted Weibull Distribution: Properties and Applications, Annals of Data Science **6**(1), 83-102 (2019).
[14] Rahman, M. M., Al-Zahrani, B., and Shahbaz, M. Q., New General Transmuted Family of Distributions with Applications, Pakistan Journal of Statistics and Operation Research **14**(4), 807–829 (2018).

[15] Ben Bolker, Package ‘bbmle’, Title: Tools for General Maximum Likelihood Estimation (2017).
[16] Rahman, M. M., Al-Zahrani, B., and Shahbaz, M. Q., A General Transmuted Family of Distributions, Pakistan Journal of Statistics and Operation Research **14**(2), 451–469 (2018).
[17] Rahman, M. M., Al-Zahrani, B., and Shahbaz, M. Q., Cubic Transmuted Pareto Distribution, Annals of Data Science **7**, 91-108 (2020).
[18] Abdul-Moniem I. B., Seham, M., Transmuted Gompertz distribution, Comput Appl Math J **1**(3), 88-96 (2015).
[19] Barlow, R. E., Toland, R. H., and Freeman, T., A Bayesian analysis of stress rupture life of Kevlar 49/epoxy spherical pressure vessels, In Proc. conference on applications of statistics, Marcel Dekker, New York (1984).
[20] Lee, E. T., Wang, J. W., Statistical Methods for Survival Data Analysis, John Wiley & Sons, USA (2003).