Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/100336

Predictive Inference from the Exponentiated Weibull Model Given Adaptive Progressive Censored Data

Abdullah Y. Al-Hossain[*]

Department of Mathematics, Faculty of Science, Jazan University, Jazan, Saudi Arabia

Received: 2 Nov. 2015, Revised: 2 Feb. 2016, Accepted: 3 Feb. 2016
Published online: 1 May 2016

Abstract: Adaptive progressive censoring schemes have been shown to be useful in striking a balance between statistical estimation efficiency and the time spent on a life-testing experiment. In this paper, the problem of predicting the future order statistics and future upper record values based on observed adaptive progressive Type-II censored samples from exponentiated Weibull (EW) distribution is addressed. Using the Bayesian approach and the two-sample scheme, the predictive and survival functions are derived and then the interval predictions of the future samples are obtained. Two-sample Bayesian predictive survival function can not be obtained in closed-form and so Gibbs sampling procedure is used to draw Markov Chain Monte Carlo (MCMC) samples, which are then used to compute the approximate predictive survival function. The paper also includes an illustration of our method in examples about breaking stress of carbon fibres.

Keywords: Exponentiated Weibull distribution; Two-sample prediction scheme; Adaptive Type-II Progressive Censoring Scheme; Markov chain Monte Carlo; Gibbs sampling; Posterior predictive density.

1 Introduction

Before a new product is launched to the market, life tests are often required to assess its reliability. During the testing, censoring is usually adopted to obtain the lifetime information within a reasonable timeframe. The traditional censoring schemes (type-I and type-II censoring) do not allow for units to be removed from the test at points other than the terminal point of the experiment. This allowance will be important when a compromise between reduced time of experimentation and the observations of at least some extreme lifetimes are sought. Moreover, it is important when some of the surviving units in the experiment that are removed earlier can be used for some other tests. To allow for more flexibility in removing surviving units from the test, more general censoring approaches are called for. The progressive Type-II right censoring scheme is an appealing one and has attracted much attention in the literature. For extensive reviews of literatures on progressive censoring, see [1] and the monograph by Balakrishnan and Aggarwala [2]. The design of the progressively Type-II censored experiment can be described as follows: Starting all n units at the same time, the first progressive censoring step takes place at the observation of the first failure time $X_{1:m:n}$. At this time, R_1 units are withdrawn from the experiment. Then, the experiment continues with the reduced sample size $n-R_1-1$. After observing the next failure at time $X_{2:m:n}$, R_2 units from the still operating units are withdrawn. We proceed with this censoring steps until the $m-th$ failure is observed. Then, the experiment ends. The observed failure times $X_{1:m:n}, \ldots, X_{m:m:n}$ are called progressively Type-II censored order statistics of size m observed from sample of size n with censoring scheme $(R_1,\ldots,R_m)$. A crucial assumption in the design of the progressively censored experiment is that the censoring scheme $(R_1,\ldots,R_m)$ is known in advance, which means that the integers $R_1,\ldots,R_m$ are prefixed. However, although this assumption is normally assumed in the literature, it may not be satisfied in real-life experiments since the experimenter may change the censoring numbers during the experiment (for some reasons). Therefore, it is desirable to have a model that takes into account such an adaption process. Such a model is the adaptive progressive censoring proposed by Ng et al. [3], who introduce a (prefixed) threshold parameter $T > 0$ as a control parameter in their life-time experiment.

[*] Corresponding author e-mail: aalhossain@jazanu.edu.sa

An important problem that may face the experimenter in life testing experiments is the prediction of unknown observations that belong to a future sample, based on current available sample, known as informative sample. For example, the experimenters or the manufacturers would like to have bounds for the life of their products so that their warranty limits could be plausibly set and customers purchasing manufactured products would like to know the bounds for the life of the product to be purchased. For different application areas, see [4] and [5]. As in the case of estimation, a predictor can be either a point or an interval predictor. Several researchers have considered Bayesian prediction for future observations based on several types of censored data; see [4], [6], [7], [8], [9], and [10]. Draper and Guttman [11] discussed the two-sample Bayesian prediction of the future lifetime of an item based on a Type-I hybrid censored data from an exponential distribution. Ebrahimi [12] developed the classical prediction intervals for future failures in the case of exponential distribution under Type-I hybrid censoring. Recently, Balakrishnan and Shafay [13] and Shafay and Balakrishnan [14] considered, respectively, a general form for the underlying distribution and a general conjugate prior and developed a general procedure for determining the Bayesian prediction intervals for future lifetimes based on Type-II and Type-I hybrid censored data.

There are a number of situations in which an observation is retained only if it is a record value (lower or upper), which include studies in industrial quality control experiments, destructive stress testing, meteorology, hydrology, seismology, athletic events and mining. An observation X_j will be called an upper (lower) record value if its value is greater (less) than that of all previous observations. Several authors have discussed prediction problems with the data involving record values and order statistics. In this context, the prediction of records based on records and of order statistics based on order statistics have been addressed. One may refer to, among others, [5], [15], [16], [17], [18], [19], [20], and the references contained therein. Recently, Ahmadi and Balakrishnan [21] discussed the prediction of future records (order statistics) based on order statistics (records), and derived several nonparametric prediction intervals for this purpose. Ahmadi and MirMostafaee [22] and Ahmadi et al. [23] obtained prediction intervals for order statistics as well as for the mean life time from a future sample based on observed usual records from an exponential distribution using the classical and Bayesian approaches, respectively.

The exponentiated Weibull (EW) distribution (which is denoted by EW$(\alpha,\ \theta)$) was introduced by Mudholkar and Srivastava [24]. This distribution is an extension of the well-known Weibull distribution by adding an additional shape parameter. The EW family contains distributions with nonmonotone failure rates besides a broader class of monotone failure rates. The EW distribution as a failure model is more realistic than that of monotone failure rates and plays an important role in the analysis many types of survival data. It has been well established in the literature that the EW distribution provides significantly better fits than traditional models based on the exponential, gamma, Weibull and log-normal distributions. A recent survey on the EW distribution can be found in the excellent review by Nadarajah et al. [25]. The form of the probability density function (pdf) and cumulative distribution function (cdf) of the EW distribution with two shape parameters, α and θ are given, respectively, by

$$f(x;\alpha,\theta)=\alpha\theta x^{\alpha-1}\exp(-x^{\alpha})(1-\exp(-x^{\alpha}))^{\theta-1}, \quad (1)$$

where $x>0,\ \alpha,\theta>0$, and

$$F(x;\alpha,\theta)=(1-\exp(-x^{\alpha}))^{\theta}, \quad (2)$$

Mudholkar and Hutson [26] showed that the density function of the EW distribution is decreasing when $\alpha\theta\le 1$ and unimodal when $\alpha\theta>1$. The applications of the EW distribution have been widespread. For examples, it's used for modeling of extreme value data using floods, tree diameters, firmware system failure, the survival pattern of test subjects after a treatment is administered to them, distribution for excess-of-loss insurance data, software reliability data, bus-motor failure data, mean residual life computation of (n - k + 1)-out-of-n systems and other models. For more details of these applications see [25].

Maximum likelihood estimations (besides testing of hypotheses) for the EW distribution using several sets of data are discussed by Mudholkar et al. [27]. Parametric characterizations of the density function are discussed by Mudholkar and Hutson [26] and Jiang and Murthy [28]. Other statistical properties of this distribution are discussed by Nassar and Eissa [29]. Nassar and Eissa [30] derived Bayes estimates of the two shape parameters, reliability and failure rate functions of the EW lifetime model, from complete and Type II censored samples. Pal et al. [31] introduced many properties and obtained some inferences for the three parameter EW distribution. Kim et al. [32] obtained the maximum likelihood and Bayes estimators for the two shape parameters and the reliability function of the EW model based on progressive Type-II censored samples. Some Bayesian inferences based on generalized order statistics from the EW distributions using Markov chain Monte Carlo (MCMC) methods are discussed by Jaheen and Al-Harby [33].

The novelty of this article is to apply the prediction procedure to the adaptive progressive-censored data taken from EW distribution and predicting both the future order statistics and future upper record values from the same distribution.

The rest of this paper is organized as follows: In Section 2, we describe the formulation of an adaptive type II progressive-censoring scheme as described by Ng et al. [3]. In Section 3, we cover Likelihood, Prior and Posterior

Functions. Bayes prediction for future order statistic and upper record values are presented in Section 4 and Section 5, respectively. Data analysis is provided in Section 6, and finally we conclude the paper in Section 7.

2 An Adaptive Type-II Progressive Scheme

Kundu and Joarder [34] proposed a censoring scheme called Type- II progressive hybrid censoring scheme, in which a life testing experiment with progressive Type-II right censoring scheme $(R_1,\dots,R_m)$ is terminated at a prefixed time T. However, the drawback of the Type-II progressive hybrid censoring, similar to the conventional Type-I censoring (time censoring), is that the effective sample size is random and it can turn out to be a very small number (even equal to zero), and therefore the standard statistical inference procedures may not be applicable or they will have low efficiency. Ng et al. [3] suggest an adaptive Type-II progressive censoring, in this censoring, a properly planned adaptive progressively censored life testing experiment can save both the total test time and the cost induced by failure of the units and increase the efficiency of statistical analysis. The adaptive type-II progressive censoring scheme works as follows: Suppose the experimenter provides a time T, which is an ideal total test time, but we may allow the experiment to run over time T. If $X_{m:m:n} < T$, the experiment proceeds with the pre-specified progressive censoring scheme $(R_1,\ R_2,\ \dots,\ R_m)$ and stops at the time $X_{m:m:n}$ (see Figure (1)). Otherwise, once the experimental time passes time T but the number of observed failures has not reached m, we would like to terminate the experiment as soon as possible for fixed value of m, then we should leave as many surviving items on the test as possible. Suppose J is the number of failures observed before time T, i.e.

$$X_{J:m:n} < T < X_{J+1:m:n},\ J = 0,1,\dots,m,$$

where $X_{0:m:n} \equiv 0$ and $X_{m+1:m:n} \equiv \infty$. After passed time T, we do not withdraw any items at all except for the time of the m^{th} failure where all remaining surviving items are removed. Therefore, we set $R_{J+1} = \dots = R_{J-1} = 0$ and $R_m = n-m-\sum_{i=1}^{j} R_i$, i.e., the effectively applied scheme with $j^* = \max\{J:\ X_{J::m:n} < T\}$ is

$$\left(R_1,\dots,R_{j^*},0,0,0,n-m-\sum_{i=1}^{j^*} R_i\right)$$

The basic idea of this scheme is to speed up the test as much as possible when the test duration exceeds a pre-determined threshold T. It illustrates how an experimenter can control the experiment. If he is interested in getting observations early, he will remove less units (or even none). If he wants to have larger observed failure times, he will remove more units. Figure (2) gives the schematic representation of this situation.

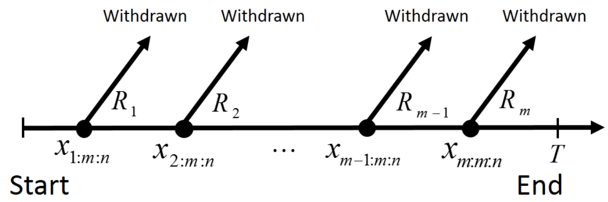

Fig. 1: Experiment terminates before time T (i.e. $X_{m:m:n} < T$)

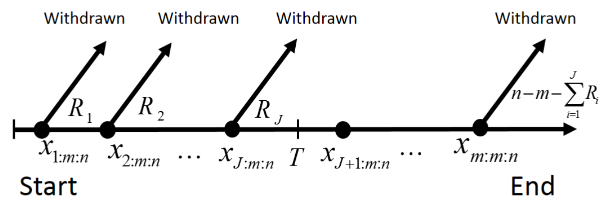

Fig. 2: Experiment terminates after time T (i.e. $X_{m:m:n} \geq T$)

The value of T plays an important role in the determination of the values of R_i and also as a compromise between a shorter experimental time and a higher chance to observe extreme failures. When $T = \infty$, the adaptive variant reduces to a progressive Type- II censoring one with censoring scheme $(R_1,\ \dots,\ R_m)$. If $T = 0$, this adaptive variant leads to a conventional Type-II censoring scheme.

Let $X_i = X^R_{i;m,n}$, $i = 1,2,\dots,m$, be an adaptive Type-II progressive censored order statistics of size m from a life test on n items whose lifetimes have distribution, with (pdf) $f(x)$, (cdf) $F(x)$ and censored scheme $\mathbf{R} = (R_1,$ $R_2,\ \dots,\ R_m)$. Given $J = j$, the likelihood function based on this data is given by

$$f_{1,2,\dots,m}(x^{\mathbf{R}}_{1;m,n}, x^{\mathbf{R}}_{2;m,n}, \dots, x^{\mathbf{R}}_{m;m,n})$$
$$= d_J \left[\prod_{i=1}^{m} f(x^{\mathbf{R}}_{i;m,n})\right] \times \left[\prod_{i=1}^{J} [1-F(x^{\mathbf{R}}_{i;m,n})]^{R_i}\right] \tag{3}$$

$$0 < x^{\mathbf{R}}_{1;m,n,k} < x^{\mathbf{R}}_{2;m,n,k} < \dots < x^{\mathbf{R}}_{m;m,n,k} < \infty,$$

where

$$d_J = \prod_{i=1}^{m} [n-i+1-\sum_{k=1}^{\min\{i-1,J\}} R_k] \text{ and } C_j = n-m-\sum_{i=1}^{J} R_i$$

3 Likelihood, Prior and Posterior Functions

Determined T and $J=j$, for the EW distribution with PDF and CDF given in (1) and (2) respectively, the likelihood function is given by

$$\ell(\alpha,\theta|\text{data}) = d_j(1-v_m^{\theta})^{C_j}\alpha^m\theta^m\prod_{i=1}^{m}u_i v_i^{\theta-1}\prod_{i=1}^{j}w_i^{R_i} \quad (4)$$

where

$$\left.\begin{array}{r} u_i(\alpha)\equiv u_i = x_i^{\alpha-1}\exp(-x_i^{\alpha})\\ v_i(\alpha)\equiv v_i = 1-\exp(-x_i^{\alpha})\\ w_i(\alpha,\theta)\equiv w_i = 1-v_i^{\theta}\end{array}\right\} \quad (5)$$

Under assumption that the two parameters α and θ are unknown, it is assumed that α and θ each have independent gamma(a,b), and gamma(c,d) priors respectively, for $a>0; b>0; c>0; d>0$, i.e.

$$\pi_1(\alpha|a,b)\propto\alpha^{a-1}e^{-b\,\alpha}\quad\text{and}\quad\pi_2(\theta|c,d)\propto\theta^{c-1}e^{-d\,\theta}.$$

The joint density function of the data, α and θ becomes:

$$\begin{aligned}\pi(\alpha,\theta|\text{data}) \propto\ & \alpha^{m+a-1}\theta^{m+c-1}\,e^{-b\,\alpha}e^{-d\theta}\\ & \times(1-v_m^{\theta})^{C_j}\prod_{i=1}^{m}u_i v_i^{\theta-1}\prod_{i=1}^{j}w_i^{R_i} \quad (6)\end{aligned}$$

The posterior density (6) can be rewritten as

$$\pi(\alpha,\theta|\text{data})\propto g_1(\alpha|\theta,\text{data})g_2(\theta|\alpha,\text{data})H(\alpha,\theta|\text{data}), \quad (7)$$

Where the conditional posterior distributions $g_1(\alpha|\theta,\text{data})$ and $g_2(\theta|\alpha,\text{data})$ of the parameters α and θ can be computed and written, respectively, as

$$g_1(\alpha|\theta,\text{data})\propto\alpha^{m+a-1}\,e^{-b\,\alpha}\prod_{i=1}^{m}u_i \quad (8)$$

$$g_2(\theta|\alpha,\text{data})\propto\theta^{m+c-1}\,e^{-d\,\theta} \quad (9)$$

and

$$H(\alpha,\theta|\text{data}) = (1-v_m^{\theta})^{C_j}\prod_{i=1}^{m}v_i^{\theta-1}\prod_{i=1}^{j}w_i^{R_i} \quad (10)$$

4 Bayesian Two-Sample Prediction for Future Order Statistics

A two-sample prediction scheme is performed as follows: Suppose that $X_{1:m:n}^{\mathbf{R}}$, $X_{2:m:n}^{\mathbf{R}},\ldots,$ $X_{m:m:n}^{\mathbf{R}}$ represents an observed informative adaptive progressively type-II censored sample of size m obtained from a sample of size n with progressive censoring $(R_1,\ \ldots,\ R_m)$ drawn from a population whose CDF is EW(α,θ) distribution (2). Suppose also that $Y_1,Y_2,\ldots,Y_n$ represents a future (unobserved) independent sample of size n drawn from the same population. Based on informative sample, the important aspect of prediction is to construct a two-sided predictive interval for the $s-th$ order statistic Y_s in the future sample, $1\le s\le n$.

The density function of Y_s for given $\alpha>0$ and $\theta>0$ is of the form

$$\begin{aligned}g_{(s)}(y_s|\alpha,\theta) =\ & D(s)\,[1-F(y_s|\alpha,\theta)]^{(n-s)}\\ & \times[F(y_s|\alpha,\theta)]^{s-1}f(y_s|\alpha,\theta), \quad (11)\end{aligned}$$

where $D(s)=\frac{n!}{(n-s)!(s-1)!}$,
here $f(.|\alpha,\theta)$ and $F(.|\alpha,\theta)$ are given respectively in (1) and (2). Substituting from (1) and (2) into (11),we obtain

$$\begin{aligned}g_{(s)}(y_s|\alpha,\theta) =\ & D(s)\alpha\theta y_s^{\alpha-1}e^{-y_s^{\alpha}}\left(1-e^{-y_s^{\alpha}}\right)^{\theta s-1}\\ & \times\left[1-\left(1-e^{-y_s^{\alpha}}\right)^{\theta}\right]^{n-s}. \quad (12)\end{aligned}$$

By using the binomial expansion, the density (12) takes the form

$$\begin{aligned}g_{(s)}(y_s|\alpha,\theta) =\ & D(s)\alpha\theta y_s^{\alpha-1}e^{-y_s^{\alpha}}\\ & \times\sum_{k=0}^{n-s}a_k(s)\left(1-e^{-y_s^{\alpha}}\right)^{\theta(k+s)-1}, \quad (13)\end{aligned}$$

where

$$a_k(s) = (-1)^k\binom{n-s}{k},\ y_s>0.$$

The Bayes predictive density function of Y_s is given by

$$g_{(s)}^{*}(y_s|\text{data}) = \int_0^{\infty}\int_0^{\infty}g_{(s)}(y_s|\alpha,\theta)\pi(\alpha,\theta|\text{data})d\alpha d\theta, \quad (14)$$

where $\pi(\alpha,\theta|\text{data})$ is the joint posterior density of α and θ as given in (7). The distribution function corresponding to the density function $g_{(s)}(y_s|\alpha,\theta)$ is

$$\begin{aligned}& G_{(s)}(y_s|\alpha,\theta)\\ & = D(s)\theta\sum_{k=0}^{n-s}a_k(s)\left[\frac{\left(1-e^{-y_s^{\alpha}}\right)^{\theta(k+s)}}{\theta(k+s)}-1\right] \quad (15)\end{aligned}$$

and the predictive distribution is

$$G_{(s)}^{*}(y_s|\text{data}) = \int_0^{\infty}\int_0^{\infty}G_{(s)}(y_s|\alpha,\theta)\pi(\alpha,\theta|\text{data})d\alpha d\theta. \quad (16)$$

It is immediate that $g_{(s)}^{*}(y_s|\text{data})$ and $G_{(s)}^{*}(y_s|\text{data})$ can not be expressed in closed form and hence it cannot be evaluated analytically.
A simulation based consistent estimator of $g_{(s)}^{*}(y_s|\text{data})$ and $G_{(s)}^{*}(y_s|\text{data})$ can be obtained by using the MCMC Gibbs sampling procedure and compute $G_{(s)}^{*}(y_s|\text{data})$ for all y. The details are explained below.

Gibbs Sampling:

We need the following theorem for further development.

Theorem 1. The density function $g_1(\alpha|\theta,\text{data})$ as given in (8) has a log-concave density function.

Proof: It's easy to proof that:

$$\frac{d^2}{d\alpha^2}\log[g_1(\alpha|\theta,data)]$$
$$= -\frac{(m+a-1)}{\alpha^2} - \sum_{i=1}^{m} x_i^{\alpha}\log^2 x_i < 0.$$

Since $g_1(\alpha|\theta,data)$ has a log-concave density, using the idea of [35], it is possible to generate a sample from $g_1(\alpha|\theta,data)$. Moreover, since $g_2(\theta|\alpha,data)$ follows $\text{gamma}(m+c,d)$, it is quite simple to generate from $g_2(\theta|\alpha,data)$.

Using Theorem 1, a simulation based consistent estimate of $G^*_{(s)}(y_s|\text{data})$ can be obtained using the following **Algorithm**:

Step 1: Generate α from $g_1(\alpha|\theta,data)$ using the method developed by [35].

Step 2: Generate θ from $g_2(\theta|\alpha,data)$.

Step 3: Repeat Step 1 and Step 2 and obtain $\{(\alpha_i,\theta_i), i=1,2,...,M\}$.

Step 4: A simulation consistent estimator of $G^*_{(s)}(y_s|\text{data})$ can be obtained as

$$\hat{G}^*_{(s)}(y_s|\text{data}) = \sum_{i=1}^{M} G_{(s)}(y_s|\alpha_i,\theta_i)W_i \tag{17}$$

where

$$W_i = \frac{H(\alpha_i,\theta_i|\text{data})}{\sum_{i=1}^{M} H(\alpha_i,\theta_i|\text{data})} \tag{18}$$

Step 5: A symmetric $100\gamma\%$ predictive interval for Y_s can be obtained by solving the non-linear equations (19) and (20), for the lower bound, L and upper bound, U:

$$P[Y_s > L|\text{data}] = 1 - G^*_{(s)}(L|\text{data}) = \frac{1+\gamma}{2}, \tag{19}$$

and

$$P[Y_s > U|\text{data}] = 1 - G^*_{(s)}(U|\text{data}) = \frac{1-\gamma}{2}, \tag{20}$$

We need to apply a suitable numerical method as they cannot be solved analytically.

5 Bayesian Prediction Interval for Future Records

Many researchers have considered the prediction of records based on records, and similarly the prediction of order statistics based on order statistics. Recently, Ahmadi and Balakrishnan [21] discussed how one can predict future usual records (order statistics) from an independent Y-sequence based on order statistics (usual records) from an independent X-sequence and developed nonparametric prediction intervals. Ahmadi and Balakrishnan [21] and Ahmadi and MirMostafaee [22] obtained prediction intervals for order statistics as well as for the mean life time from a future sample based on observed usual records from an exponential distribution using the classical and Bayesian approaches, respectively. Here, we consider the case of records and order statistics jointly and discuss the construction of prediction intervals for future records based on observed informative adaptive progressively type-II censored sample. We are interested in two-sided prediction intervals of the future records.

We assume that $(X^R_{1;m,n}, X^R_{2;m,n}, \dots, X^R_{m;m,n})$ are the observed adaptive progressively type-II censored order statistics from a population whose CDF is $\text{EW}(\alpha,\theta)$ distribution (2), and $(Z_{U(1)}, Z_{U(2)}, \dots, Z_{U(r)})$ are the first r upper records from a future sequence from the same distribution. Suppose that we are interested in the predictive density of the upper record $Z_{U(s)}, 1 \le s \le r$. The probability density function of the s^{th} upper record is given by

$$h_{(s)}(z_s|\alpha,\theta) = \frac{1}{(s-1)!}\{-\log(1-F(z_s))\}^{s-1} f(z_s), \tag{21}$$

by using (1) and (2), $h_{(s)}(z_s|\alpha,\theta)$ can be written as

$$h_{(s)}(z_s|\alpha,\theta) = \frac{1}{(s-1)!}\alpha\theta z_s^{\alpha-1}e^{-z_s^{\alpha}}\left(1-e^{-z_s^{\alpha}}\right)^{\theta-1}$$
$$\times[-\log(1-(1-e^{-z_s^{\alpha}})^{\theta})]^{s-1}, \tag{22}$$

and the distribution function corresponding to the density function $h_{(s)}(z_s|\alpha,\theta)$, is given by

$$H_{(s)}(z_s|\alpha,\theta)$$
$$= \frac{1}{(s-1)!}\int_0^{z_s}\alpha\theta t_s^{\alpha-1}e^{-t_s^{\alpha}}\left(1-e^{-t_s^{\alpha}}\right)^{\theta-1}$$
$$\times[-\log(1-(1-e^{-t_s^{\alpha}})^{\theta})]^{s-1}dt.$$
$$= -\frac{1}{(s-1)!}\int_1^{(1-(1-e^{-z_s^{\alpha}})^{\theta})}(-\log(w))^{(s-1)}dw.$$
$$= \frac{1}{(s-1)!}\left[\Gamma(s) - \Gamma(s,-\log(1-(1-e^{-z_s^{\alpha}})^{\theta}))\right]. \tag{23}$$

Using Equations (6) and (22), the Bayes predictive density of $Z_{(s)}$ is given by

$$h^*_{(s)}(z_s|\alpha,\theta) = \int_0^{\infty}\int_0^{\infty} h_{(s)}(z_s|\alpha,\theta)\pi(\alpha,\theta|\text{data})d\alpha d\theta, \tag{24}$$

and predictive distribution of $Z_{(s)}$ is then given by

$$H^*_{(s)}(z_s|\alpha,\theta) = \int_0^{\infty}\int_0^{\infty} H_{(s)}(z_s|\alpha,\theta)\pi(\alpha,\theta|\text{data})d\alpha d\theta, \tag{25}$$

Table 1: 100 observations on breaking stress of carbon fibres

3.7	2.74	2.73	2.5	3.6
3.11	3.27	2.87	1.47	3.11
4.42	2.41	3.19	3.22	1.69
3.28	3.09	1.87	3.15	4.9
3.75	2.43	2.95	2.97	3.39
2.96	2.53	2.67	2.93	3.22
3.39	2.81	4.2	3.33	2.55
3.31	3.31	2.85	2.56	3.56
3.15	2.35	2.55	2.59	2.38
2.81	2.77	2.17	2.83	1.92
1.41	3.68	2.97	1.36	0.98
2.76	4.91	3.68	1.84	1.59
3.19	1.57	0.81	5.56	1.73
1.59	2	1.22	1.12	1.71
2.17	1.17	5.08	2.48	1.18
3.51	2.17	1.69	1.25	4.38
1.84	0.39	3.68	2.48	0.85
1.61	2.79	4.7	2.03	1.8
1.57	1.08	2.03	1.61	2.12
1.89	2.88	2.82	2.05	3.65

where $\pi(\alpha,\theta|\text{data})$ is the joint posterior density of α and θ as given in (6). Since (24) and (25) do not permit explicit solutions for the prediction bounds on $Z_{(s)}$, then, as in the previuse section, using MCMC samples $\{(\alpha_i,\theta_i), i = 1,2,...,M\}$, a simulation consistent estimators of $H^*_{(s)}(z_s|\alpha,\theta)$ can be obtained as

$$\hat{H}^*_{(s)}(z_s|\text{data}) = \sum_{i=1}^{M} H_{(s)}(z_s|\alpha_i,\theta_i)W_i. \tag{26}$$

Moreover, a symmetric 100% predictive interval for $Z_{(s)}$ can be obtained by solving the non-linear equations (26), for the lower bound L and upper bound U

$$\hat{H}^*_{(s)}(L|\text{data}) = \frac{\gamma}{2}, \quad \hat{H}^*_{(s)}(U|\text{data}) = 1-\frac{\gamma}{2} \tag{27}$$

In this case it is also not possible to obtain the solutions analytically, and one needs a suitable numerical technique for solving these non-linear equations.

6 Illustrative Examples

In this section we consider a real life data set and illustrate the methods proposed in the previous sections. The data set is from [36]. A complete sample from the data gives 100 observations on breaking stress of carbon fibres (in Gba) are given in Table 1.

Qian [37] used the standard likelihood ratio test to show that the EWdistribution is acceptable for modeling the breaking stress.

Table 2: Two-sample prediction for the future order statistics Y_S

90% prediction intervals for Y_S		
Y_S	[Lower,Upper]	Length
Y_1	[0.6391,1.1815]	0.5424
Y_2	[0.9035,1.3084]	0.4049
Y_3	[1.0745,1.4287]	0.3542
Y_4	[1.2166,1.5518]	0.3352
Y_5	[1.3610,1.6964]	0.3354
95% prediction intervals for Y_S		
Y_S	[Lower,Upper]	Length
Y_1	[0.5593,1.1887]	0.6294
Y_2	[0.8400,1.3151]	0.47510
Y_3	[1.0162,1.4348]	0.4186
Y_4	[1.1674,1.5633]	0.3959
Y_5	[1.3009,1.7022]	0.4013

Example 1. In this example we consider the case when the data are adaptive Type-II progressive censored. It is assumed that we observe only the $(m = 60)$ data points and the rest are progressive censored. In this case we take $m = 60$, $T = 1.4$ and $\mathbf{R} = (\ 20,\ 0^{58}, 20)$, where the notation: $(2,3^0) = \{2,0,0,0\}$.Thus, the adaptive progressive censored sample is: 0.39, 0.85, 0.98, 1.12, 1.17, 1.18, 1.22, 1.36, 1.41, 1.57, 1.57, 1.59, 1.61, 1.61, 1.69, 1.69, 1.71, 1.73, 1.80, 1.84, 1.84, 1.87, 1.92, 2.03, 2.03, 2.12, 2.17, 2.17, 2.17, 2.35, 2.38, 2.41, 2.48, 2.48, 2.5, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.74, 2.77, 2.79, 2.81, 2.82, 2.83, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19 with $(j = 8)$. It is assumed that both the parameters are unknown. Since we do not have any prior information available, we use non-informative priors on both α and θ, $(a = b = c = d = 0.001)$. Suppose we put 50 new carbon fibres on the same test, and we wish to find predictive interval of the future sample, based on the observed sample. Now using Algorithm in Section 3 , we generate 1000 MCMC samples $\{(\alpha_i,\theta_i), i = 1,2,...,1000\}$ and based on them we compute a symmetric 90% and 95% predictive intervals for Y_s , by solving the non-linear equations (19) and (20). The results are listed in Table 2 for the future order statistics Y_s and in Table 3 for the future upper records Z_s, $s = 1,...,5$.

Example 2. In this example, we consider that $m = 60$ and $T = 3.33$ and R_i's are same as in Example 1. In this case the adaptive progressive censored sample is : 0.39, 0.81, 0.98, 1.08, 1.12, 1.18, 1.22, 1.25, 1.36, 1.47, 1.57, 1.57, 1.59, 1.61, 1.61, 1.69, 1.69, 1.71, 1.73, 1.84, 1.84, 1.87, 2.0, 2.03, 2.03, 2.05, 2.12, 2.17, 2.17, 2.35, 2.38, 2.43, 2.48, 2.48, 2.5, 2.53, 2.55, 2.55, 2.56, 2.59, , 2.73, 2.74, 2.76, , 2.81, 2.81, 2.82, 2.83, 2.87, 2.88, 2.93, 2.95, 2.97, 2.97, 3.09, 3.11, 3.15, 3.19, 3.19, 3.22, 3.22, with

Table 3: Two-sample prediction for the future upper record values Z_S

90% (HPD) credible intervals for Z_S		
Z_S	[Lower,Upper]	Length
Z_1	[1.1245,4.5879]	3.4634
Z_2	[1.8556,6.1634]	4.3078
Z_3	[2.4624,7.6056]	5.1432
Z_4	[3.0289,8.8879]	5.8590
Z_5	[3.5874,10.1545]	6.5671
95% (HPD) credible intervals for Z_S		
Z_S	[Lower,Upper]	Length
Z_1	[0.9598,5.2428]	4.2830
Z_2	[1.6539,6.8960]	5.2421
Z_3	[2.2220,8.4183]	6.1963
Z_4	[2.7516,9.8634]	7.1118
Z_5	[3.2637,11.1584]	7.8947

Table 4: Two-sample prediction for the future order statistics Y_S

90% prediction intervals for Y_S		
Y_S	[Lower,Upper]	Length
Y_1	[0.6406,1.2105]	0.5699
Y_2	[0.9223,1.3478]	0.4255
Y_3	[1.1001,1.4704]	0.3703
Y_4	[1.2550,1.6086]	0.3536
Y_5	[1.4017,1.7584]	0.3567
95% prediction intervals for Y_S		
Y_S	[Lower,Upper]	Length
Y_1	[0.5586,1.2173]	0.6587
Y_2	[0.8525,1.3526]	0.5001
Y_3	[1.0439,1.4835]	0.4396
Y_4	[1.1950,1.6125]	0.4175
Y_5	[1.3355,1.7611]	0.4256

$(j = 60,\ i.e.\ X_{m:m:n} < T)$.
Using Algorithm in Section 3 , we generate 1000 MCMC samples $\{(\alpha_i, \theta_i), i = 1, 2, ..., 1000\}$ and based on them we compute a symmetric 90% and 95% predictive intervals for Y_s , by solving the non-linear equations in (27). The results are listed in Table 4 for the future order statistics Y_s and in Table 5 for the future upper records $Z_s,\ s = 1, ..., 5$.

7 Conclusion

In this paper we have considered the Bayesian two-sample prediction problem of the exponentiated Weibull distribution based on adaptive progressive Type-II censored data. The prior belief of the model is represented by the independent gamma priors on the two shape parameters. It is observed that when the two shape parameters are unknown the prediction intervals can not be obtained in closed form. We used the Gibbs sampling technique to generate MCMC samples and obtained the predictive intervals for the future order statistics and future upper records. The details have been explained using a real life example.

Table 5: Two sample prediction for the future upper record values Z_S

90% prediction intervals for Z_S		
Z_S	[Lower,Upper]	Length
Z_1	[1.1420,4.9406]	3.7986
Z_2	[1.9251,6.6788]	4.7537
Z_3	[2.5800,8.3013]	5.7213
Z_4	[3.2124,9.7846]	6.5722
Z_5	[3.8193,11.2480]	7.4287
95% prediction intervals for Z_S		
Z_S	[Lower,Upper]	Length
Z_1	[0.9775,5.6563]	4.6813
Z_2	[1.7015,7.5947]	5.8932
Z_3	[2.3221,9.2557]	6.9336
Z_4	[2.8996,10.8781]	7.9785
Z_5	[3.4567,12.4636]	9.0067

References

[1] N. Balakrishnan, "Progressive censoring methodology: An appraisal", TEST, 16(2), 211-259 (2007).
[2] N. Balakrishnan and R. Aggarwala, "Progressive Censoring Theory, Methods and Applications", Birkhäuser: M A. Boston, (2000).
[3] H. K. T. Ng, D. Kundu, and P. S. Chan, "Statistical Analysis of Exponential Lifetimes under an Adaptive Type-II Progressive Censoring Scheme", Naval Research Logistics. 56, 687-698 (2009).
[4] E. K. Al-Hussaini, "Predicting observables from a general class of distributions", Journal of Statistical Planning and Inference, 79, 79-91 (1999).
[5] E. K. Al-Hussaini and A. A. Ahmad, "On Bayesian interval prediction of future records", TEST 12(1), 79-99 (2003).
[6] E. K. Al-Hussaini, A. M. Nigm and Z. F. Jaheen, "Bayesian prediction based on finite mixtures of Lomax components model and type I censoring", Statistics, 35(3), 259-268 (2001).
[7] I. R. Dunsmore, "The Bayesian predictive distribution in life testing models", Technometrics 16, 455-460 (1974).
[8] A. M. Nigm and H. I. Hamdy, "Bayesian prediction bounds for the Pareto lifetimemodel", Communications in Statistics-Theory and Methods, 16, 1761-1772 (1987).
[9] E. K. Al-Hussaini and Z. F. Jaheen, "Bayesian prediction bounds for the Burr type XII model", Communications in Statistics-Theory and Methods, 24, 1829-1842 (1995).
[10] M. Z. Raqab and M. T. Madi, "Bayesian inference for the generalized exponential distribution", Journal of Statistics Computations Simulation, 69, 109-124 (2005).
[11] N. Draper and I. Guttman, "Bayesian analysis of hybrid life tests with exponential failure times", Annals of the Institute of Statistical Mathematics, 39, 219-225 (1987).
[12] N. Ebrahimi, "Prediction intervals for future failures in the exponential distribution under hybrid censoring", IEEE Transactions on Reliability, 41, 127-132 (1992).
[13] N. Balakrishnan and A. R. Shafay, "One- and two-sample Bayesian prediction intervals based on Type-II hybrid censored data", Communications in Statistics-Theory and Methods, 41(9), 231-242 (2011).

[14] A. R. Shafay and N. Balakrishnan, ”One- and Two-sample Bayesian prediction intervals based on type-I hybrid censored data”, Communications in Statistics-Simulation and Computation, 41(1), 65-88 (2012).
[15] M. Ahsanullah, ”Linear prediction of record values for the two parameter exponential distribution”, Annals of the Institute of of Statistical Mathematics, 32, 363-368 (1980).
[16] I. R. Dunsmore, ”The future occurrence of records”, Annals of the Institute of Statistical Mathatics, 35, 276-277 (1983).
[17] K. S. Kaminsky and P. I. Nelson, ”Prediction of Order Statistics”, Handbook of Statistics, 17, 431-450 (1998).
[18] J. Ahmadi and M. Doostparast, ”Bayesian estimation and prediction for some life distributions based on record values”, Statistical Papers, 47, 373-392 (2006).
[19] M. Raqab and N. Balakrishnan, ”Prediction intervals for future records”, Statistical Probability Letters, 78, 1955-1963 (2008).
[20] M. Z. Raqab, ”Distribution-free prediction intervals for the future current record statistics”, Statistical Papers, 53, 429-439 (2009).
[21] J. Ahmadi and N. Balakrishnan, ”Prediction of order statistics and record values from two independent sequences”, Statistics, 44(4), 417-430 (2010).
[22] J. Ahmadi and S. M. T. K. MirMostafaee, ”Prediction intervals for future records and order statistics coming from two parameter exponential distribution”, Statistics & Probability Letters, 79(7), 977-983 (2009).
[23] J. Ahmadi, S. M. T. K. MirMostafaee and N. Balakrishnan, ”Nonparametric prediction intervals for future record intervals based on order statistics”, Statistical & Probability Letters, 80(21-22), 1663-1672 (2010).
[24] G. S. Mudholkar and D. K. Srivastava, ”Exponentiated Weibull family for analyzing bathtub failure-ratedata”, IEEE Transactions on Reliability, 42(2), 299-302 (1993).
[25] S. Nadarajah, G. M. Cordeiro and E. M. M. O. Ortega, ”The exponentiated Weibull distribution: a survey”, Statistical Papers, 54(3), 839-877 (2013).
[26] G. S. Mudholkar and A. D. Hutson, ”The exponentiated Weibull family: some properties and a flood data application”, Communications in Statistics-Theory and Methods, 25(12), 3059-3083 (1996).
[27] G. S. Mudholkar, D. K. Srivastava and M. Freimer, ”The exponentiated Weibull family: a reanalysis of the bus-motor-failure data”, Technometrics, 37(4), 436-445 (1995).
[28] R. Jiang and D. N. P. Murthy, ”The exponentiated Weibull family: a graphical approach”, IEEE Trans Reliab, 48, 68-72 (1999).
[29] M. M. Nassar and F. H. Eissa, ”On the exponentiated Weibull distribution”, Communications in Statistics-Theory and Methods, 32, 131-1336 (2003).
[30] M. M. Nassar and F. H. Eissa, ”Bayesian estimation for the exponentiated Weibull model”, Communications in Statistics-Theory and Methods, 33(10), 2343-2362 (2005).
[31] M. Pal, M. M. Ali and J. Woo, ”Exponentiated Weibull distribution”, Statistica, 66(2), 139-147 (2006).
[32] C. Kim, J. Jung and Y. Chung, ”Bayesian estimation for the exponentiated Weibull model under Type II progressive censoring”, Statistical Papers, 52, 53-70 (2011).
[33] Z. F. Jaheen and M. M. Al-Harbi, ”Bayesian Estimation for the Exponentiated Weibull Model via Markov Chain Monte Carlo Simulation”, Communications in Statistics-Theory and Methods, 40 (4), 532-543 (2011).
[34] D. Kundu and A. Joarder, ”Analysis of Type-II progressively hybrid censored data”, Computational Statistics & Data Analysis. 50(10), 2509-2528 (2006).
[35] L. Devroye, ”A simple algorithm for generating random variates with a log-concave density”, Computing, 33(3-4), 247-257 (1984).
[36] M. D. Nichols and W. J. Padgett, ”A bootstrap control chart for Weibull percentiles”, Quality and Reliability Engineering International, 22(2), 141-151 (2006).
[37] L. Qian, ”The Fisher information matrix for a three-parameters exponentiated Weibull distribution under type II censoring”, Statistical Methodology, 9(3), 320-329 (2012).

Abdullah Y. Al-Hossain received his BSc in Mathematics from KSA, MSc (in mathematical statistics) from Iowa State University, USA in 1992 and PhD (in applied statistics) from Lancaster University, UK in 2001. He is currently an Associate Professor, in Mathematics Department, Faculty of Science, Jazan University. He authored and co-authored about 35 conference and journal papers. He served at King Khalid University. He moved to Jazan University. He was the Dean of Faculty of Engineering and the Acting-Dean for The Computer Science Faculty. Then the Dean of Faculty of science and the Acting-Dean of Academic Development Deanship at Jazan University. He is currently the Head of Mathematics Department and the Dean of Faculty of science at Jazan University, Saudi Arabia.