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On A New Weibull Burr XII Distribution for Lifetime Data

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Abstract: In this article, we propose a new Weibull Burr XII distribution based on the quantile function approach given by Aljarrah et al.(2014). The new distribution is very flexible. We study a few of its statistical properties, such as, shapes and asymptotes, rth moments, sth incomplete moments, moment generating function, quantile function, median, mode, mean deviations. Stochastic ordering, moments of residual and reversed residual life are also given. The Shannon entropy, expression of ith order statistic and model parameters are estimated by maximum likelihood method. Simulation is done to see the behavior of maximum likelihood estimates. Application is given on two real life data sets to compare our model with other computative models.

Keywords: Burr XII Distribution, Quantile Function, Maximum Likelihood Estimation.

1 Introduction

In the last few years, several ways of generating new probability distribution from classic ones were developed and discussed by using a function of baseline cdf. Some well-known generators are the beta-G by Eugene et al. (2002), Kumaraswamy-G by Cordeiro and de Castro (2011), McDonald-G by Alexander et al. (2012), Kummer beta-G by Pescim et al. (2012), gamma-G by Zografos and Balakrishnan (2009), Ristic and Balakrishnan (2012) and Torabi and Montazari (2012), log-gamma-G by Amini et al. (2012), logistic-G by Torabi and Montazari (2014), beta extended Weibull-G by Cordeiro et al. (2012), exponentiated generalized-G by Cordeiro et al. (2013), Transformed-Transformer by Alzaatreh et al. (2013), exponentiated Transformed-Transformer by Alzaghal et al. (2013), Weibull-G by Bourguignon et al. (2014) and exponentiated half-logistic-G by Cordeiro et al. (2014).

Recently, Arslan et al. (2017) and Jamal et al. (2017) introduced two new families of distributions using the function of cdf $-\log[1-G(x)]$ and $\frac{G(x)}{1-G(x)}$, where $G(x)$ and $g(x)$ are baseline cdf and pdf, respectively. Now, we introduce Weibull Burr XII (WBXII) distribution using Aljarrah et al. (2014) frame work, he define the cdf of the new family as

$$F_X(x) = \int_0^{Q_Y\{F_R(x)\}} f_T(t)\,dt, \tag{1}$$

The pdf corresponding to (1.1) is given by

$$F_X(x) = f_R(x)\,Q_Y'\{F_R(x)\}\; f_T\left(Q_Y\{F_R(x)\}\right). \tag{2}$$

Let the random variables R, Y and T follows the Burr(c,k), Exponential(1) and Weibull(α,β) distributions, with cdf's $F_R(x) = 1-(1+x^c)^{-k}$, $F_Y(x) = 1-e^{-x}$ and $F_T(x) = 1-e^{-\alpha x^\beta}$. Then, we define the Weibull Burr XII distribution with cdf as under

$$F_X(x) = 1-\exp\left[-\alpha\left\{k\log(1+x^c)\right\}^\beta\right]. \tag{3}$$

The pdf corresponding to (1.3) is given by

$$f_X(x) = \frac{\alpha\beta ckx^{c-1}}{1+x^c}\left\{k\log(1+x^c)\right\}^{\beta-1} \exp\left[-\alpha\left\{k\log(1+x^c)\right\}^\beta\right]. \tag{4}$$

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Henceforth, a random variable with density (1.4) is denoted by $X \sim WBXII(\alpha,\beta,c,k)$. The survival function and hazard rate functions are

$$S_X(x) = \exp\left[-\alpha\left\{k\log(1+x^c)\right\}^{\beta}\right] \qquad (5)$$

and

$$h_X(x) = \frac{\alpha\beta c k x^{c-1}}{1+x^c}\left\{k\log(1+x^c)\right\}^{\beta-1}. \qquad (6)$$

The goal of this study is to develop a model which gives a better fit on the survival data as compared to other models, because of the fact that there is need to develop such models for life time survival data analysis. Both Weibull and Burr XII models are lifetime distributions and the new model based on the composition on these two model will have the characteristics of both models and will be more appropriate. This study generalizes the Weibull distribution with two additional parameters. Further motivations for the new model are: (i) the cdf of the new model is quite simple, which implies simple expressions for the pdf, sf and hrf; (ii) the new model is flexible with respect to the density and hazard rate shapes. The possible density shapes are right-skewed, left-skewed, symmetrical, J and reverse J shapes. This means that the WBXII density can show suitable fit to those data sets, whose histograms are similar to the new density shapes. Further, the WBXII distribution exhibits monotone [increasing (IFR) and decreasing (DFR)] and non-monotone [upside-down bathtub (UBT)] failure rate shapes to cope with almost all types of lifetime data sets; (iii) the WBXII the sub-model of WBXII given in Table 1.

The paper is presented as follows. In section 2, we give main properties of WBXII distribution, including, asymptotes and shapes, rth moment, sth incomplete moment, moment generating function, mode, quantile function, median, mean deviations, skewness and kurtosis, stochastic ordering, moments of residual and reversed residual life, Shannon entropy, distribution function and probability density function of ith order statistic, quantile spread order is given. In section 3 estimation of parameters of WBXII distribution is given by maximum likelihood method. In section 4, simulation is carried out to see the behavior of ML estimates and application on two real data sets is given in section 5.

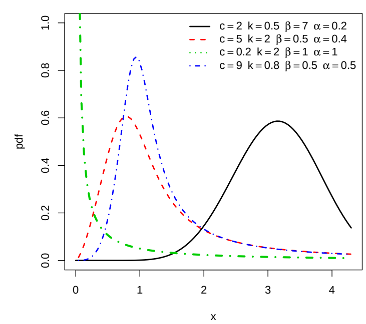
2 Main properties

In this section, we will discuss some Statistical properties of the WBXII distribution.

Table 1: A set of sub-models of the WBXII distributions

Sr. no.	α	β	c	k	Distribution
1.	-	-	1	-	Weibull Lomax distribution
2.	-	-	-	1	Weibull Log logistic distribution
3.	-	1	-	-	Exponential Burr distribution
4.	-	1	1	-	Exponential Lomax distribution
5.	-	1	-	1	Exponential Log logistic distribution
6.	-	1	1	1	Lomax distribution
7.	1	1	-	-	Burr distribution

(a)



(b)

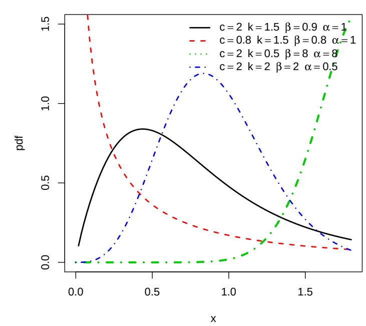


Fig. 1: Density plots of WBXII distribution.

2.1 Shapes

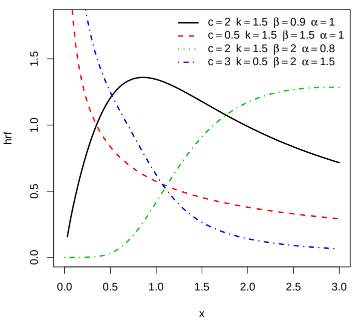
The shapes of the density and hazard rate functions can be described as.

$$\lim_{x\to 0} f_X(x) = \begin{cases} \infty & if \quad c<1 \\ k^{\beta}\alpha\beta\, e^{-\alpha k^{\beta}} & if \quad c=1 \\ 0 & if \quad c>1 \end{cases}$$

and

$$\lim_{x\to 0} h_X(x) = \begin{cases} \infty & if \quad c<1 \\ k^{\beta}\alpha\beta & if \quad c=1 \\ 0 & if \quad c>1 \end{cases}.$$

(a)



(b)

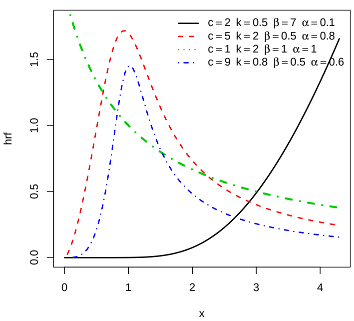


Fig. 2: hazard rate plots of WBXII distribution.

For $x \sim \infty$, we get

$$\lim_{x\to\infty} f_X(x) = \begin{cases} \infty & if \quad c<1 \\ k^{\beta}\alpha\beta\, e^{-\alpha k^{\beta}} & if \quad c=1 \\ 0 & if \quad c>1 \end{cases}$$

and

$$\lim_{x\to\infty} h_X(x) = \begin{cases} \infty & if \quad c<1 \\ k^{\beta}\alpha\beta & if \quad c=1 \\ 0 & if \quad c>1 \end{cases}.$$

Critical points of the WBXII density function are the roots of the equation:

$$\frac{c-1}{x} - \frac{cx^{c-1}}{1+x^{c}} + \frac{(\beta-1)x^{c-1}}{(1+x^{c})\,[k\log(1+x^{c})]} - \frac{\alpha\,\beta\,c\,k\,c^{c-1}}{1+x^{c}}\,[k\,\log(1+x^{c})] = 0. \quad (7)$$

The critical point of the hazard rate function are obtained from the equation:

$$\frac{c-1}{x} - \frac{cx^{c-1}}{1+x^{c}} + \frac{(\beta-1)x^{c-1}}{(1+x^{c})\,[k\log(1+x^{c})]} = 0. \quad (8)$$

Note that there may be more than one root to (2.1) and (2.2).

2.2 Useful expansion

First, we consider the following three expansions, which are useful to determine some structural properties of the WBXII distribution.
Expansion 1:
Power series expansion

$$e^{-z} = \sum_{i=0}^{\infty} \frac{(-1)^{i}}{i!} z^{i}. \quad (9)$$

Expansion 2:
Generalized binomial expansion.

$$(a-b)^{n} = \sum_{i=0}^{n} \binom{n}{i} a^{n-j}\,(-b)^{j}. \quad (10)$$

Expansion 3:
Using transformation, if $T \sim k\log(1+x^{c})$, then $X \sim \left[e^{\frac{t}{k}}-1\right]^{\frac{1}{c}}$. then we get

$$\left[e^{\frac{t}{k}}-1\right]^{\frac{1}{c}} = \sum_{i=0}^{n} \binom{\frac{1}{c}}{i} e^{\frac{t}{k}(\frac{1}{c}-i)}\,(-1)^{i}. \quad (11)$$

Equations (9)-(11) are the results to obtain some statistical properties of the WBXII distribution. Such as ordinary and incomplete moments and mean deviations.

2.3 Moments and moment generating function

In this subsection, we will discuss the r^{th} moment, s^{th} incomplete moment, quantile function, median, skewness, kurtosis and mean deviations of the WBXII distribution.
The r^{th} moments of the WBXII distribution can be obtained by using the following expression

$$E(X^{r}) = \int_{0}^{\infty} x^{r}\, f_T(t)\, dt,$$

Using the transformation, we obtain

$$E(X^{r}) = \int_{0}^{\infty} \left[e^{\frac{t}{k}}-1\right]^{\frac{r}{c}} f_T(t)\, dt, \quad (12)$$

Using the series expansion (11), we obtain

$$E(X^{r}) = \sum_{i=0}^{\infty} \binom{\frac{r}{c}}{i} (-1)^{i} \int_{0}^{\infty} e^{\frac{t}{k}(\frac{r}{c}-j)}\,\alpha\,\beta\,t^{\beta}\, e^{-\alpha t^{\beta}}\, dt,$$

Using series expansion in (9), we obtain

$$E(X^r) = \alpha\beta \sum_{i=0}^{n} \binom{\frac{r}{c}}{i} (-1)^i \sum_{j=0}^{\infty} \frac{(-\alpha)^j}{j!} \quad (13)$$

$$\int_0^{\infty} e^{\frac{t}{k}(\frac{r}{c}-j)} t^{\beta(j+1)-1} dt, \quad (14)$$

Using gamma function $\Gamma(a)b^a = \int_0^{\infty} x^{a-1} e^{-\frac{x}{b}} dx$, we obtain

$$E(X^r) = \alpha\beta \sum_{i=0}^{n} \binom{\frac{r}{c}}{i} (-1)^i \sum_{j=0}^{\infty} \frac{(-\alpha)^j}{j!} \quad (15)$$

$$\Gamma[\beta(j+1)] \left(\frac{-1}{\frac{r}{ck} - \frac{j}{k}} \right)^{\beta(j+1)}. \quad (16)$$

Similarly, the s^{th} incomplete moment of the WBXII distribution can be obtained as

$$T_s'(x) = \int_0^x \left[e^t - 1\right]^{\frac{s}{c}} f_T(t)\, dt. \quad (17)$$

Following the similar algebra as above, we have

$$T_s'(x) = \alpha\beta \sum_{i=0}^{n} \binom{\frac{s}{c}}{i} (-1)^i \sum_{j=0}^{\infty} \frac{(-\alpha)^j}{j!} \quad (18)$$

$$\gamma\left[\beta(j+1), \frac{-k\log(1+x^c)}{\frac{s}{ck} - \frac{j}{k}}\right] \left(\frac{-1}{\frac{s}{ck} - \frac{j}{k}} \right)^{\beta(j+1)}, \quad (19)$$

where $\gamma(a,x)b^a = \int_0^x x^{a-1} e^{-\frac{x}{b}} dx$.

The mode of the WBXII distribution is given by

$$\frac{d}{dx} \log f_X(x) = \frac{c-1}{x} - \frac{cx^{c-1}}{1+x^c} + \frac{(\beta-1)cx^{c-1}}{[k\log(1+x^c)](1+x^c)} \quad (20)$$

$$- \frac{\alpha\beta ckx^{c-1}[k\log(1+x^c)]^{\beta-1}}{1+x^c}. \quad (21)$$

By setting the above equation equal to zero and solving the equation numerically we can find the mode(s). The quantile function of WBXII distribution is given by

$$Q_X(u) = \left[\exp\left\{ \frac{\left[-\frac{1}{\alpha}\log(1-u)\right]^{\frac{1}{\beta}}}{k} \right\} - 1\right]^{\frac{1}{c}}. \quad (22)$$

Put $u = 0.5$, to obtain median

$$Q_X(0.5) = \left[\exp\left\{ \frac{\left[-\frac{1}{\alpha}\log 0.5\right]}{k} \right\} - 1\right]^{\frac{1}{c}}. \quad (23)$$

The mean deviations of WBXII distribution about mean and median can be obtained by using following relations.

$$D_\mu = E(|X-\mu|) = 2\mu F(\mu) - 2\mu^1(\mu) \quad (24)$$

and

$$D_M = E(|X-M|) = \mu - 2\mu^1(M), \quad (25)$$

where $\mu = E(X)$ is given by (16), $F(\mu)$ is easily calculated from (3) and $\mu^1(M)$ is obtained from (19) with $M = F^{-1}(\frac{1}{2})$, using (23).

We consider the measures of skewness and kurtosis based on quantiles to check the effects of shape parameters.

The Bowley's skewness is based on quartiles

$$B = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)}$$

and the Moor's kurtosis is based on octiles.

$$M = \frac{Q(7/8) - Q(5/8) - Q(3/8) + Q(1/8)}{Q(6/8) - Q(2/8)}$$

Where $Q(.)$ is the quantile function. The Moor's and Bowley's measure are less sensitive for outliers and exists for distribution with out moments.

2.4 Stochastic ordering

Stochastic ordering has been recognized as an important tool to reliability theory and other fields to assess comparative behaviors. Let $X_1 \sim \text{WBXII}(c,k,\alpha_1,\beta_1)$ and $X_2 \sim \text{WBXII}(c,k,\alpha_2,\beta 2)$ with c,k as the common parameters. Then, the density functions of $f(x)$ and $g(x)$ are, respectively, given by

$$f(x) = \frac{ck\alpha_1\beta_1 x^{c-1}}{1+x^c}[k\log(1+x^c)]^{\beta_1-1}$$
$$\exp\left\{-\alpha_1[k\log(1+x^c)]^{\beta_1}\right\},$$

$$g(x) = \frac{ck\alpha_2\beta_2 x^{c-1}}{1+x^c}[k\log(1+x^c)]^{\beta_2-1}$$
$$\exp\left\{-\alpha_2[k\log(1+x^c)]^{\beta_2}\right\},$$

Then, from the likelihood ratio ordering $\frac{f(x)}{g(x)}$, we have

$$\frac{f(x)}{g(x)} = \frac{\alpha_1\beta_1}{\alpha_2\beta_2}[k\log(1+x^c)]^{\beta_1-\beta_2} \exp\left[-\alpha_1 z_1^{\beta} + \alpha_2 z_2^{\beta}\right],$$

where $z = k\log(1+x^c)$ Taking derivative with respect to x, we obtain

$$\frac{d}{dx}\frac{f(x)}{g(x)} = (\beta_1-\beta_2)\frac{\alpha_1\beta_1 ckx^{c-1}}{\alpha_2\beta_2(1+x^c)}$$
$$\frac{[k\log(1+x^c)]^{\beta_1-\beta_2-1}}{\exp\left\{-\left[-\alpha_1 z_1^{\beta} + \alpha_2 z_2^{\beta}\right]\right\}}\left[-\alpha_1\beta_1 z^{\beta_1} + \alpha_2\beta_2 z^{\beta_2}\right].$$

From the above equation, we observe that if $\beta_2 < \beta_1$ then $\frac{d}{dx}\frac{f(x)}{g(x)} < 0$ and, if $\beta_1 = \beta_2 = \beta$ and $\alpha_1 < \alpha_2$, then, this implies that likelihood ratio exists between $X \leq_{lr} Y$.

2.5 Moments of residual and reversed residual life

Moments of residual life.

$$M_n(s) = \frac{1}{R(x)} \int_x^\infty (x-s)^n f_X(x)\, dx,$$

Using series expansion in (11) and the transformation, we obtain

$$M_n(t) = \frac{1}{R(t)} \sum_{i=0}^\infty \binom{n}{j} \binom{\frac{n-j}{c}}{i} (-1)^{i+j} s^j \int_t^\infty e^{\left(\frac{n-j}{ck}-\frac{i}{k}\right)t} \alpha\beta t^{\beta-1} e^{\alpha t^\beta}\, dt,$$

Using series expansion in (9), we obtain

$$M_n(t) = \frac{1}{R(t)} \sum_{i=0}^\infty \binom{n}{j} \binom{\frac{n-j}{c}}{i} (-1)^{i+j} s^j \sum_{l=0}^\infty \frac{(-\alpha)^l}{l!} \int_t^\infty e^{\left(\frac{n-j}{ck}-\frac{i}{k}\right)t} \alpha\beta t^{\beta-1} t^{\beta l}\, dt,$$

After some simplification, we obtain

$$M_n(t) = \frac{1}{R(t)} \sum_{i=0}^\infty \binom{n}{j} \binom{\frac{n-j}{c}}{i} (-1)^{i+j} s^j \sum_{l=0}^\infty \frac{(-\alpha)^l}{l!} \alpha\beta \int_t^\infty e^{\left(\frac{n-j}{ck}-\frac{i}{k}\right)t} t^{\beta(l+1)-1}\, dt,$$

Using upper gamma function, we obtain

$$M_n(t) = \frac{1}{R(t)} \sum_{i=0}^\infty \binom{n}{j} \binom{\frac{n-j}{c}}{i} (-1)^{i+j} s^j \sum_{l=0}^\infty \frac{(-\alpha)^l}{l!} \alpha\beta\, \Gamma\left(\beta(l+1), \frac{t}{-\frac{n-j}{ck}+\frac{i}{k}}\right) \left[\frac{-1}{\frac{n-j}{ck}-\frac{i}{k}}\right]^{\beta(l+1)}.$$

Moments of reversed residual life.

$$m_n(s) = \frac{1}{F(x)} \int_0^t (s-x)^n f_X(x)\, dx,$$

Using series expansion in (11) and the transformation, we obtain

$$m_n(s) = \frac{1}{F(t)} \sum_{j=0}^\infty \sum_{i=0}^\infty \binom{n}{j} \binom{\frac{j}{c}}{i} (-1)^{i+j} s^{n-j} \int_0^t e^{t\left(\frac{j}{ck}-\frac{i}{k}\right)} \alpha\beta t^{\beta-1} e^{-\alpha t^\beta}\, dt,$$

Using series expansion in (9), we obtain

$$m_n(s) = \frac{1}{F(t)} \sum_{j=0}^\infty \sum_{i=0}^\infty \binom{n}{j} \binom{\frac{j}{c}}{i} (-1)^{i+j} s^{n-j} \sum_{l=0}^\infty \alpha\beta \frac{(-\alpha)^l}{l!} \int_0^t t^{\beta(l+1)-1} e^{t\left(\frac{j}{ck}-\frac{i}{k}\right)}\, dt.$$

Using incomplete gamma function, we obtain

$$m_n(s) = \frac{1}{F(t)} \sum_{j=0}^\infty \sum_{i=0}^\infty \binom{n}{j} \binom{\frac{j}{c}}{i} (-1)^{i+j} s^{n-j} \sum_{l=0}^\infty \alpha\beta \times \frac{(-\alpha)^l}{l!} \gamma\left(\beta(l+1), \frac{t}{\left(\frac{j}{ck}-\frac{i}{k}\right)}\right) \left(\frac{1}{\frac{j}{ck}-\frac{i}{k}}\right)^{\beta(l+1)}.$$

2.6 Shannon entropy

Entropy refers to disorder or uncertainty. Shannon entropy provides an absolute limit on the best possible average length of lossless encoding or compression of an information source. Using Theorem 2 of Aljarrah et al.(2104), we have

$$\eta_x = \eta_R - E\left[\log Q'_Y\{F_R(x)\}\right] - E\{\log F_T(t)\},$$

science $R \sim$ Burr(c,k), then

$$Q'_Y\{F_R(x)\} = \frac{d}{dx}\left[k\log(1+x^c)\right] = \frac{ckx^{c-1}}{1+x^c}$$

and

$$-E\left[\log Q'_Y\{F_R(x)\}\right] = -E\left[\log\frac{ckx^{c-1}}{1+x^c}\right] = -\log(ck) - (c-1)E(\log x) + E\left[\log(1+x^c)\right].$$

Consider $-E\{\log F_T(t)\}$ is the Shannon entropy of Weibull distribution because $T \sim \text{Weibull}(\alpha,\beta)$, then

$$\eta_T = 1 - \log(\alpha\beta) - \left(1-\frac{1}{\beta}\right)\left[\Gamma'(1) - \log\alpha\right]$$

and η_R is the Shannon entropy of Burr distribution. because $R \sim$ Burr(c,k), then

$$\eta_R = 1 + \frac{1}{k} = \log(ck) - \left(1-\frac{1}{c}\right)\left[\Gamma'(1) - \frac{\Gamma'(k)}{\Gamma(k)}\right].$$

Using the quantities η_T and η_R, we obtain the Shannon entropy of WBXII distribution is

$$\eta_x = \eta_R + \eta_T - \log(ck) - (c-1)E(\log x) + E\left[\log(1+x^c)\right].$$

2.7 Order statistics

Let $X_1, X_2, \ldots, X_n$ is an random sample of size 'n' of X and $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n} \leq$ denotes the corresponding order statistics obtained from a sample, then the cdf and pdf of ith order statistic are given by.

$$F_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} \frac{(-1)^j}{j+i} F(x)^{j+i},$$

and

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j f(x) F(x)^{j+i-1}.$$

Using (3) and (4) we get

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j \frac{\alpha \beta c k x^{c-1}}{1+x^c}$$
$$\{k \log(1+x^c)\}^{\beta-1} \exp\left[-\alpha \{k \log(1+x^c)\}^{\beta}\right]$$
$$\left[1-\exp\left[-\alpha \{k \log(1+x^c)\}^{\beta}\right]\right]^{j+i-1}.$$

Using binomial expansion in (2.3), we obtain

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \sum_{l=0}^{j+i-1} \binom{n-i}{j} \binom{j+i-1}{l}$$
$$(-1)^{j+l} \frac{\alpha \beta c k x^{c-1}}{1+x^c} \{k \log(1+x^c)\}^{\beta-1} \exp[-\alpha(l+1)$$
$$\{k \log(1+x^c)\}^{\beta}\Big],$$

rewriting the above expression, we have

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{l=0}^{j+i-1} a_l \, g(x; \alpha(l+1), \beta, c, k),$$

similarly

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{l=0}^{j+i-1} b_l \, G(x; \alpha l, \beta, c, k),$$

where the constants a_l and b_l are given by

$$a_l = \sum_{j=0}^{n-i} \binom{n-i}{j} \binom{j+i-1}{l} (-1)^{j+l}$$

and

$$b_l = \sum_{j=0}^{n-i} \binom{n-i}{j} \binom{j+i-1}{l} \frac{(-1)^{j+l}}{j+i},$$

where

$$g(x; \alpha(l+1), \beta, c, k) = \frac{\alpha \beta c k x^{c-1}}{1+x^c} \{k \log(1+x^c)\}^{\beta-1}$$
$$\exp\left[-\alpha(l+1) \{k \log(1+x^c)\}^{\beta}\right]$$

and

$$G(x; \alpha, \beta, c, k) = \exp\left[-\alpha \{k \log(1+x^c)\}^{\beta}\right].$$

2.8 Quantile spread order

The quantile spread (QS) of random variable $X \sim$WBXII(α, β, c, k) with cdf (1.3) is given by

$$QS_X(p) = \left[F^{-1}(p)\right] - \left[F^{-1}(1-p)\right] \ \forall \ p \in (0.5, 1)$$

and this implies

$$QS_X(p) = \left[S^{-1}(1-p)\right] - \left[S^{-1}(p)\right],$$

where

$$F^{-1}(p) = S^{-1}(1-p)$$

and $S = 1 - F$ is the survival function. The QS of a probability distribution describes how the probability mass is placed symmetrically about its median and hence can be used to formalize concepts such as peakedness and tail weight traditionally associated with kurtosis. This way, it allows us to separate concepts of kurtosis and peakedness for asymmetric models. Let X_1 and X_2 be two random variables follow WBXII distribution with quantile spreads QS_{X_1} and QS_{X_2}, respectively. Then X_1 is called smaller than X_2 in quantile spread order, denoted as $X_1 \leq_{QS} X_2$, if

$$QS_{X_1}(p) \leq QS_{X_2}(p), \forall p \in (0.5, 1).$$

The following properties of the quantile spread order can be determined

- The order $\leq_{QS}$ is *location-free*, i.e.,

$$X_1 \leq_{QS} X_2 \text{ if } (X_1 + c) \leq_{QS} X_2 \text{ for any real } c.$$

- The order $\leq_{QS}$ is *dilative*, i.e,

$$X_1 \leq_{QS} aX_1 \text{ whenever } a \geq 1 \text{ and } X_2 \leq_{QS} aX_2, \ \forall \ a \geq 1.$$

- Assume F_{X_1} and F_{X_2} are symmetric, then

$$X_1 \leq_{QS} X_2 \text{ if, and only if } F_{X_1}^{-1}(p) \leq F_{X_2}^{-1}(p), \ \forall \ p \in (0.5, 1).$$

- The order $\leq_{QS}$implies ordering of *the mean absolute deviation around the median*, MAD,

$$MAD(X_1) = E\left[|X_1 - \text{Median}(X_1)|\right]$$
and
$$MAD(X_2) = E\left[|X_2 - \text{Median}(X_2)|\right],$$

i.e.,

$$X_1 \leq_{QS} X_2 \text{ implies } MAD(X_1) \leq_{QS} MAD(X_2).$$

- Finally

$$X_1 \leq_{QS} X_2 \text{ if, and only if } -X_1 \leq_{QS} -X_2.$$

3 Estimation

We determine the maximum likelihood estimates (MLEs) of the model parameters of the WBXII distribution for complete samples only. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the WBXII distribution. First, we express the log-likelihood function for complete samples with the vector of parameter $\Theta = (c, k, \alpha, \beta)^T$ as

$$l(\Theta) = n\log(ck\alpha\beta) + (c-1)\sum_{i=1}^{n}\log x_i - \sum_{i=1}^{n}\log(1+x_i^c)$$
$$+(\beta-1)\sum_{i=1}^{n}\log\left[k\log(1+x_i^c)\right] - \alpha\sum_{i=1}^{n}\left[k\log(1+x_i^c)\right]^{\beta}.$$

The associated nonlinear log-likelihood system $\frac{\partial}{\partial\Theta} l(\Theta) = 0$ is

$$\frac{\partial}{\partial\alpha} l(\Theta) = \frac{n}{\alpha} - \sum_{i=1}^{n}\left[k\log(1+x_i^c)\right]^{\beta},$$
$$\frac{\partial}{\partial\beta} l(\Theta) = \frac{n}{\beta} + \sum_{i=1}^{n}\log\left[k\log(1+x_i^c)\right]$$
$$-\alpha\sum_{i=1}^{n}\left[k\log(1+x_i^c)\right]^{\beta}\log\left[k\log(1+x_i^c)\right],$$
$$\frac{\partial}{\partial k} l(\Theta) = \frac{n}{k} + (\beta-1)\frac{n}{k} - \alpha\beta\sum_{i=1}^{n}\left[k\log(1+x_i^c)\right]^{\beta-1}$$
$$\log(1+x_i^c)$$

and

$$\frac{\partial}{\partial c} l(\Theta) = \frac{n}{c} + \sum_{i=1}^{n}\log x_i - \sum_{i=1}^{n}\frac{x_i^c\log x_i}{1+x_i^c}$$
$$+(\beta-1)\sum_{i=1}^{n}\frac{x_i^c\log x_i}{\left[k\log(1+x_i^c)\right](1+x_i^c)}$$
$$-\alpha\beta k\sum_{i=1}^{n}\left[k\log(1+x_i^c)\right]^{\beta-1}\frac{x_i^c\log x_i}{1+x_i^c}.$$

Setting these equations to zero and solving them simultaneously yields the MLEs of the model parameters.

Under standard regularity conditions, the multivariate normal $N_4(0, J(\hat{\Theta})^{-1})$ distribution, where $J(\hat{\Theta})^{-1}$ is the observed information evaluated at $\hat{\Theta}$, can be used to construct approximate confidence intervals for the model parameters. Further, we can compare the WBXII model with any of its special models using likelihood ratio (LR) statistics.

4 Simulation study

In this Section we perform simulation study using the WBXII distribution. To see the performance of MLE's of this distribution, we generate 1,000 samples of sizes 20, 50 and 100 from the WBXII distribution using its quantile function. We use the R program the for random generation and use the optim-CG routine to obtain MLEs. The results of the simulation are reported in Table 1. From this Table we observe that the estimates approach true values as the sample size increases whereas the standard deviations of the estimates decrease.

5 Application

In this Section we present two applications of the new WBXII distribution to real data for illustrative purposes. These applications will show the flexibility of the new distribution in modeling positive data. All the computations presented in this section were done using maxLike routine in R program. The first real data set represents the remission times (in months) of a random sample of 128 bladder cancer patients (Lee and Wang, 2003): 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69. This data set is well-known as unimodal hrf shaped and this data set analyzied by Lemonte (2013) and Cordeiro et al. (2016).

Secondly, we consider the number of failures for the air conditioning system of jet airplanes. This data set were analyzied by Cordeiro and Lemonte (2011), Huang and Oluyede (2014) and, de Andrade et al. (2016). The data are: 194, 413, 90, 74, 55, 23, 97, 50, 359, 50, 130, 487, 57, 102, 15, 14, 10, 57, 320, 261, 51, 44, 9, 254, 493, 33, 18, 209, 41, 58, 60, 48, 56, 87, 11, 102, 12, 5, 14, 14, 29, 37, 186, 29, 104, 7, 4, 72, 270, 283, 7, 61, 100, 61, 502, 220, 120, 141, 22, 603, 35, 98, 54, 100, 11, 181, 65, 49, 12, 239, 14, 18, 39, 3, 12, 5, 32, 9, 438, 43, 134, 184, 20, 386, 182, 71, 80, 188, 230, 152, 5, 36, 79, 59, 33, 246, 1, 79, 3, 27, 201, 84, 27, 156, 21, 16, 88, 130, 14, 118, 44, 15, 42, 106, 46, 230, 26, 59, 153, 104, 20, 206, 5, 66, 34, 29, 26, 35, 5, 82, 31, 118, 326, 12, 54, 36, 34, 18, 25, 120, 31, 22, 18, 216, 139, 67, 310, 3, 46, 210, 57, 76, 14, 111, 97, 62, 39, 30, 7, 44, 11, 63, 23, 22, 23, 14, 18, 13, 34, 16, 18, 130, 90, 163, 208, 1, 24, 70, 16, 101, 52, 208, 95, 62, 11, 191, 14, 71.

By using these data set, we fit the WBXII, beta Burr XII (BBXII) (Paranaiba et al., 2011), Kumaraswamy BXII (KwBXII) (Paraniaba et al., 2013), another Weibull BXII distribution (WBXII-2) (Bourguignon et al., 2014) and BXII models. The model selection is applied using

Table 2: Emprical mean, SD for the selected WBXII distributions

Parameter				N=20				N=50				N=100			
$\hat{\alpha}$	$\hat{\beta}$	$\hat{c}$	$\hat{k}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{c}$	$\hat{k}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{c}$	$\hat{k}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{c}$	$\hat{k}$
1	0.5	0.5	0.5	1.043	0.614	0.516	0.545	1.012	0.531	0.510	0.512	1.003	0.517	0.501	0.502
				(0.148)	(0.270)	(0.264)	(0.149)	(0.084)	(0.112)	(0.146)	(0.082)	(0.055)	(0.072)	(0.095)	(0.054)
1	1	1	1	1.047	1.506	1.007	1.062	1.017	1.293	0.958	1.025	1.010	1.095	0.997	1.013
				(0.136)	(0.832)	(0.680)	(0.139)	(0.088)	(0.574)	(0.397)	(0.096)	(0.055)	(0.312)	(0.292)	(0.058)
1	2	2	0.5	1.001	2.285	2.240	0.513	1.002	2.143	2.066	0.512	1.002	2.096	2.014	0.510
				(0.040)	(0.650)	(1.034)	(0.156)	(0.028)	(0.431)	(0.625)	(0.114)	(0.020)	(0.328)	(0.446)	(0.083)
0.5	1	2	0.5	0.509	1.199	2.187	0.512	0.498	1.041	2.128	0.499	0.499	1.023	2.061	0.500
				(0.120)	(0.364)	(1.070)	(0.137)	(0.066)	(0.165)	(0.667)	(0.068)	(0.044)	(0.116)	(0.462)	(0.045)
5	0.5	5	0.5	5.058	0.576	4.811	0.711	5.019	0.536	4.887	0.577	5.008	0.517	4.942	0.536
				(0.129)	(0.178)	(0.725)	(0.420)	(0.051)	(0.099)	(0.650)	(0.212)	(0.028)	(0.058)	(0.401)	(0.136)
2	2	0.5	0.5	1.998	2.172	0.647	0.492	1.999	2.079	0.555	0.498	1.999	2.043	0.525	0.500
				(0.022)	(0.682)	(0.401)	(0.180)	(0.015)	(0.482)	(0.213)	(0.126)	(0.011)	(0.357)	(0.141)	(0.094)
2	2	2	2	2.030	2.481	1.965	2.028	2.030	2.206	2.081	2.027	2.021	2.145	2.061	2.016
				(0.158)	(0.777)	(0.789)	(0.282)	(0.140)	(0.675)	(0.785)	(0.231)	(0.109)	(0.600)	(0.659)	(0.186)
1	2	3	4	1.277	2.455	3.314	4.148	0.967	2.172	2.955	3.986	1.019	2.053	3.034	4.008
				(0.826)	(0.674)	(1.489)	(0.493)	(0.294)	(0.374)	(0.576)	(0.151)	(0.177)	(0.246)	(0.309)	(0.088)
4	3	2	1	4.000	3.064	2.217	1.011	3.999	3.004	2.115	0.995	3.999	2.969	2.078	0.993
				(0.020)	(0.677)	(0.750)	(0.129)	(0.007)	(0.494)	(0.451)	(0.085)	(0.005)	(0.358)	(0.346)	(0.060)

Table 3: ℓ, MLEs and their standard errors (in parentheses) for the remission data.

Distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{c}$	$\hat{k}$	$\hat{\ell}$
WBXII	0.020	3.407	0.612	2.011	-410.7942
	(0.285)	(0.835)	(0.226)	(8.403)	
KwBXII	12.022	41.101	0.284	1.261	-410.9813
	(6.837)	(6.848)	(0.088)	(0.632)	
BBXII	69.574	72.467	0.169	0.789	-415.2973
	(0.472)	(0.356)	(0.011)	(0.015)	
WBXII-2	4989.657	2.460	0.968	0.014	-415.6878
	(4.203)	(0.257)	(0.187)	(0.006)	
BXII			2.334	0.233	-453.5166
			(0.355)	(0.040)	

the estimated log-likelihood , Kolmogorov-Smirnov (K-S) statistics, Akaike information criterion (AIC), Consistent Akaike information criteria (CAIC), Bayesian information criterion (BIC), and Hannan-Quinn information criterion (HQIC). AIC, CAIC, BIC and HQIC are by given by$AIC = -2\widehat{\ell} + 2p$, $CAIC = -2\widehat{\ell} + 2pn/(n-p-1)$, $HQIC = -2\widehat{\ell} + 2p\log[\log(n)]$ and $BIC = -2\widehat{\ell} + p\log(n)$, p is the number of parameters and n is the sample size. The results of these applications are listed in Tables 1-4. These results show that the WBXII distribution has the lowest AIC, CAIC, BIC, HQIC and K-S values and has the biggest estimated log-likelihood and p-value of the K-S statistics among all the fitted models. So it could be chosen as the best model under these criteria.

The histograms of these datas and the estimated pdfs and cdfs of the application models are displayed in Figures 1 and 2. It is clear from the these figures we show that the WBXII model provides the good fit to these data set.

Table 4: Goodness-of-t statistics of the remission data

Distribution	AIC	CAIC	BIC	HQIC	K-S	K-S P-value
WBXII	829.588	829.913	840.996	834.223	0.047	0.937
KwBXII	829.962	830.287	841.370	834.597	0.0503	0.9023
BBXII	838.594	838.919	850.002	843.229	0.0628	0.6932
WBXII-2	839.375	839.700	850.783	844.010	0.0575	0.7910
BXII	911.033	911.129	916.737	913.350	0.2507	0.0000

Table 5: ℓ, MLEs and their standard errors (in parentheses) for the air conditioning system data.

Distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{c}$	$\hat{k}$	$\hat{\ell}$
WBXII	1.0319	4.3412	0.5620	0.3909	-1032.971
	(2.6695)	(0.7168)	(0.2419)	(0.2811)	
BBXII	92.0291	38.0973	0.1414	1.2278	-1036.3830
	(0.2065)	(0.0854)	(0.0085)	(0.0284)	
WBXII-2	2552.175	3.2464	1.119	0.0176	-1049.9730
	(2.1016)	(0.1924)	(0.2924)	(0.0058)	
BXII			14.5113	0.0177	-1173.0030
			(7.6410)	(0.0094)	

Table 6: Goodness-of-fit statistics of the remission data

Distribution	AIC	CAIC	BIC	HQIC	K-S	K-S P-value
WBXII	2073.9410	2074.1600	2086.8870	2079.1860	0.0438	(0.8639)
BBXII	2080.7660	2080.9850	2093.7120	2086.0110	0.0448	(0.8449)
WBXII-2	2107.9460	2108.1650	2120.8920	2113.1910	0.0494	(0.7495)
BXII	2350.0060	2350.0710	2356.4790	2352.6290	0.3708	(0.0000)

6 Conclusion

We propose and study WBXII distribution and obtain some mathematical properties, such as quantile function, ordinary and incomplete moment, mean deviations, stochastic ordering, Moments of residual and reversed residual life, Shannon entropy, quantile spread order. The model parameters are estimated by the method of maximum likelihood. Simulations are performed to check the asymptotic properties of the estimates. Two applications on real data set are presented to illustrate the

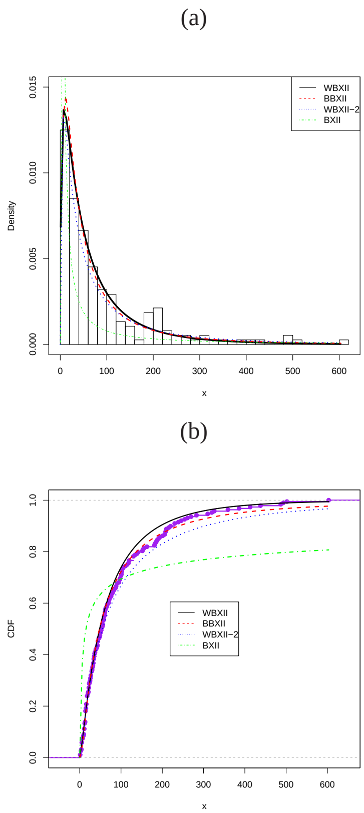


Fig. 3: Estimated pdf and cdf for data set 1.

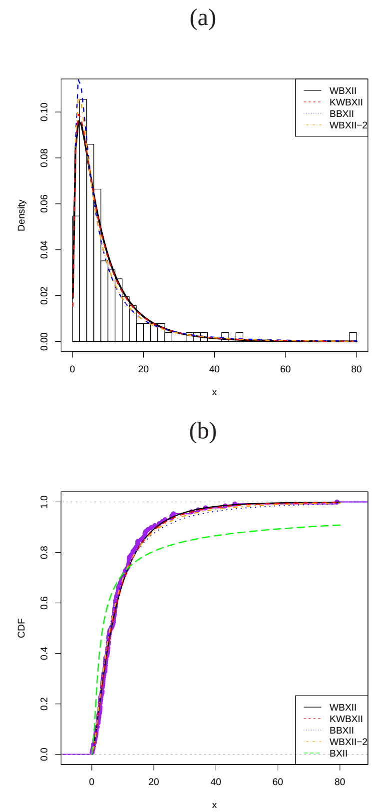


Fig. 4: Estimated pdf and cdf for data set 2.

potentiality of the proposed model. We expect the utility of the proposed model in different fields especially in lifetime and reliability.

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