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Roman and Inverse Roman Domination in Hexagonal Systems

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Abstract: In the recent past the field graph theory have grown exponential due to its various application in the real life systems. Among the various sub field in graph theory domination theory in graphs has its special place for its interesting and vast application in networking and other advance field of sciences. Among various types of domination, Roman dominating function is defined as $f : V(G) \to \{0,1,2\}$ satisfying the condition that $\forall u \in V, f(u) = 0,\ \exists v \in V,\ f(v) = 2$ and $d(u\ v) = 1$. If $V - D$ contains a Roman dominating function f^1. Where "D" is the set of all vertices v for which $f(v) > 0$. Then f^1 is called the Inverse Roman dominating function on a graph $G(V,E)$ with respect to Roman dominating function f, in this paper Roman Domination Number and Inverse Roman Dominating Number in hexagonal system is obtained.
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Keywords: Domination number,Inverse domination number, Roman domination number.

1 Introduction

A graph is a set of object and the relation between them. Objects and relations between them are represented by nodes and edges denoted as $G(V,E)$. Claude Berge [1] introduced the concept of domination theory in graphs. This concept was introduced in line with the game of chess and its covering pawns. Ore [2] formally published a book on graph theory in which for the first time dominating set and domination number was formally introduced. Next 15 years nothing much has been done in this concept. Only in the year 1977 Cockayne and Hedetniemi [3] published a survey paper using the notation $\gamma(G)$ for domination number which subsequently became the accepted notation. This survey paper seems to have set the motion to modern study of domination theory in graphs. Now there are thousands of research papers on this concept and is steadily growing. Many researchers have defined various types of dominating functions which accounts to be more than 100 different types among them Roman domination function is one.

A British mathematician, Ian Stewart [4] in his article titled "Defend the Roman empire" analyzed Emperor Constantine strategy to defend his empire. Emperor Constantine the great ruled the Roman Empire in 4^{th} century CE. During this period the empire was under severe attack due to various conflicts both internally and externally. In order to protect his empire the challenge faced by the emperor was to place the limited available legion in a specific location so that the entire empire is secured from the enemy's attack. Hence the emperor came up with an innovative idea to place these four legions to secure eight locations, by placing two legions in a location such that in case of conflict one legion will protect the stationed location and other will protect the adjacent location. Inspired by this article Henning and Hedetniemi [3] formally introduced the concept of Roman dominating function. Dominating function on a graph $G(V,E)$ is defined to be function $f : V(G) \to \{0,1,\}$ satisfying the condition that for every vertex $u \in V, f(u) = 0$ is adjacent to at least one vertex $v \in V$ such that $f(v) = 1$. For a real valued function $f : V \to R$, the weight of the f is $w(f) = \sum_{v \in V} f(v)$. The domination number denoted by $\gamma(G)$ is the minimum weight among all the dominating function in graph G.

Roman Dominating Function (RDF) on a graph $G(V,E)$ is defined to be function $f : V(G) \to \{0,1,2\}$ satisfying the condition that for every vertex $u \in V$, $f(u) = 0$ is adjacent to at least one vertex $v \in V$ such that

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$f(v) = 2$. For a real valued function $f : V \rightarrow R$, the weight of the f is $w(f) = \sum_{v \in V} f(v)$. The Roman Domination Number (RDN) denoted by $\gamma_R(G)$ is the minimum weight among all the RDF in graph G. If $V - D$ contains a Roman dominating function $f^1 : V(G) \rightarrow \{0,1,2\}$. "$D$" is the set of vertices v for which $f(v) > 0$. Then f^1 is called the Inverse Roman Dominating Function (IRDF) on a graph G with respect to Roman dominating function f. The Inverse Roman Domination Number (IRDN) denoted by $\gamma_R^1(G)$ is the minimum weight among all the IRDF in graph G.

ReVelle and Rosing studied the deployment of the legions through a form of zero-one integer programming [5]. The assigning of legions location is not the only problem related with Roman domination but also this concept is applicable to many similar problems in the modern world like finding the optimal location of setting up of hospitals, restaurants, fire stations, mobile towers, police stations etc.

Hexagonal systems are widely available in the nature such as primary structure in crystalline solid. Benzenoid hydro carbon, insect eyes, basalt columns, honey comb, snowflakes etc. In modern world various hexagon interconnecting network topology are studied [6,7,8,9,10,11]. The covering area of a cellular network is often visualized and approximated as a hexagonal cell network. For any undefined terms and notations, we refer Harary [12]. For preliminary results we refer [13,14,15,16,17].

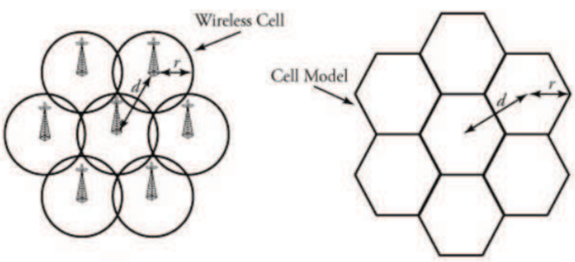
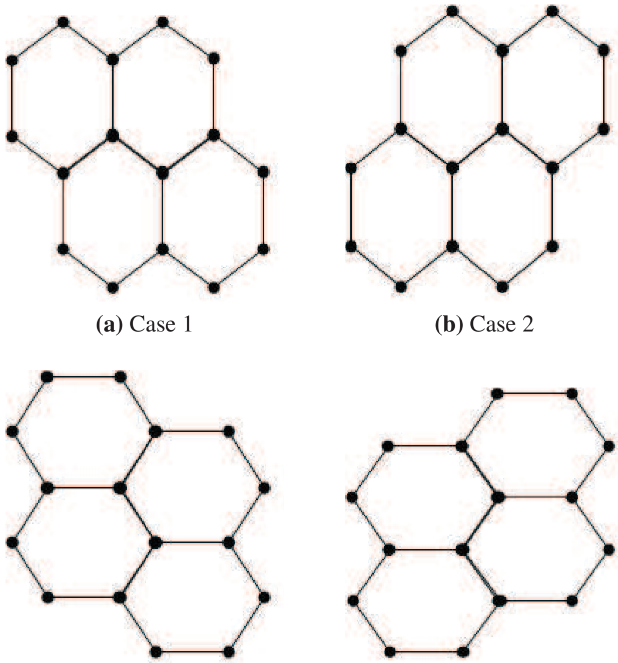


Fig. 1: Cellular networks modelled into a hexagonal system

2 Isomorphic of hexagonal system

Hexagonal system are isomorphic i.e., $H(a \times b) = H(b \times a)$.

Illustration:



(a) Case 1 **(b)** Case 2 **(c)** Case 3 **(d)** Case 4

Fig. 2: Isomorphism of Hexagonal Grid $H_{2,2}$

Rotation of case 1 to 90° results in case 4. Rotation of case 1 to 180° results in case 2. Rotation of case 2 to 90° results in case 3. Hence we can conclude that all the above four cases are isomorphic to each other.

3 Labelling of hexagonal system

Hexagonal system is denoted $H_{m,n}$, m hexagonal rows and n hexagonal column. It's also called as Hexagonal Grid with m rows and n columns. For n =1, we have a linear hexagonal chain. The zigzag line with no vertical edges is called horizontal zigzag line. These lines are denoted by L_i, $(1 \le i \le m+1)$. Vertices of L_i, $i = 1,\ m+1$ are of the form $v_{i,1},\ v_{i,2},\ \ldots,\ v_{i,2n},\ v_{i,2n+1}$. Otherwisefor each L_i,$2 \le i \le m$ we have the vertices of the form $v_{i,1},\ v_{i,2},\ \ldots,\ v_{i,2n+1},\ v_{i,2n+2}$. Therefore $H_{m,n}$ has $2(2n+1)+(m-1)(2n+2) = 2(mn+m+n)$ vertices. For $n = 2,3,\ldots$ we have double triple hexagonal chain $H_{4,3}$, with 4 hexagonal rows and 3 hexagonal columns is illustrated in Fig. 3.

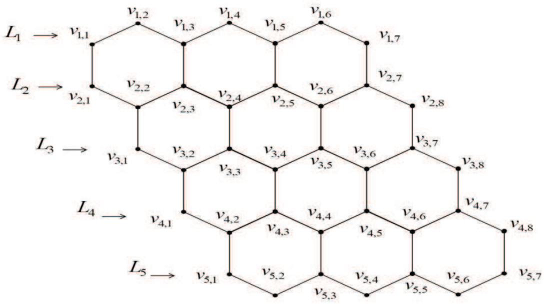


Fig. 3: Labeling for the Hexagonal Grid $H_{4,3}$

4 Main Result

Theorem 4.1. *If $G = H_{1,n}$, then $\gamma_R(G) = \gamma_R^1(G) = 2n+2$.*

Proof. The vertices are labeled as given in the Fig. 4. The linear chain hexagon has n hexagon. Without loss of generality $f(v_{i,j}) = 2$, $i = 1$ and $i = 2$ alternate labeling, $j = 2k+1$, $0 \le k \le n$. Hence the cardinality of minimal dominating set is given by $|D| = n+1$. Interchanging the value of $i = 1\ 2$ as $i = 2\ 1$ in the RDFf, we get $f^1(v_{i,j}) = 2$, $i = 2\ 1$, $j = 2k+1, 0 \le k \le n$. Hence the cardinality of minimal inverse dominating set is given by $\left|D^1\right| = n+1$.

Hence $|D| = \left|D^1\right|$, $\gamma_R(G) = \gamma_R^1(G) = 2n+2$.

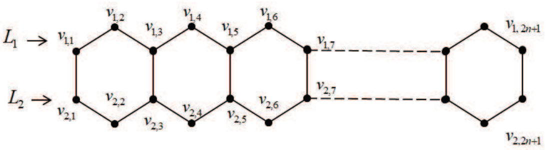


Fig. 4: Labeling for the Hexagonal Grid $H_{1,n}$

Theorem 4.2. *If $G = H_{2,n}$, then*

$$\gamma_R(G) = \begin{cases} 2n+2+\left\lceil \frac{4n}{3} \right\rceil & n = Odd \\ 2n+2+\left\lceil \frac{4n+2}{3} \right\rceil & n = Even \end{cases} \quad and$$

$$\gamma_R^1(G) = \begin{cases} 2n+3+\left\lceil \frac{4n+2}{3} \right\rceil & n = Odd \\ 2n+2+\left\lceil \frac{4n+2}{3} \right\rceil & n = Even \end{cases} .$$

Proof. Label the vertices as given in Fig. 5. In a given $H_{m,n}$, if $m = 2$, then the graph is called double hexagon with three levels.

Case 1: n is Odd:- L_1 and L_3 has $2n+1$ zig zag vertices whereas L_2 has $2n+2$ zig zag vertices. We split the graph G as $H_{1,n}$ and path graph P_n. Without loss of generality $f(v_{1,2}) = f(v_{2,2}) = 2$, $f(v_{ij}) = 2$, $i = 1$ and $i = 2$ alternate labeling, $j = 2k+1$, $2 \le k \le n-2$. We have $\gamma_R(H_{1,n}) = 2n+2$. Now the function f is only for the vertices in L_3 which is a path graph P_n. Since $f(v_{2,2}) = f(v_{2,2n+1}) = 2$, then the path P_n will have $2n$ vertices. Hence $\gamma_R(P_n) = \left\lceil \frac{2(2n)}{3} \right\rceil = \left\lceil \frac{4n}{3} \right\rceil$. For Inverse Roman domination also $\gamma_R^1(H_{1,n}) = 2n+2$, whereas now the path P_n will have $2n+1$ vertices. This can be considered as $P_1 \cup P_{2n+1}$, hence $f(P_1) = 1$ and $\gamma_R^1(P_{2n+1}) = \left\lceil \frac{2(2n+1)}{3} \right\rceil = \left\lceil \frac{4n+2}{3} \right\rceil$. Hence for $n =$ odd is proved.

Case 2: n is even:- similar to the above case we can obtain $\gamma_R(H_{1,n}) = 2n+2$, P_n will have $2n+1$ vertices. Therefore $\gamma_R(G) = \gamma_R^1(G) = 2n+2+\left\lceil \frac{4n+2}{3} \right\rceil$.

Theorem 4.3. *If $G = H_{m,n}$, then*
for, m is odd: $\gamma_R(G) = (m+1)(n+1)$,
$\gamma_R^1(G) \le (m+1)(n+1)+\min(m,n)-1$,
for, m,n even: $\gamma_R(G) \le m(n+1)+\left\lceil \frac{4n+2}{3} \right\rceil$,

$$\gamma_R^1(G) \le \begin{cases} m(n+1)+1+\left\lceil \frac{4n+2}{3} \right\rceil, & 2n+1 \cong 0 \bmod 3 \\ m(n+1)+\left\lceil \frac{4n+2}{3} \right\rceil, & 2n+1 \ne 0 \bmod 3. \end{cases}$$

Proof. Case 1: if m is odd, n = even or odd, then we have total $\frac{m+1}{2}$ linear chain of $H_{1,n}$ for the Roman dominating function f it's found that $\gamma_R(H_{1,n}) = 2n+1$. Therefore $\left(\frac{m+1}{2}\right) \times 2n+2$, $\gamma_R(G) = (m+1)(n+1)$.

For Inverse Roman dominating function f^1. For each $\left(\frac{m+1}{2}\right) \times 2n+2$ linear chain of $H_{1,n}$ in the function f^1 there are $m-1$ vertices left out. hence $f^1(v_{i,j}) = 1$. Hence $\gamma_R^1(G) \le (m+1)(n+1)+\min(m,n)-1$.

Case 2: if m, n is even, then we have total $\frac{m}{2}$ linear chain of $H_{1,n}$ for the Roman dominating function f also given $\gamma_R(H_{1,n}) = 2n+1$.
Therefore $\left(\frac{m}{2}\right) \times 2n+2 = m(n+1)$. The last row is the path with 2n + 1 vertex hence $\left\lceil \frac{4n+2}{3} \right\rceil$. Hence $\gamma_R(G) \le m(n+1)+\left\lceil \frac{4n+2}{3} \right\rceil$. As the last row is a path with $2n+1$ vertices. Hence for IRDF with $2n+1$ vertex we have the following result

$$\gamma_R^1(G) \le \begin{cases} m(n+1)+1+\left\lceil \frac{4n+2}{3} \right\rceil, & 2n+1 \cong 0 \bmod 3 \\ m(n+1)+\left\lceil \frac{4n+2}{3} \right\rceil, & 2n+1 \ne 0 \bmod 3. \end{cases}$$

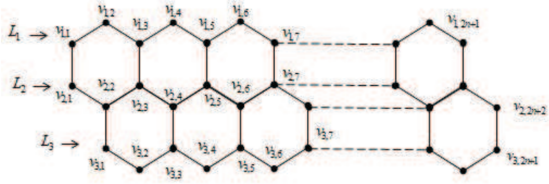


Fig. 5: Labeling for the Hexagonal Grid $H_{2,n}$

5 Honey Comb

Definition 5.1. *$H_C(n)$ is obtained from $H_C(n-1)$ by adding a layer of hexagon to the boundary of $H_C(n-1)$. Hence generating the honey comb graph $H_C(n)$ is a successive repetition of the adding one hexagonal to the boundary. $H_C(1)$, $H_C(2)$ and $H_C(3)$ are given in the below Fig. 6. If $G = H_C(n)$, then $V(G) = 6n^2$ and $E(G) = 9n^2 - 3n$, diameter of $H_C(n) = 4n-1$.*

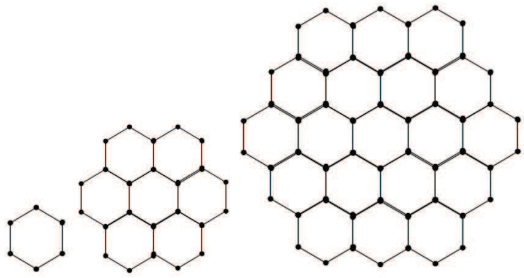


Fig. 6: Honey Comb graph $H_C(1)$, $H_C(2)$ and $H_C(3)$

Theorem 5.1. *If $G = H_C(n)$, then*
$\gamma_R(G) = \gamma_R^1(G) \leq 12n^2 - 12n + 4$.

Proof. Given $\gamma_R(H_C(1)) = 4$, rotating the Roman dominating function for $f(v) > 0$ by 90° we get,$\gamma_R^1(H_C(1)) = 4$. $H_C(n)$ is obtained from $H_C(n-1)$ by adding a layer of hexagon to the boundary of $H_C(n-1)$. Hence the following equations could be obtained as a property of sub graph $H_C(2) = H_C(1) + 6H_C(1)$, the equation is obtained considering the fact that $H_C(2)$ has one $H_C(1)$ in the middle and six $H_C(1)$ in the border of the centre $H_C(1)$. Similarly
$H_C(3) = H_C(1) + 6H_C(1) + 12H_C(1)$,
$H_C(4) = H_C(1) + 6H_C(1) + 12H_C(1) + 18H_C(1)$,
$H_C(5) = H_C(1) + 6H_C(1) + 12H_C(1) + 18H_C(1) + 24H_C(1)$. Therefore using the progression formulas, this equation can be generalised as
$H_C(n) = 1 + \left[\left(\frac{n-1}{2}\right)(6 + (n-1)6)\right] H_C(1)$.
$\gamma_R(G) = 2n^2 - 12n + 4$. Similarly rotating the Roman dominating function for $f(v) > 0$ by 90° we get Inverse Roman dominating function.

Also $G = H_C(n)$, has many over lapping vertices of the hexagon, hence we get the upper bound. Hence the result,$\gamma_R(G) = \gamma_R^1(G) \leq 12n^2 - 12n + 4$.

The above bound is not strong, hence we find an alternate bound for honey comb graph.

Theorem 5.2. *If $G = H_C(n)$, then*
$\gamma_R(H_C(n)) \leq 2 \times \gamma_R(H_{m.,n}), \gamma_R^1(H_C(n)) \leq 2 \times \gamma_R^1(H_{m.,n})$.

Proof. Here we divide the Honey Comb graph $H_C(n)$ into to equal graph by cutting in the middle as shown in the Fig. 7. This division will be more convenient to find Roman and Inverse Roman domination function. If $H_C(n)$ is divided in the middle and made into two equal parts. Then we have $H_{1,2n-1}$ linear hexagonal chain in the middle, followed by $H_{1,j}$, $k \leq j \leq 2k-1$, $n \leq k \leq 2n-1$ linear hexagonal chain above and below. Hence if we consider $H_{n,n}$ is above the middle $H_{1,2n-1}$ linear chain we get lower bound and if we consider $H_{2n-1,2n-1}$ is above the middle $H_{1,2n-1}$ linear chain we will get upper bound. From Theorem 4.3. We had already obtained the result as follows. If $G = H_{m,n}$, then for, m is odd,
$\gamma_R(G) = (m+1)(n+1)$,
$\gamma_R^1(G) = (m+1)(n+1) + \min(m,n) - 1$,
for, m,n is even, $\gamma_R(G) = m(n+1) + \left\lceil \frac{4n+2}{3} \right\rceil$,

$$\gamma_R^1(G) = \begin{cases} m(n+1) + 1 + \left\lceil \frac{4n+2}{3} \right\rceil, & 2n+1 \cong 0 \bmod 3 \\ m(n+1) + \left\lceil \frac{4n+2}{3} \right\rceil, & 2n+1 \neq 0 \bmod 3. \end{cases}$$

Hence
$\gamma_R(H_C(n)) \leq 2 \times \gamma_R(H_{m,n}), \gamma_R^1(H_C(n)) \leq 2 \times \gamma_R^1(H_{m,n})$.

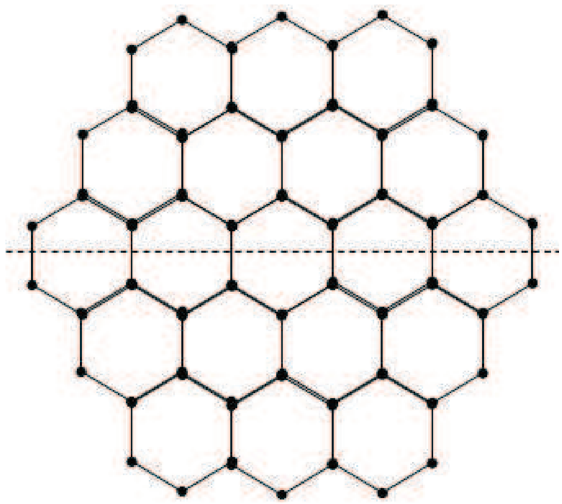


Fig. 7: Honey Comb graph $H_C(3)$

6 Conclusion

The hexagonal systems are one of the most familiar networks available in the nature. The hexagonal system is well known for its stability. In this paper labelling of hexagonal system is obtained in a unique way so that in future study, same kind of labelling could be used. Also Roman and Inverse Roman domination of Hexagonal Grid and Honey Comb graph is obtained. As these are very familiar networks, hence there are lots of scopes of the further research. Finding the application of above work in combating the network threats is still open.

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