Numerical Investigation of Fractional Order Buruli Ulcer Model

Mohamed A. Hafez [1,2], *Mohamed Qasymeh* [3], *Ali Akgül* [4,5,*], *Zafar Iqbal* [6], *Nauman Ahmed* [6], *Umaima Akhtar*[6], and *Betty Wan Voon* [7]

[1] Faculty of Engineering and Quantity Surviving, INTI International University Colleges, Nilai, Malaysia
[2] Faculty of Management, Shinawatra University, Pathum Thani, Thailand
[3] Electrical and Computer Engineering Department, Abu Dhabi University, Abu Dhabi, United Arab Emirates
[4] Department of Electronics and Communication Engineering, Saveetha School of Engineering, SIMATS, Chennai, Tamilnadu, India
[5] Department of Mathematics, Art and Science Faculty, Siirt University, 56100 Siirt, Turkey
[6] Department of Mathematics and Statistics, The University of Lahore, Lahore, Pakistan
[7] Department of Civil Engineering, (COE), Universiti Tenaga Nasional, Kajang, Malaysia

Abstract: In this study, the transmission of buruli ulcer disease was examined. For this purpose, a classical Buruli ulcer model is converted into a fractional-order epidemic model by introducing the Caputo fractional differential operator. It was found that the system has two equilibrium points: disease-free and endemic equilibrium. Furthermore, the stability of the model is observed using a Jacobian matrix. Subsequently, the Grunwald Letnikov approximation is hybridized with a non-standard finite difference design to solve the problem. Because the state variables describe the number of individuals, they cannot be negative. The main properties of the numerical design, that is, the positivity, boundedness, and convergence towards the true equilibrium points, were investigated via simulations. Numerical graphs reflect the reliability and efficacy of the proposed numerical template.

Keywords: Buruli ulcer model, Caputo fractional differential operator, Grunwald Letnikov approximation.

1 Introduction

Clancey discovered recurrent skin ulcers in Uganda in 1964 that was caused by a unique *M.ulcerans*-like mycobacterium termed "*Mycobacterium Buruli*". However, it was later shown that *Mycobacterium ulcerans*, which was first described in Australia, contributes to buruli ulcers [1]. The British physician Albert Cook remarked on the first case of persistent skin ulcers in *The Mengo Hospital Notes* at the end of the nineteenth century, which was consistent with the pathophysiology of *Mycobacterium ulcerans* infection [2]. MacCallum and his colleagues determined the characteristics of related skin ulcers in six individuals in Australia in 1948 [3]. *M. ulcerans*, an acid-fast mycobacterium, was ultimately identified as the causative agent of these ulcers [4]. The corrosive fast mycobacterium *M. ulcerans* was ultimately shown to be the culprit after isolation from these ulcers. The main separation of the extremely slowly growing mycobacterium in culture was accomplished by concurrently hatching culture plates in a malfunctioning hatchery at 30–33 °C, which is the microorganism's low ideal development temperature. This temperature increase is believed to be the key factor in skin tropism and the restricted fundamental dissemination of *M. ulcerans* contamination. In the Democratic Republic of the Congo, a substantial case series of 170 persons with necrotic skin ulcers caused by a corrosive rapid mycobacterium was discovered in the 1940s and 1950s [5]. Due to the high prevalence of *M. ulcerans* contaminations that were discovered in the 1950s and 1960s in a geologically exceptionally restricted area of the populated Buruli County near the Nile River in Uganda, the disease came to be more commonly known as Buruli ulcer (BU) [6,7]. Unlike infectious diseases, such as leprosy and tuberculosis, which are transmitted from person to person, *M. ulcerans* is thought to be spread by ambient contact. It is rare for *M. ulcerans* to be directly transmitted to humans. The only thing recorded to have occurred was a

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* Corresponding author e-mail: aliakgul00727@mail.com

human bite. The hypothesis in this case is that *M. ulcerans* was already on the patient's skin surface from an environmental source and had been injected into the skin by the bite of the playmate [8].

The symptoms of Buruli ulcer include swelling of the skin and destruction of soft tissues [9]. One or more slow-growing, generally painless ulcers. The signs that gradually appear are as follows: a spot of skin that resembles a spider or mosquito bite appears (most commonly on the limbs) [10,11], over the course of a few days or weeks, the spot gets bigger and a crusty, non-healing scab may develop in the area. People who are sick should visit a doctor and receive antibiotics (medicines that can help stop the disease). If these antibiotics are not administered soon after getting sick, the disease can sometimes lead to deformity, functional disability (such as limited joint movement), bone infection, secondary bacterial infection of the skin lesions. To fight the infection, the immune system goes into overdrive, releasing chemicals into the bloodstream. This sets off a chain reaction resulting in serious inflammatory reaction throughout the body [12,13,14,33].

The suggested antibiotic therapy includes rifampicin and clarithromycin as oral medication [15]. Up to 20% of patients may experience paradoxical effects, however, this does not always indicate that antibiotic therapy failed. To treat considerable tissue necrosis brought on by severe paradoxical reactions, oral corticosteroids (prednisolone 0.5–1.0 mg/kg daily, tapered over 4–8 weeks) may be utilized [16,17]. Instead of the standard 8 weeks, antibiotics may be provided for up to 12 weeks if the lesion affects deeper structures (such as a bone or joint) or is linked to prednisolone therapy for a severe paradoxical reaction. Antibiotics need to be taken for at least 4 weeks before surgery to close the wound permanently [18].

2 Preliminaries

This section is devoted to the basic definitions and lemmas that were helpful in this study.

Definition 1.*The fractional derivative in Caputo sense of order $\delta \in (0,1)$ is given by*

$$_{0}^{c}D_{t}^{\delta}x(t) = \frac{1}{\Gamma(m-\delta)} \int_{0}^{t} \frac{g^{(m)}(\delta)}{(t-\delta)^{\delta-m+1}} d\delta, \quad m = [\delta]+1$$

where x is an integrable function and [0] *is the extreme fraction function [19].*

In addition, the corresponding fractional integral of order δ with $Re(\delta) > 0$ is given by

$$_{0}^{c}I_{t}^{\delta}x(t) = \frac{1}{\Gamma(\delta)} \int_{0}^{t} (t-\delta)^{\delta-1}x(\delta)d\delta.$$

Definition 2.*The Laplace transform of the fractional differential operator in Caputo sense [20] of order δ is specified by*

$$\mathscr{L}\{_{0}^{c}D_{t}^{\delta}x(t)\} = s^{\delta}\mathscr{L}(x(t)) - \sum_{i=0}^{m-1} s^{\delta-i-1}x^{(i)}(0), \quad m-1 < \delta \le m \in \mathbb{N}.$$

Definition 3.*Let $f(t)$ be a given function of t defined for all $t \ge 0$ then the Laplace transform of $f(t)$ denoted by $\mathscr{L}\{f(t)\}$ or $F(s)$ or $\bar{f}(s)$ is defined as $\phi(s)$*

$$\mathscr{L}\{f(t)\} = \bar{f}(s) = F(s) = \phi(s) = \int_{0}^{\infty} e^{-st}f(t)dt$$

as long as the integral exists, where s is a real or complex factor, is called an Inverse Laplace transformation [21].

Lemma 1(Generalized mean-value theorem [22]). *Assume that $f(t) \in C[a,b]$ for $\delta \in (0,1]$, then we get*

$$f(t) = f(a) + \frac{1}{\Gamma(\delta)} {}_{0}^{c}D_{t}^{\delta}f(\xi)(t-a)^{\delta}, \quad a < \xi < t, \forall t \in (a,b].$$

Lemma 2.*Suppose that $f(t) \in C[a,b]$ and $_{0}^{c}D_{t}^{\delta}f(t) \in C[a,b]$ for $\delta \in (0,1)$. If $_{0}^{c}D_{t}^{\delta}f(t) \le 0$, $f(t)$ is non-increasing on $[a,b]$ for each $t \in [a,b]$. If $_{0}^{c}D_{t}^{\delta}f(t) \ge 0$, $f(t)$ is non-decreasing on $[a,b]$ for each $t \in [a,b]$ [23].*

3 Mathematical model and its analysis

Mathematical modeling is necessary in many fields, including science, engineering, economics, and social sciences. This includes the use of mathematical equations and computer models to calculate the activities of complex systems. The significance of numerical modeling lies in its capability to provide vision for complex systems. Mathematical modeling allows scientists and scholars to study complex systems that are difficult to study experimentally. By using computer models, they can increase visions into the behavior of these systems and calculate how they will be good under different conditions [24]-[32].

We transformed the integer order Buruli ulcer model existing in [33] into a fractional-order Buruli ulcer model by changing integer order derivatives with fractional order Caputo derivatives and the factors involved in the model.

So, we have the following fractional order model:

$$
{}_0^c D_t^\delta S_H(t) = \mu_H^\delta P_H + \theta^\delta R_H - \frac{\beta_H^\delta S_H I_V}{P_H} - \frac{\beta_H^\delta S_H I_{FF}}{P_H} - \mu_H^\delta S_H, \tag{1}
$$

$$
{}_0^c D_t^\delta I_H(t) = \frac{\beta_H^\delta S_H I_V}{P_H} + \frac{\beta_H^\delta S_H I_{FF}}{P_H} - (\gamma^\delta + \mu_H^\delta) I_H, \tag{2}
$$

$$
{}_0^c D_t^\delta R_H(t) = \gamma^\delta I_H - (\theta^\delta + \mu_H^\delta) R_H, \tag{3}
$$

$$
{}_0^c D_t^\delta S_V(t) = \mu_V^\delta P_V - \frac{\beta_V^\delta S_V I_{FF}}{P_V} - \frac{\eta \beta_V^\delta S_V U_M}{K} - \mu_V^\delta S_V, \tag{4}
$$

$$
{}_0^c D_t^\delta I_V(t) = \frac{\beta_V^\delta S_V I_{FF}}{P_V} + \frac{\eta \beta_V^\delta S_V U_M}{K} - \mu_V^\delta I_V, \tag{5}
$$

$$
{}_0^c D_t^\delta S_{FF}(t) = \mu_{FF}^\delta P_{FF} - \frac{\beta_{FF}^\delta I_{FF} U_M}{K} - \mu_{FF}^\delta S_{FF}, \tag{6}
$$

$$
{}_0^c D_t^\delta I_{FF}(t) = \frac{\beta_{FF}^\delta I_{FF} U_M}{K} - \mu_{FF}^\delta I_{FF}, \tag{7}
$$

$$
{}_0^c D_t^\delta U_M(t) = \sigma_{FF}^\delta I_{FF} + \sigma_V^\delta I_V - \mu_E^\delta U_M. \tag{8}
$$

The initial conditions were selected as nonnegative.
The following parameters and variables were measured:
The parameters used in the model are:

- β_H^δ: The actual interaction proportion among the vectors, fish frogs, and susceptible humans.
- β_V^δ: The actual interaction proportion among the fish frog and susceptible vector.
- β_{FF}^δ: The actual interaction proportion among susceptible fish frogs and Mycobacterium ulcerans.
- γ^δ: The recovery rate of ill humans.
- θ^δ: Interaction of damage with the exception of improved humans.
- μ_H^δ: Birth rates in the human population.
- μ_V^δ: Birth rate of the vector population.
- μ_{FF}^δ: Birth rate of the frog population.
- K: Environmental carrying capacity of the bacterial population.
- σ_{FF}^δ: Rate of shedding of Mycobacterium ulcerans into the environment by fish frogs.
- σ_V^δ: Rate of shedding of Mycobacterium ulcerans into the environment by vector.
- μ_E^δ: Rate at which Mycobacterium ulcerans is cleared from the environment.

The variables of the model are:

- S_H: The number of susceptible humans.
- I_H: Number of infected individuals.
- R_H: The number of recovered humans.
- S_V: The number of susceptible vectors.
- I_V: The number of infected vectors.
- S_{FF}: The number of susceptible fish-frog.
- I_{FF}: The number of infected fish-frog.
- U_M: The number of Mycobacterium ulcerans.

3.1 Positivity and boundedness of the fractional model

The positivity and boundedness are important features of the epidemic models. Here, we will investigate the physical features of the Buruli ulcer model. In the perspective of the fractional order tobacco smoking model, make sure that the state variables remain positive and bounded is important for maintaining the model's significance to the behavior of individuals in the population and for generating consistent numerical results. It is a key aspect of both the mathematical rigor and the practical applicability of your research.

Theorem 1. *For the given initial conditions of Buruli ulcer model, the solution of $S_H, I_H, R_H, S_V, I_V, S_{FF}, I_{FF}, U_M \in \mathbb{R}$. The initial conditions are $S_H \geq 0$, $I_H \geq 0$, $R_H \geq 0$, $S_V \geq 0$, $I_V \geq 0$, $S_{FF} \geq 0$, $I_{FF} \geq 0$ and $U_M \geq 0$.*

Proof. For this, we shall define the norm

$$\|f\|_\infty = \sup_{t \in D_f} |f(t)|$$

$$
{}_0^c D_t^\delta S_H(t) = \mu_H^\delta P_H + \theta^\delta R_H - \frac{\beta_H^\delta S_H I_V}{P_H} - \frac{\beta_H^\delta S_H I_{FF}}{P_H} - \mu_H^\delta S_H,
$$

$$
{}_0^c D_t^\delta S_H(t) \geq - \left[\frac{\beta_H^\delta I_V}{P_H} + \frac{\beta_H^\delta I_{FF}}{P_H} + \mu_H^\delta \right] S_H, \tag{9}
$$

Let $\frac{\beta_H^\delta I_V}{P_H} + \frac{\beta_H^\delta I_{FF}}{P_H} + \mu_H^\delta = M$,
Then equation (9) becomes

$$
{}_0^c D_t^\delta S_H(t) + M S_H(t) \geq 0.
$$

By applying Laplace Transformation, we get the following expression

$$
\mathscr{L}\{{}_0^c D_t^\delta S_H(t)\} + \mathscr{L}\{M S_H(t)\} \geq 0,
$$

$$
\mathscr{L}\{S_H(t)\} \geq \frac{s^{\delta-1} S(0)}{s^\delta + M}.
$$

By applying inverse Laplace, we derive the positivity as follows

$$
S_H(t) \geq \mathscr{L}^{-1}\left\{ \frac{s^{\delta-1} S(0)}{s^\delta + M} \right\},
$$

$$
S_H(t) \geq 0 \quad \forall t \geq 0.
$$

Similarly, the other equations of the model imply the positivity

$$
I_H(t) \geq 0 \,\forall t \geq 0, \quad R_H(t) \geq 0 \,\forall t \geq 0, \quad S_V(t) \geq 0 \,\forall t \geq 0, \quad I_V(t) \geq 0 \,\forall t \geq 0.
$$

$$
S_{FF}(t) \geq 0 \,\forall t \geq 0, \quad I_{FF}(t) \geq 0 \,\forall t \geq 0, \quad U_M(t) \geq 0 \,\forall t \geq 0.
$$

Theorem 2. *For the given initial conditions of Buruli ulcer model are $S_H \geq 0$, $I_H \geq 0$, $R_H \geq 0$, $S_V \geq 0$, $I_V \geq 0$, $S_{FF} \geq 0$, $I_{FF} \geq 0$ and $U_M \geq 0$, the solution of the system is uniformly bounded.*

Proof.

$$
\begin{aligned}
{}_0^c D_t^\delta N(t) &= {}_0^c D_t^\delta S_H(t) + {}_0^c D_t^\delta I_H(t) + {}_0^c D_t^\delta R_H(t) + {}_0^c D_t^\delta S_V(t) \\
&\quad + {}_0^c D_t^\delta I_V(t) + {}_0^c D_t^\delta S_{FF}(t) + {}_0^c D_t^\delta I_{FF}(t) + {}_0^c D_t^\delta U_M(t), \\
{}_0^c D_t^\delta N(t) &= \mu_H^\delta P_H - \mu_H^\delta S_H - \mu_H^\delta R_H - \mu_V^\delta P_V - \mu_V^\delta S_V \\
&\quad - \mu_V^\delta I_V - \mu_{FF}^\delta P_{FF} - \mu_{FF}^\delta S_{FF} - \mu_{FF}^\delta I_{FF} \\
&\quad + \sigma_{FF}^\delta I_{FF} + \sigma_V^\delta I_V - \mu_E^\delta U_M, \\
{}_0^c D_t^\delta N(t) &= \mu_H^\delta I_H + \sigma_{FF}^\delta I_{FF} + \sigma_V^\delta I_V - \mu_E^\delta U_M, \\
{}_0^c D_t^\delta N(t) &\leq \mu_H^\delta P_H + \sigma_{FF}^\delta P_{FF} + \sigma_V^\delta P_V - \mu_E^\delta U_M.
\end{aligned}
$$

Let $\mu_H^\delta P_H + \sigma_{FF}^\delta P_{FF} + \sigma_V^\delta P_V = M$,

$$^c_0 D_t^\delta N(t) \leq M - \mu_E^\delta U_M,$$
$$^c_0 D_t^\delta N(t) \leq M - \mu_E^\delta N(t),$$
$$^c_0 D_t^\delta N(t) + \mu_E^\delta N(t) \leq M.$$

By applying Laplace transformation as the above expression, we reach at

$$\mathscr{L}\{N(t)\} \leq \frac{s^{\delta-1}N(0)}{(s^\delta + \mu_E^\delta)} + \frac{M}{s(s^\delta + \mu_E^\delta)}.$$

By Laplace inverse

$$N(t) \leq Mt^\delta E_{\delta,1+\delta}(-\mu_E^\delta t^\delta) + N(0)E_{\delta,1}(-\mu_E^\delta t^\delta),$$

Let $M_1 = \max(N(0), \frac{t^\delta}{\mu_E^\delta})$,

$$N(t) \leq Mt^\delta E_{\delta,1+\delta}(-\mu_E^\delta t^\delta) + N(0)E_{\delta,1}(-\mu_E^\delta t^\delta)(-\mu_E^\delta t^\delta) + \frac{1}{\Gamma(1)}.$$

After some straight forward calculations, we conclude that

$$N(t) \leq M_1.$$

3.2 Existence and uniqueness

In this work, establishing the existence and uniqueness of solutions through Laplace transformations is essential for authorizing the model, ensuring its analytical accuracy, improving mathematical accuracy, and simplifying its practical application in addressing the complex issue of tobacco smoking popularity and control.

Theorem 3.*For every time t, the solution of the fractional model will exist and solution will also be unique. Now, we will prove existence and uniqueness of model. Existence of a solution means that the assumptions are consistent. Uniqueness means that they are sufficient to determine the system behavior.*

*Proof.*Let

$$K(S_H) = \mu_H^\delta P_H + \theta^\delta R_H - \frac{\beta_H^\delta S_H I_V}{P_H} - \frac{\beta_H^\delta S_H I_{FF}}{P_H} - \mu_H^\delta S_H,$$

$$\|K(S_{1H}) - K(S_{2H})\| = \left\| \mu_H^\delta P_H + \theta^\delta R_H - \frac{\beta_H^\delta S_{1H} I_V}{P_H} - \frac{\beta_H^\delta S_{1H} I_{FF}}{P_H} - \mu_H^\delta S_{1H} \ -\mu_H^\delta P_H - \theta^\delta R_H + \frac{\beta_H^\delta S_{2H} I_V}{P_H} + \frac{\beta_H^\delta S_{2H} I_{FF}}{P_H} + \mu_H^\delta S_{2H} \right\|,$$

$$= \left\| \frac{\beta_H^\delta I_V}{P_H}[S_{2H} - S_{1H}] + \left[\frac{\beta_H^\delta I_{FF}}{P_H} + \mu_H^\delta \right][S_{2H} - S_{1H}] \right\|,$$

$$\|K(S_{1H}) - K(S_{2H})\| \leq \left[\frac{\beta_H^\delta I_V}{P_H} + \frac{\beta_H^\delta I_{FF}}{P_H} + \mu_H^\delta \right] \|S_{2H} - S_{1H}\|.$$

Therefore $K(S_H)$ satisfies Lipschitz condition for contraction mapping

$$\frac{\beta_H^\delta I_V}{P_H} + \frac{\beta_H^\delta I_{FF}}{P_H} + \mu_H^\delta < 1.$$

Now, consider equation (2) Let

$$L(I_H) = \frac{\beta_H^\delta S_H I_V}{P_H} + \frac{\beta_H^\delta S_H I_{FF}}{P_H} - (\gamma^\delta + \mu_H^\delta)I_H,$$

$$\|L(I_{1H}) - L(I_{2H})\| = \left\| \frac{\beta_H^\delta S_H I_V}{P_H} + \frac{\beta_H^\delta S_H I_{FF}}{P_H} - (\gamma^\delta + \mu_H^\delta)I_{1H} \ -\frac{\beta_H^\delta S_H I_V}{P_H} - \frac{\beta_H^\delta S_H I_{FF}}{P_H} + (\gamma^\delta + \mu_H^\delta)I_{2H} \right\|,$$

$L(I_H)$ satisfies Lipschitz condition as before with

$$\gamma^\delta + \mu_H^\delta < 1.$$

Similarly, it can be shown that all other equations (3)-(8) satisfy the Lipschitz condition.
Let

$$f_1 = \frac{\beta_H^\delta I_V}{P_H} + \frac{\beta_H^\delta I_{FF}}{P_H} + \mu_H^\delta,$$

$$f_2 = \gamma^\delta + \mu_H^\delta,$$

$$f_3 = \theta^\delta + \mu_H^\delta,$$

$$f_4 = \frac{\beta_V^\delta I_{FF}}{P_V} + \frac{\eta \beta_V^\delta U_M}{K} + \mu_V^\delta,$$

$$f_5 = \mu_V^\delta,$$

$$f_6 = \mu_{FF}^\delta,$$

$$f_7 = \frac{\beta_{FF}^\delta U_M}{K} + \mu_{FF}^\delta,$$

$$f_8 = \mu_E^\delta.$$

Also let $F = \max\{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8\}$,
Finally, we arrive at

$$\|K(S_{1H}) - K(S_{2H})\| \le F \|S_{1H} - S_{2H}\|,$$
$$\|L(I_{1H}) - L(I_{2H})\| \le F \|I_{1H} - I_{2H}\|,$$
$$\|N(R_{1H}) - N(R_{2H})\| \le F \|R_{1H} - R_{2H}\|,$$
$$\|G(S_{1V}) - G(S_{2V})\| \le F \|S_{1V} - S_{2V}\|,$$
$$\|P(I_{1V}) - P(I_{2V})\| \le F \|I_{1V} - I_{2V}\|,$$
$$\|Y(S_{1FF}) - Y(S_{2FF})\| \le F \|S_{1FF} - S_{2FF}\|,$$
$$\|Q(I_{1FF}) - Q(I_{2FF})\| \le F \|I_{1FF} - I_{2FF}\|,$$
$$\|X(U_{1M}) - X(U_{2M})\| \le F \|U_{1M} - U_{2M}\|.$$

For $F < 1$, $K(S_H), L(I_H), N(R_H), G(S_V), P(I_V), Y(S_{FF}), Q(I_{FF}), X(U_M)$ satisfy Lipschitz condition.
Hence, the system (1)-(8) has unique solutions.

4 Stability of the model

In this work, the fractional order tobacco smoking model, defining the stability of the model is important for understanding how the tobacco-smoking dynamics develop over time. It can help answer questions about the long-term behavior of smoking occurrence and the efficiency of interventions. This knowledge is not only valuable for academic purposes but can also inform public health strategies and schemes for tobacco control.

4.1 Stability at disease free point

The equilibrium points of the system can be calculated by setting ${}_0^c D_t^\delta I_H(t) = {}_0^c D_t^\delta I_H(t) = {}_0^c D_t^\delta R_H(t) = {}_0^c D_t^\delta S_V(t) = {}_0^c D_t^\delta I_V(t) = {}_0^c D_t^\delta S_{FF}(t) = {}_0^c D_t^\delta I_{FF}(t) = {}_0^c D_t^\delta U_M(t) = 0.$

This system has two steady state points, namely disease free and endemic. The disease free steady state (DFSS) is $(S_H^0, 0, 0, S_V^0, 0, S_{FF}^0, 0, 0)$ and the endemic steady state (ESS) is $(s_H^*, I_H^*, R_H^*, S_V^*, S_{FF}^*, I_{FF}^*, U_M^*)$ where:

$$s_H^* = \frac{\mu_H^\delta P_H + \theta^\delta R_H^* - (\gamma^\delta + \mu_H^\delta) I_H^*}{\mu_H^\delta},$$

$$I_H^* = \frac{\frac{\beta_H^\delta}{P_H} S_H^* I_V^* + \frac{\beta_H^\delta}{P_H} S_H^* I_{FF}^*}{\theta^\delta + \mu_H^\delta},$$

$$R_H^* = \frac{\gamma^\delta I_H^*}{\theta^\delta + \mu_H^\delta},$$

$$S_V^* = P_V - I_V^*,$$

$$I_V^* = \frac{\frac{\beta_V^\delta}{P_V} S_V^* I_{FF}^* + \frac{\eta \beta_V^\delta}{K} S_V^* U_M^*}{P_V},$$

$$S_{FF}^* = \frac{\mu_{FF}^\delta P_{FF} - \frac{\beta_{FF}^\delta I_{FF}^* U_M^*}{K}}{P_{FF}},$$

$$I_{FF}^* = \frac{U_M^* \mu_E - I_V^* \sigma_V}{\sigma_{FF}},$$

$$U_M^* = \frac{P_{FF} K}{\beta_{FF}^\delta},$$

which is unique and positive. The basic reproduction number of the said system is given as

$$R_0 = \frac{\eta \beta_V P_V \sigma_V}{K \mu_V \mu_E}.$$

We check the stability of the system of equations by putting the equilibrium points such as disease free point and endemic point where disease free equilibrium point is $(S_H^0, 0, 0, S_V^0, 0, S_{FF}^0, 0, 0)$. Now, by putting values of partial derivatives in above matrix we get

$$J = \begin{pmatrix}
-\frac{\beta_H^\delta I_V}{P_H} - \frac{\beta_H^\delta I_{FF}}{P_H} - \mu_H^\delta & 0 & \theta^\delta & 0 & -\frac{\beta_H^\delta S_H}{P_H} & 0 & -\frac{\beta_H^\delta S_H}{P_H} & 0 \\
\frac{\beta_H^\delta I_V}{P_V} + \frac{\beta_H^\delta I_{FF}}{P_H} & -(\gamma^\delta + \mu_H^\delta) & 0 & 0 & \frac{\beta_H^\delta S_H}{P_H} & 0 & \frac{\beta_H^\delta S_H}{P_H} & 0 \\
0 & \gamma^\delta & -(\theta^\delta + \mu_H^\delta) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{\beta_V^\delta I_{FF}}{P_V} - \frac{\eta \beta_V^\delta U_M}{K} - \mu_V^\delta & 0 & 0 & -\frac{\beta_V^\delta S_V}{P_V} & -\frac{\eta \beta_V^\delta S_V}{K} \\
0 & 0 & 0 & \frac{\beta_V^\delta I_{FF}}{P_V} + \frac{\eta \beta_V^\delta U_M}{K} & -\mu_V^\delta & 0 & \frac{\beta_V^\delta S_V}{P_V} & \frac{\eta \beta_V^\delta S_V}{K} \\
0 & 0 & 0 & 0 & 0 & -\mu_{FF}^\delta & -\frac{\beta_{FF}^\delta U_M}{K} & -\frac{\beta_{FF}^\delta I_{FF}}{K} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\beta_{FF}^\delta U_M}{K} - \mu_{FF}^\delta & \frac{\beta_{FF}^\delta I_{FF}}{K} \\
0 & 0 & 0 & 0 & \sigma_V^\delta & 0 & \sigma_{FF}^\delta & -\mu_E^\delta
\end{pmatrix}$$

Now, we put disease free point $(S_H^0, 0, 0, S_V^0, 0, S_{FF}^0, 0, 0)$ in the Jacobian matrix:

$$J = \begin{pmatrix}
-\mu_H^\delta & 0 & \theta^\delta & 0 & -\frac{\beta_H^\delta S_H^0}{P_H} & 0 & -\frac{\beta_H^\delta S_H^0}{P_H} & 0 \\
0 & -(\gamma^\delta + \mu_H^\delta) & 0 & 0 & \frac{\beta_H^\delta S_H^0}{P_H} & 0 & \frac{\beta_H^\delta S_H^0}{P_H} & 0 \\
0 & \gamma^\delta & -(\theta^\delta + \mu_H^\delta) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\mu_V^\delta & 0 & 0 & -\frac{\beta_V^\delta S_V^0}{P_V} & -\frac{\eta \beta_V^\delta S_V^0}{K} \\
0 & 0 & 0 & 0 & -\mu_V^\delta & 0 & \frac{\beta_V^\delta S_V^0}{P_V} & \frac{\eta \beta_V^\delta S_V^0}{K} \\
0 & 0 & 0 & 0 & 0 & -\mu_{FF}^\delta & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\mu_{FF}^\delta & 0 \\
0 & 0 & 0 & 0 & \sigma_V^\delta & 0 & \sigma_{FF}^\delta & -\mu_E^\delta
\end{pmatrix}$$

To detect the eigenvalues, we solve $|J - \lambda I| = 0$ and reach at:

$$\lambda_1 = -\mu_H^\delta < 0,$$
$$\lambda_2 = -(\gamma^\delta + \mu_H^\delta) < 0,$$
$$\lambda_3 = -(\theta^\delta + \mu_H^\delta) < 0,$$
$$\lambda_4 = -\mu_V^\delta < 0,$$
$$\lambda_5 = -\mu_{FF}^\delta < 0,$$
$$\lambda_6 = -\mu_{FF}^\delta < 0,$$
$$\lambda_7 = \frac{-(\mu_V^\delta + \mu_E^\delta) + \sqrt{(\mu_V^\delta)^2 + (\mu_E^\delta)^2 - 2\mu_V^\delta\mu_E^\delta + 4\mu_V^\delta\mu_E^\delta R_0}}{2} < 0,$$
$$\lambda_8 = \frac{-(\mu_V^\delta + \mu_E^\delta) - \sqrt{(\mu_V^\delta)^2 + (\mu_E^\delta)^2 - 2\mu_V^\delta\mu_E^\delta + 4\mu_V^\delta\mu_E^\delta R_0}}{2} < 0.$$

As all the eigenvalues are less than 0, the system is stable at the disease free equilibrium point.

4.2 Stability at endemic point

The Endemic equilibrium point of the underlying system is calculated and described as:

$$S_H^e = \frac{P_H^2 \sigma_v \beta_{FF}(\gamma + \mu_H)(\theta + \mu_H)}{(P_H \sigma_V \beta_{FF})(\gamma + \mu_F) + (\theta + \mu_H + \gamma)(\beta_H K \mu_E \mu_{FF})}$$

$$R_H^e = \frac{\gamma P_H(\beta_H K \mu_E \mu_{FF})}{(P_H \sigma_V \beta_{FF})(\gamma + \mu_H) + (\theta + \mu_H + \gamma)(\beta_H K \mu_E \mu_{FF})}$$

$$I_H^e = \frac{P_H(\theta + \mu_H)(\beta_H K \mu_E \mu_{FF})}{(P_H \sigma_V \beta_{FF})(\gamma + \mu_H) + (\theta + \mu_H + \gamma)(\beta_H K \mu_E \mu_{FF})}$$

$$S_V^e = \frac{k \mu_v \mu_E}{\eta \sigma_v \beta_v}$$

$$I_V^e = \frac{\mu_E k \mu_{FF}}{\sigma_v \beta_{FF}}$$

$$P_{FF}^e = S_{FF}^e$$

$$I_{FF}^e = 0$$

$$U_M^e = \frac{k \mu_{FF}}{\beta_{FF}}$$

The characteristic equation is given by:

$$\begin{vmatrix} -\beta_H I_v - \mu_H - \lambda & 0 & 0 & 0 & \frac{-\beta_H S_H}{P_H} & 0 & \frac{-\beta_H S_H}{P_H} & 0 \\ \frac{\beta_H I_v}{P_H} & -(\gamma + \mu) - \lambda & 0 & 0 & \frac{\beta_H S_H}{P_H} & 0 & \frac{\beta_H S_H}{P_H} & 0 \\ 0 & \gamma & -(\theta + \mu_H) - \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\eta \beta_V U_M}{K} - \mu_V - \lambda & 0 & 0 & \frac{-\beta_V S_V}{P_V} & \frac{-\eta \beta_V S_V}{K} \\ 0 & 0 & 0 & \frac{\eta \beta_V U_M}{K} & -\mu_V - \lambda & 0 & \frac{\beta_V S_V}{P_V} & \frac{\eta \beta_V S_V}{K} \\ 0 & 0 & 0 & 0 & 0 & -(\mu_{FF}) - \lambda & \frac{-\beta_{FF} U_M}{K} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\beta_{FF} U_M - \mu_{FF} - \lambda}{K} & 0 \\ 0 & 0 & 0 & 0 & \sigma_V & 0 & \sigma_{FF}^\delta & -\mu_E - \lambda \end{vmatrix} = 0$$

$$\lambda_1 = -\mu_{FF} - \lambda$$

$$\begin{vmatrix} -\beta_H I_v - \mu_H - \lambda & 0 & 0 & 0 & \frac{-\beta_H S_H}{P_H} & \frac{-\beta_H S_H}{P_H} & 0 \\ \frac{\beta_H I_v}{P_H} & -(\gamma+\mu)-\lambda & 0 & 0 & \frac{\beta_H S_H}{P_H} & \frac{\beta_H S_H}{P_H} & 0 \\ 0 & \gamma & -(\theta+\mu_H)-\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\eta\beta_V U_M}{K} - \mu_V - \lambda & 0 & \frac{-\beta_V S_V}{P_V} & \frac{-\eta\beta_V S_V}{K} \\ 0 & 0 & 0 & \frac{\eta\beta_V U_M}{K} & -\mu_V - \lambda & \frac{\beta_V S_V}{P_V} & \frac{\eta\beta_V S_V}{K} \\ 0 & 0 & 0 & 0 & 0 & \frac{\beta_{FF} U_M - \mu_{FF} - \lambda}{K} & 0 \\ 0 & 0 & 0 & 0 & \sigma_V & \sigma_{FF}^{\delta} & -\mu_E - \lambda \end{vmatrix} = 0$$

$$\lambda_1 = -\beta_H I_v - \mu_H - \lambda$$

$$\begin{vmatrix} -(\gamma+\mu)-\lambda & 0 & 0 & \frac{\beta_H S_H}{P_H} & \frac{\beta_H S_H}{P_H} & 0 \\ \gamma & -(\theta+\mu_H)-\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\eta\beta_V U_M}{K} - \mu_V - \lambda & 0 & \frac{-\beta_V S_V}{P_V} & \frac{-\eta\beta_V S_V}{K} \\ 0 & 0 & \frac{\eta\beta_V U_M}{K} & -\mu_V - \lambda & \frac{\beta_V S_V}{P_V} & \frac{\eta\beta_V S_V}{K} \\ 0 & 0 & 0 & 0 & \frac{\beta_{FF} U_M - \mu_{FF} - \lambda}{K} & 0 \\ 0 & 0 & 0 & \sigma_V & \sigma_{FF}^{\delta} & -\mu_E - \lambda \end{vmatrix} = 0$$

$$-\frac{\beta_H I_v}{P_H}$$

$$\begin{vmatrix} 0 & 0 & 0 & \frac{\beta_H S_H}{P_H} & \frac{\beta_H S_H}{P_H} & 0 \\ \gamma & -(\theta+\mu_H)-\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\eta\beta_V U_M}{K} - \mu_V - \lambda & 0 & \frac{-\beta_V S_V}{P_V} & \frac{-\eta\beta_V S_V}{K} \\ 0 & 0 & \frac{\eta\beta_V U_M}{K} & -\mu_V - \lambda & \frac{\beta_V S_V}{P_V} & \frac{\eta\beta_V S_V}{K} \\ 0 & 0 & 0 & 0 & \frac{\beta_{FF} U_M - \mu_{FF} - \lambda}{K} & 0 \\ 0 & 0 & 0 & \sigma_V & \sigma_{FF}^{\delta} & -\mu_E - \lambda \end{vmatrix} = 0$$

$$= (-\beta_H I_v - \mu_H - \lambda)(-(\gamma+\mu)-\lambda)$$

$$\begin{vmatrix} -(\theta+\mu_H)-\lambda & 0 & 0 & 0 & 0 \\ 0 & -\frac{\eta\beta_V U_M}{K} - \mu_V - \lambda & 0 & \frac{-\beta_V S_V}{P_V} & \frac{-\eta\beta_V S_V}{K} \\ 0 & \frac{\eta\beta_V U_M}{K} & -\mu_V - \lambda & \frac{\beta_V S_V}{P_V} & \frac{\eta\beta_V S_V}{K} \\ 0 & 0 & 0 & \frac{\beta_{FF} U_M - \mu_{FF} - \lambda}{K} & 0 \\ 0 & 0 & \sigma_V & \sigma_{FF}^{\delta} & -\mu_E - \lambda \end{vmatrix}$$

$$+ \frac{\gamma\beta_H I_v}{P_H}$$

$$\begin{vmatrix} 0 & 0 & \frac{\beta_H S_H}{P_H} & \frac{\beta_H S_H}{P_H} & 0 \\ 0 & -\frac{\eta\beta_V U_M}{K} - \mu_V - \lambda & 0 & \frac{-\beta_V S_V}{P_V} & \frac{-\eta\beta_V S_V}{K} \\ 0 & \frac{\eta\beta_V U_M}{K} & -\mu_V - \lambda & \frac{\beta_V S_V}{P_V} & \frac{\eta\beta_V S_V}{K} \\ 0 & 0 & 0 & \frac{\beta_{FF} U_M - \mu_{FF} - \lambda}{K} & 0 \\ 0 & 0 & \sigma_V & \sigma_{FF}^{\delta} & -\mu_E - \lambda \end{vmatrix} = 0$$

$$= (-\beta_H I_v - \mu_H - \lambda)(-(\gamma + \mu) - \lambda)(-(\theta + \mu_H) - \lambda)$$

$$\begin{vmatrix} -\frac{\eta\beta_V U_M}{K} - \mu_V - \lambda & 0 & \frac{-\beta_V S_V}{P_V} & \frac{-\eta\beta_V S_V}{K} \\ \frac{\eta\beta_V U_M}{K} & -\mu_V - \lambda & \frac{\beta_V S_V}{P_V} & \frac{\eta\beta_V S_V}{K} \\ 0 & 0 & \frac{\beta_{FF} U_M - \mu_{FF} - \lambda}{K} & 0 \\ 0 & \sigma_V & \sigma_{FF}^{\delta} & -\mu_E - \lambda \end{vmatrix} = 0$$

The eigenvalues are determined as:

$$\lambda_2 = -\beta_H I_v - \mu_H$$
$$\lambda_3 = -(\gamma + \mu)$$
$$\lambda_4 = -(\theta + \mu_H)$$

The characteristic equation expands to:

$$\left(-\frac{\eta\beta_V U_M}{K} - \mu_V - \lambda\right) \begin{vmatrix} -\mu_V - \lambda & \frac{\beta_V S_V}{P_V} & \frac{\eta\beta_V S_V}{K} \\ 0 & \frac{\beta_{FF} U_M - \mu_{FF} - \lambda}{K} & 0 \\ \sigma_V & \sigma_{FF}^{\delta} & -\mu_E - \lambda \end{vmatrix}$$

$$- \left(\frac{\eta\beta_V U_M}{K}\right) \begin{vmatrix} 0 & \frac{-\beta_V S_V}{P_V} & \frac{-\eta\beta_V S_V}{K} \\ 0 & \frac{\beta_{FF} U_M - \mu_{FF} - \lambda}{K} & 0 \\ \sigma_V & \sigma_{FF}^{\delta} & -\mu_E - \lambda \end{vmatrix} = 0$$

This further simplifies to:

$$\left(-\frac{\eta\beta_V U_M}{K} - \mu_V - \lambda\right)(-\mu_V - \lambda) \begin{vmatrix} \frac{\beta_{FF} U_M - \mu_{FF} - \lambda}{K} & 0 \\ \sigma_{FF}^{\delta} & -\mu_E - \lambda \end{vmatrix}$$

$$+ \sigma_V \begin{vmatrix} \frac{\beta_V S_V}{P_V} & \frac{\eta\beta_V S_V}{K} \\ \frac{\beta_{FF} U_M - \mu_{FF} - \lambda}{K} & 0 \end{vmatrix}$$

$$- \left(\frac{\eta\beta_V U_M}{K}\right) \begin{vmatrix} \frac{-\beta_V S_V}{P_V} & \frac{-\eta\beta_V S_V}{K} \\ \frac{\beta_{FF} U_M - \mu_{FF} - \lambda}{K} & 0 \end{vmatrix} = 0$$

The final characteristic equation becomes:

$$\left(-\frac{\eta\beta_V U_M}{K} - \mu_V - \lambda\right)(-\mu_V - \lambda)(-\mu_E - \lambda)\left(\frac{\beta_{FF} U_M - \mu_{FF} - \lambda}{K}\right)$$

$$- \sigma_V \left(\frac{\eta\beta_V S_V}{K}\right)\left(\frac{\beta_{FF} U_M - \mu_{FF} - \lambda}{K}\right)$$

$$+ \left(\frac{\eta\beta_V U_M}{K}\right)\left(\frac{\eta\beta_V S_V}{K}\right)\left(\frac{\beta_{FF} U_M - \mu_{FF} - \lambda}{K}\right) = 0$$

This can be factored as:

$$\left(\frac{\beta_{FF} U_M - \mu_{FF} - \lambda}{K}\right)\left[\left(1 - \sigma_V\left(\frac{\eta\beta_V S_V}{K}\right)\right)\right.$$

$$\left. + \left(\frac{\eta\beta_V U_M \sigma_V}{K}\right)\left(\frac{\eta\beta_V S_V}{K}\right)\right] = 0$$

The remaining eigenvalues are:

$$\lambda_5 = -\frac{\eta\beta_V U_M}{K} - \mu_V$$
$$\lambda_6 = \mu$$
$$\lambda_7 = \mu_E$$
$$\lambda_8 = -\mu_{FF} + \frac{\beta_{FF} U_M}{K}$$

The stability condition requires:

$$\lambda_8 = -\mu_{FF} + \frac{\beta_{FF}U_M}{K} < 0 \quad \text{if and only if} \quad U_M \leq 0$$

5 Numerical Modeling

In order to get numerical solutions for the $S_H, I_H, R_H, S_V, I_V, S_{FF}, I_{FF}$, and U_M fractional-order epidemic models, we develop the Grunwald-Letnikov non-standard finite difference (GL-NSFD) scheme. We apply the Grunwald-Letnikov technique on the Caputo operator to estimate the fractional order epidemic model. We develop the GL-NSFD scheme to numerically investigate the fractional-order epidemic model under Mickens rules.

The fractional derivative equation:

$$_0^c D_t^\delta S_H = \mu_H^\delta P_H + \theta^\delta R_H - \frac{\beta_H^\delta S_H I_V}{P_H} - \frac{\beta_H^\delta S_H I_{FF}}{P_H} - \mu_H^\delta S_H$$

Applying the GL-NSFD scheme gives:

$$\frac{1}{\varphi(h)^\delta}\left(S_H^{(n+1)} - \sum_{i=1}^{n+1} e_i' S_H^{(n+1-i)} - \gamma_{n+1}' S_H^{(0)}\right) = \mu_H^\delta P_H + \theta^\delta R_H^{(n)} - \frac{\beta_H^\delta S_H^{(n+1)} I_V^{(n)}}{P_H} - \frac{\beta_H^\delta S_H^{(n+1)} I_{FF}^{(n)}}{P_H} - \mu_H^\delta S_H^{(n+1)}$$

$$S_H^{(n+1)} = \frac{\sum_{i=1}^{n+1} e_i' S_H^{(n+1-i)} + \gamma_{n+1}' S_H^{(0)} + \varphi(h)^\delta \mu_H^\delta P_H + \varphi(h)^\delta \theta^\delta R_H^{(n)}}{1 + \varphi(h)^\delta \frac{\beta_H^\delta I_V^{(n)}}{P_H} + \varphi(h)^\delta \frac{\beta_H^\delta I_{FF}^{(n)}}{P_H} + \varphi(h)^\delta \mu_H^\delta} \tag{10}$$

Similarly, from equation (2):

$$I_H^{(n+1)} = \frac{\sum_{i=1}^{n+1} e_i' I_H^{(n+1-i)} + \gamma_{n+1}' I_H^{(0)} + \varphi(h)^\delta \frac{\beta_H^\delta S_H^{(n+1)} I_V^{(n)}}{P_H} + \varphi(h)^\delta \frac{\beta_H^\delta S_H^{(n+1)} I_{FF}^{(n)}}{P_H}}{1 + \varphi(h)^\delta(\gamma^\delta + \mu_H^\delta)} \tag{11}$$

For recovered population:

$$R_H^{(n+1)} = \frac{\sum_{i=1}^{n+1} e_i' R_H^{(n+1-i)} + \gamma_{n+1}' R_H^{(0)} + \varphi(h)^\delta \gamma^\delta I_H^{(n+1)}}{1 + \varphi(h)^\delta(\theta^\delta + \mu_H^\delta)} \tag{12}$$

For susceptible vectors:

$$S_V^{(n+1)} = \frac{\sum_{i=1}^{n+1} e_i' S_V^{(n+1-i)} + \gamma_{n+1}' S_V^{(0)} + \varphi(h)^\delta \mu_V^\delta P_V}{1 + \varphi(h)^\delta \frac{\beta_V^\delta I_{FF}^{(n)}}{P_V} + \varphi(h)^\delta \frac{\eta \beta_V^\delta U_M^{(n)}}{K} + \varphi(h)^\delta \mu_V^\delta} \tag{13}$$

For infected vectors:

$$I_V^{(n+1)} = \frac{\sum_{i=1}^{n+1} e_i' I_V^{(n+1-i)} + \gamma_{n+1}' I_V^{(0)} + \varphi(h)^\delta \frac{\beta_V^\delta S_V^{(n+1)} I_{FF}^{(n)}}{P_V} + \varphi(h)^\delta \frac{\eta \beta_V^\delta S_V^{(n+1)} U_M^{(n)}}{K}}{1 + \varphi(h)^\delta \mu_V^\delta} \tag{14}$$

For susceptible fish-frogs:

$$S_{FF}^{(n+1)} = \frac{\sum_{i=1}^{n+1} e_i' S_{FF}^{(n+1-i)} + \gamma_{n+1}' S_{FF}^{(0)} + \varphi(h)^\delta \mu_{FF}^\delta P_{FF} - \varphi(h)^\delta \frac{\beta_{FF}^\delta I_{FF}^{(n)} U_M^{(n)}}{K}}{1 + \varphi(h)^\delta \mu_{FF}^\delta} \tag{15}$$

For infected fish-frogs:

$$I_{FF}^{(n+1)} = \frac{\sum_{i=1}^{n+1} e_i' I_{FF}^{(n+1-i)} + \gamma_{n+1}' I_{FF}^{(0)} + \varphi(h)^\delta \frac{\beta_{FF}^\delta I_{FF}^{(n)} U_M^{(n)}}{K}}{1 + \varphi(h)^\delta \mu_{FF}^\delta} \tag{16}$$

For bacteria population:

$$U_M^{(n+1)} = \frac{\sum_{i=1}^{n+1} e_i' U_M^{(n+1-i)} + \gamma_{n+1}' U_M^{(0)} + \varphi(h)^\delta \sigma_{FF}^\delta I_{FF}^{(n+1)} + \varphi(h)^\delta \sigma_V^\delta I_V^{(n+1)}}{1 + \varphi(h)^\delta \mu_E^\delta} \tag{17}$$

5.1 Positivity of the Numerical Technique

Since the model deals with human population, all compartment populations must remain positive at any time t.

Theorem 1

Suppose all variables and control parameters are positive, i.e., $S_H^0 > 0, I_H^0 > 0, R_H^0 > 0, S_V^0 > 0, I_V^0 > 0, S_{FF}^0 > 0, I_{FF}^0 > 0, U_M^0 > 0$ and $\beta_H^\delta > 0, \beta_V^\delta > 0, \beta_{FF}^\delta > 0, \gamma^\delta > 0, \theta^\delta > 0, \mu_H^\delta > 0, \mu_V^\delta > 0, \mu_{FF}^\delta > 0, \sigma_{FF}^\delta > 0, \sigma_V^\delta > 0$ and μ_E^δ are all positive, then $S_H^{(n+1)} > 0, I_H^{(n+1)} > 0, R_H^{(n+1)} > 0, S_V^{(n+1)} > 0, I_V^{(n+1)} > 0, S_{FF}^{(n+1)} > 0, I_{FF}^{(n+1)} > 0, U_M^{(n+1)} > 0$, for all $n = 0, 1, 2, 3, \ldots$, subject to the condition that $K\mu_{FF}^\delta > \beta_{FF}^\delta U_M$.

*Proof.*consider

$$S_H^{(n+1)} = \frac{\sum_{i=1}^{(n+1)} e_i' S_H^{(n+1-i)} + \gamma_{(n+1)}' S_H^{(0)} + \varphi(h)^\delta \mu_H^\delta P_H + \varphi(h)^\delta \theta^\delta R_H^{(n)}}{1 + \varphi(h)^\delta \frac{\beta_H^\delta I_V^{(n)}}{P_H} + \varphi(h)^\delta \frac{\beta_H^\delta I_{FF}^{(n)}}{P_H} + \varphi(h)^\delta \mu_H^\delta}.$$

For $n = 0$, we get

$$S_H^{(1)} = \frac{\sum_{i=1}^{(1)} e_i' S_H^{(1-i)} + \gamma_{(1)}' S_H^{(0)} + \varphi(h)^\delta \mu_H^\delta P_H + \varphi(h)^\delta \theta^\delta R_H^{(0)}}{1 + \varphi(h)^\delta \frac{\beta_H^\delta I_V^{(0)}}{P_H} + \varphi(h)^\delta \frac{\beta_H^\delta I_{FF}^{(0)}}{P_H} + \varphi(h)^\delta \mu_H^\delta},$$

$$S_H^{(1)} = \frac{e_1' S_H^{(0)} + \gamma_{(1)}' S_H^{(0)} + \varphi(h)^\delta \mu_H^\delta P_H + \varphi(h)^\delta \theta^\delta R_H^{(0)}}{1 + \varphi(h)^\delta \frac{\beta_H^\delta I_V^{(0)}}{P_H} + \varphi(h)^\delta \frac{\beta_H^\delta I_{FF}^{(0)}}{P_H} + \varphi(h)^\delta \mu_H^\delta} > 0.$$

Similarly, for $n = 1$ and $n = 2$, it can easily be proved that $S_H^{(2)}$ and $S_H^{(3)}$ are positive.

Now, suppose that the scheme preserves the positivity for $n \in \{3, 4, \ldots, n-1\}$. Now, we will prove it for some positive integer $n \in \mathbb{Z}^+$.

For this, consider

$$S_H^{(n+1)} = \frac{\sum_{i=1}^{(n+1)} e_i' S_H^{(n+1-i)} + \gamma_{(n+1)}' S_H^{(0)} + \varphi(h)^\delta \mu_H^\delta P_H + \varphi(h)^\delta \theta^\delta R_H^{(n)}}{1 + \varphi(h)^\delta \frac{\beta_H^\delta I_V^{(n)}}{P_H} + \varphi(h)^\delta \frac{\beta_H^\delta I_{FF}^{(n)}}{P_H} + \varphi(h)^\delta \mu_H^\delta},$$

$$S_H^{(n+1)} = \frac{e_1' S_H^{(n)} + e_2' S_H^{(n-1)} + e_3' S_H^{(n-2)} \cdots + e_n' S_H^{(1)} + e_{n+1}' S_H^{(0)} + \gamma_{(n+1)}' S_H^{(0)} + \varphi(h)^\delta \mu_H^\delta P_H + \varphi(h)^\delta \theta^\delta R_H^{(n)}}{1 + \varphi(h)^\delta \frac{\beta_H^\delta I_V^{(n)}}{P_H} + \varphi(h)^\delta \frac{\beta_H^\delta I_{FF}^{(n)}}{P_H} + \varphi(h)^\delta \mu_H^\delta}.$$

As all the state variables and parameters involved in the expression are positive. Therefore, $S_H^{(n+1)} > 0 \ \forall \ n \in \mathbb{Z}^+$.

On the same lines, it can be shown that all the other discretized state variables are positive, which complete the proof.

5.2 Boundedness of the numerical scheme

Since the model deals with the human population so, at any time t, sum of the populations of all the compartments must not be greater than the whole population.

Theorem 2

Suppose that $S_H^0 > 0, I_H^0 > 0, R_H^0 > 0, S_V^0 > 0, I_V^0 > 0, S_{FF}^0 > 0, I_{FF}^0 > 0, U_M^0 > 0$ are finite also $\beta_H^\delta > 0, \beta_V^\delta > 0, \beta_{FF}^\delta > 0, \gamma^\delta > 0, \theta^\delta > 0, \mu_H^\delta > 0, \mu_V^\delta > 0, \mu_{FF}^\delta > 0, \sigma_{FF}^\delta > 0, \sigma_V^\delta > 0$ and μ_E^δ are all positive then there is a constant M_{n+1} such that $S_H^{(n+1)} \leq M_{n+1}, I_H^{(n+1)} \leq M_{n+1}, R_H^{(n+1)} \leq M_{n+1}, S_V^{(n+1)} \leq M_{n+1}, I_V^{(n+1)} \leq M_{n+1}, S_{FF}^{(n+1)} \leq M_{n+1}, I_{FF}^{(n+1)} \leq M_{n+1}, U_M^{(n+1)} \leq M_{n+1} \ \forall \ n = 0, 1, 2, 3 \ldots$.

Proof. Adding equations (10) - (17), we get

$$S_H^{(n+1)}[1+\varphi(h)^\delta \mu_H^\delta] + I_H^{(n+1)}[1+\varphi(h)^\delta \mu_H^\delta] + R_H^{(n+1)}[1+\varphi(h)^\delta (\theta^\delta + \mu_H^\delta)]$$
$$+ S_V^{(n+1)}[1+\varphi(h)^\delta \mu_V^\delta] + I_V^{(n+1)}[1+\varphi(h)^\delta \mu_V^\delta] + S_{FF}^{(n+1)}[1+\varphi(h)^\delta \mu_{FF}^\delta]$$
$$+ I_{FF}^{(n+1)}[1+\varphi(h)^\delta \mu_{FF}^\delta] + U_M^{(n+1)}[1+\varphi(h)^\delta \mu_E^\delta]$$
$$= \sum_{i=1}^{(n+1)} e_i'[S_H^{(n+1-i)} + I_H^{(n+1-i)} + R_H^{(n+1-i)} + S_V^{(n+1-i)} + I_V^{(n+1-i)}$$
$$+ S_{FF}^{(n+1-i)} + I_{FF}^{(n+1-i)} + U_M^{(n+1-i)}] + \gamma_{(n+1)}'(S_H^{(0)} + I_H^{(0)}$$
$$+ R_H^{(0)} + S_V^{(0)} + I_V^{(0)} + S_{FF}^{(0)} + I_{FF}^{(0)} + U_M^{(0)}) + \varphi(h)^\delta(\mu_H^\delta P_H$$
$$+ \theta^\delta R_H + \mu_V^\delta P_V + \mu_{FF}^\delta P_{FF} + \sigma_{FF}^\delta I_{FF}^{(n+1)} + \sigma_V^\delta I_V^{(n+1)}).$$

For $n = 0$, we get the following expression:

$$S_H^{(1)}[1+\varphi(h)^\delta \mu_H^\delta] + I_H^{(1)}[1+\varphi(h)^\delta \mu_H^\delta] + R_H^{(1)}[1+\varphi(h)^\delta (\theta^\delta + \mu_H^\delta)]$$
$$+ S_V^{(1)}[1+\varphi(h)^\delta \mu_V^\delta] + I_V^{(1)}[1+\varphi(h)^\delta \mu_V^\delta] + S_{FF}^{(1)}[1+\varphi(h)^\delta \mu_{FF}^\delta]$$
$$+ I_{FF}^{(1)}[1+\varphi(h)^\delta \mu_{FF}^\delta] + U_M^{(1)}[1+\varphi(h)^\delta \mu_E^\delta]$$
$$= e_1'(S_H^{(0)} + I_H^{(0)} + R_H^{(0)} + S_V^{(0)} + I_V^{(0)} + S_{FF}^{(0)} + I_{FF}^{(0)} + U_M^{(0)})$$
$$+ \gamma_{(1)}'(S_H^{(0)} + I_H^{(0)} + R_H^{(0)} + S_V^{(0)} + I_V^{(0)} + S_{FF}^{(0)} + I_{FF}^{(0)} + U_M^{(0)})$$
$$+ \varphi(h)^\delta(\mu_H^\delta P_H + \theta^\delta R_H + \mu_V^\delta P_V + \mu_{FF}^\delta P_{FF} + \sigma_{FF}^\delta I_{FF}^{(1)} + \sigma_V^\delta I_V^{(1)}).$$

Here $(S_H^{(0)} + I_H^{(0)} + R_H^{(0)} + S_V^{(0)} + I_V^{(0)} + S_{FF}^{(0)} + I_{FF}^{(0)} + U_M^{(0)}) = N^{(0)}$,
$\Rightarrow S_H^{(1)}[1+\varphi(h)^\delta \mu_H^\delta] \le e_1' N^{(0)} + \gamma_{(1)}' N^{(0)} + \varphi(h)^\delta(\mu_H^\delta P_H + \theta^\delta R_H + \mu_V^\delta P_V + \mu_{FF}^\delta P_{FF} + \sigma_{FF}^\delta I_{FF}^{(1)} + \sigma_V^\delta I_V^{(1)})$,
$\Rightarrow S_H^{(1)}[1+\varphi(h)^\delta \mu_H^\delta] \le (e_1' + \gamma_{(1)}') N^{(0)} + \varphi(h)^\delta(\mu_H^\delta P_H + \theta^\delta R_H + \mu_V^\delta P_V + \mu_{FF}^\delta P_{FF} + \sigma_{FF}^\delta I_{FF}^{(1)} + \sigma_V^\delta I_V^{(1)})$,
$(e_1' + \gamma_{(1)}') N^{(0)} + \varphi(h)^\delta(\mu_H^\delta P_H + \theta^\delta R_H + \mu_V^\delta P_V + \mu_{FF}^\delta P_{FF} + \sigma_{FF}^\delta I_{FF}^{(1)} + \sigma_V^\delta I_V^{(1)}) = M_1$ (say),
$\Rightarrow S_H^{(1)}[1+\varphi(h)^\delta \mu_H^\delta] \le M_1$,
$\Rightarrow S_H^{(1)} \le \frac{M_1}{1+\varphi(h)^\delta \mu_H^\delta}$,
$\Rightarrow S_H^{(1)} \le M_1$.
Similarly, $I_H^{(1)} \le M_1, R_H^{(1)} \le M_1, S_V^{(1)} \le M_1, I_V^{(1)} \le M_1, S_{FF}^{(1)} \le M_1, I_{FF}^{(1)} \le M_1$ and $U_M^{(1)} \le M_1$.
Now,

$$S_H^{(n+1)}[1+\varphi(h)^\delta \mu_H^\delta] + I_H^{(n+1)}[1+\varphi(h)^\delta \mu_H^\delta] + R_H^{(n+1)}[1+\varphi(h)^\delta (\theta^\delta + \mu_H^\delta)]$$
$$+ S_V^{(n+1)}[1+\varphi(h)^\delta \mu_V^\delta] + I_V^{(n+1)}[1+\varphi(h)^\delta \mu_V^\delta] + S_{FF}^{(n+1)}[1+\varphi(h)^\delta \mu_{FF}^\delta]$$
$$+ I_{FF}^{(n+1)}[1+\varphi(h)^\delta \mu_{FF}^\delta] + U_M^{(n+1)}[1+\varphi(h)^\delta \mu_E^\delta]$$
$$= e_1'[S_H^{(n)} + I_H^{(n)} + R_H^{(n)} + S_V^{(n)} + I_V^{(n)} + S_{FF}^{(n)} + I_{FF}^{(n)} + U_M^{(n)}]$$
$$+ e_2'[S_H^{(n-1)} + I_H^{(n-1)} + R_H^{(n-1)} + S_V^{(n-1)} + I_V^{(n-1)} + S_{FF}^{(n-1)} + I_{FF}^{(n-1)} + U_M^{(n-1)}]$$
$$+ e_3'[S_H^{(n-2)} + I_H^{(n-2)} + R_H^{(n-2)} + S_V^{(n-2)} + I_V^{(n-2)} + S_{FF}^{(n-2)} + I_{FF}^{(n-2)} + U_M^{(n-2)}]$$
$$+ \cdots + e_{n+1}'(S_H^{(0)} + I_H^{(0)} + R_H^{(0)} + S_V^{(0)} + I_V^{(0)} + S_{FF}^{(0)} + I_{FF}^{(0)} + U_M^{(0)})$$
$$+ \gamma_{(n+1)}'(S_H^{(0)} + I_H^{(0)} + R_H^{(0)} + S_V^{(0)} + I_V^{(0)} + S_{FF}^{(0)} + I_{FF}^{(0)} + U_M^{(0)})$$
$$+ \varphi(h)^\delta(\mu_H^\delta P_H + \theta^\delta R_H + \mu_V^\delta P_V + \mu_{FF}^\delta P_{FF} + \sigma_{FF}^\delta I_{FF}^{(n+1)} + \sigma_V^\delta I_V^{(n+1)}),$$

$[S_H^{(n)} + I_H^{(n)} + R_H^{(n)} + S_V^{(n)} + I_V^{(n)} + S_{FF}^{(n)} + I_{FF}^{(n)} + U_M^{(n)}] = M_n$ (say),

Now, consider the following relation:

$$S_H^{(n+1)}[1+\varphi(h)^\delta \mu_H^\delta]+I_H^{(n+1)}[1+\varphi(h)^\delta \mu_H^\delta]+R_H^{(n+1)}[1+\varphi(h)^\delta(\theta^\delta+\mu_H^\delta)]$$
$$+S_V^{(n+1)}[1+\varphi(h)^\delta \mu_V^\delta]+I_V^{(n+1)}[1+\varphi(h)^\delta \mu_V^\delta]+S_{FF}^{(n+1)}[1+\varphi(h)^\delta \mu_{FF}^\delta]$$
$$+I_{FF}^{(n+1)}[1+\varphi(h)^\delta \mu_{FF}^\delta]+U_M^{(n+1)}[1+\varphi(h)^\delta \mu_E^\delta]$$
$$=e_1'(8M^{(n)})+e_2'(8M^{(n-1)})+e_3'(8M^{(n-2)})+\cdots+e_{n+1}'N^{(0)}$$
$$+\gamma_{(n+1)}'N^{(0)}+\varphi(h)^\delta(\mu_H^\delta P_H+\theta^\delta R_H+\mu_V^\delta P_V+\mu_{FF}^\delta P_{FF}+\sigma_{FF}^\delta I_{FF}^{(n+1)}+\sigma_V^\delta I_V^{(n+1)}).$$

Consequently

$$S_H^{(n+1)} \leq M_{n+1},$$
$$I_H^{(n+1)} \leq M_{n+1},$$
$$R_H^{(n+1)} \leq M_{n+1},$$
$$S_V^{(n+1)} \leq M_{n+1},$$
$$I_V^{(n+1)} \leq M_{n+1},$$
$$S_{FF}^{(n+1)} \leq M_{n+1},$$
$$I_{FF}^{(n+1)} \leq M_{n+1},$$
$$U_M^{(n+1)} \leq M_{n+1}.$$

Hence the proof.

6 Simulations

Simulations are important because they provide a dynamic and useful tool for authorizing the model, understanding disease dynamics, calculating policies, and generating data. They contribute to the reliability and relevance of your research, making it a valuable resource for addressing the complex issue of tobacco smoking and its significances.

Now, we will perform the simulations for the numerical graphs. These graphs will help us to study the disease dynamics for the Buruli ulcer model. To this end, we select the following parametric values for simulations are $\beta_H^\delta = 0.1$, $\beta_V^\delta = 0.1$ (DFE), $\beta_V^\delta = 10.1$ (EE), $\beta_{FF}^\delta = 0.45$, $\gamma^\delta = 0.11$, $\theta^\delta = 0.12$, $\mu_H^\delta = 0.5$, $P_V = 1$, $\mu_V^\delta = 0.5$, $\mu_{FF}^\delta = 0.13$, $\sigma_{FF}^\delta = 0.5$, $\sigma_V^\delta = 0.5$, $\mu_E^\delta = 0.5$, η, $K = 1$, $P_{FF} = 1$ and $P_H = 1$.

The initial conditions of underlying state variables of Buruli ulcer model are $S_H^0 = 0.5$, $I_H^0 = 0.3$, $R_H^0 = 0.3$, $S_V^0 = 0.5$, $I_V^0 = 0.5$, $S_{FF}^0 = 0.5$, $I_{FF}^0 = 0.5$ and $U_M^0 = 1$.

The numerical values of the disease free equilibrium point by using the parametric values mentioned in table 1 is described as, $(S_H, I_H, R_H, S_V, I_V, S_{FF}, I_{FF}, U_M) = (1, 0, 0, 1, 0, 1, 0, 0)$, while the endemic equilibrium point is $(s_H^*, I_H^*, R_H^*, s_V^*, I_V^*, s_{FF}^*, I_{FF}^*, U_M^*) = (0.852, 0.126, 0.0223, 0.099, 0.9, 1, 0.000121, 0.9)$.
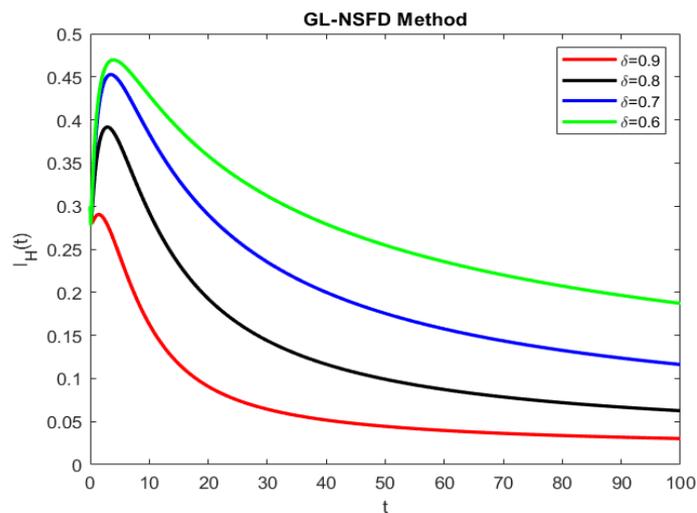
Moreover, the step size h is taken as 0.1 for all the graphs.

6.1 Description of Graphs at DFE

The Figures 1 to 5 show the numerical behavior of the susceptible populace, infected populace, recovered populace, susceptible vectors and infected vectors, due to Buruli ulcer at disease free equilibrium point. All four graphs follow different trajectories to attain steady state. It is mentionable that every graph has a specific rate of convergence to touch the equilibrium point, depending upon the fractional order parameter δ. Therefore, every graph reflects the reliability and efficacy of the proposed GL-NSFD scheme. Moreover, role of fractional order parameter δ is very crucial in deciding the path of the trajectory and speed of path for attaining the disease free equilibrium point. So, this decisive parameter plays a significant role in the rate of convergence towards the fixed point of the model. Because of this reason, this parameter may capture many real world phenomenon.
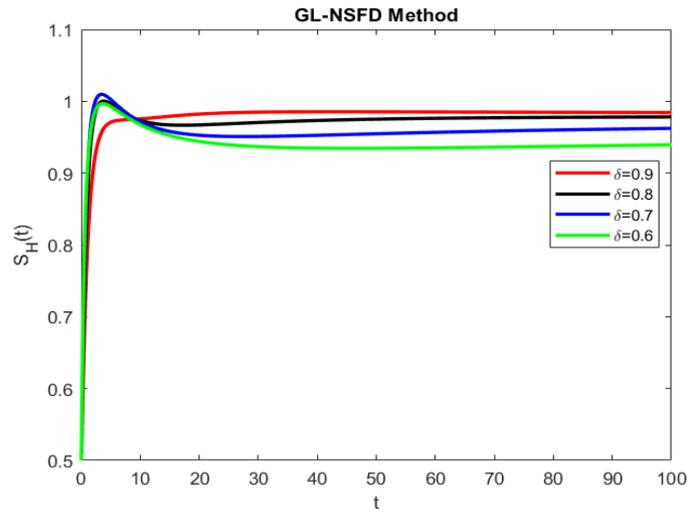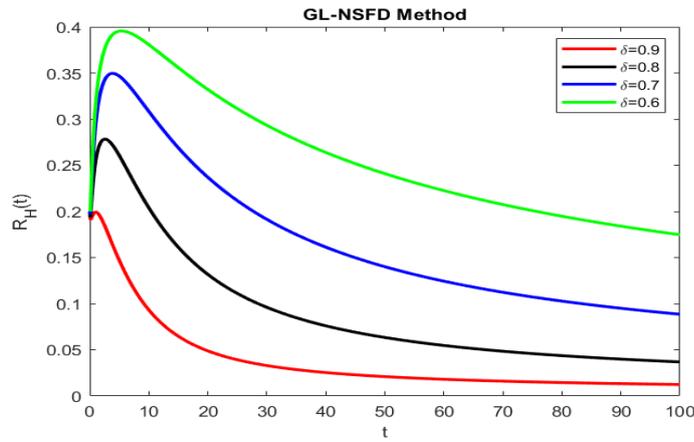
Fig. 1: Numerical graph of susceptible individuals at DFE point.

Fig. 2: Numerical graph of infected individuals at DFE point.

6.2 Description of Graphs at DEE

The Figures 6 to 10 show the numerical behavior of the susceptible populace, infected populace, recovered populace, susceptible vectors and infected vectors, due to Buruli ulcer at endemic equilibrium point. All four graphs follow different trajectories to attain steady state. It is mentionable that every graph has a specific rate of convergence to touch the equilibrium point, depending upon the fractional order parameter δ. Therefore, every graph reflects the reliability and efficacy of the proposed GL-NSFD scheme. Moreover, role of fractional order parameter δ is very crucial in deciding the path of the trajectory and speed of path for attaining the endemic equilibrium point. So, this decisive parameter plays a significant role in the rate of convergence towards the fixed point of the model. Because of this reason, this parameter may capture many real world phenomena.

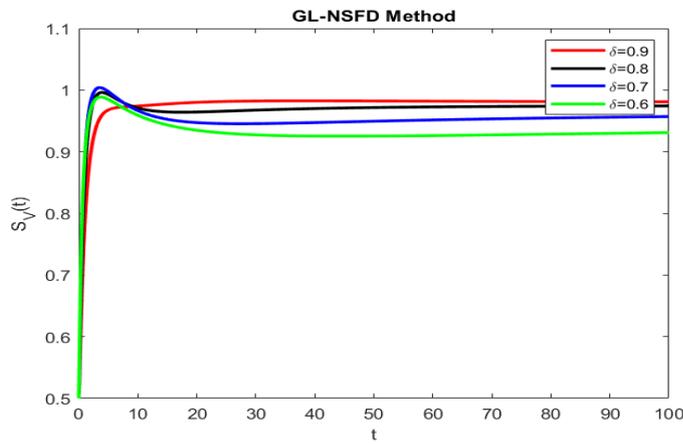**Fig. 3:** Numerical graph of recovered individuals at DFE point.

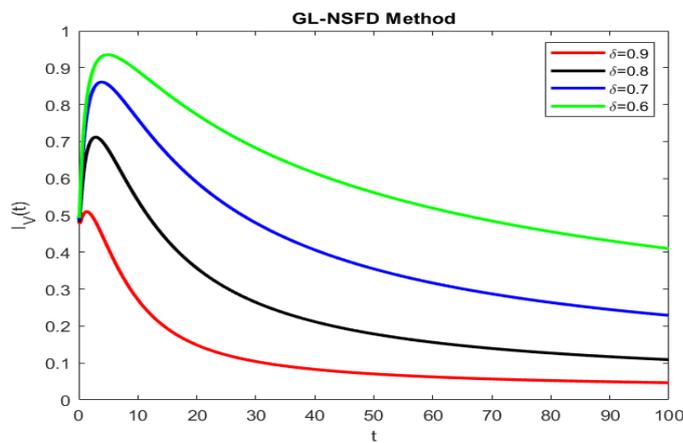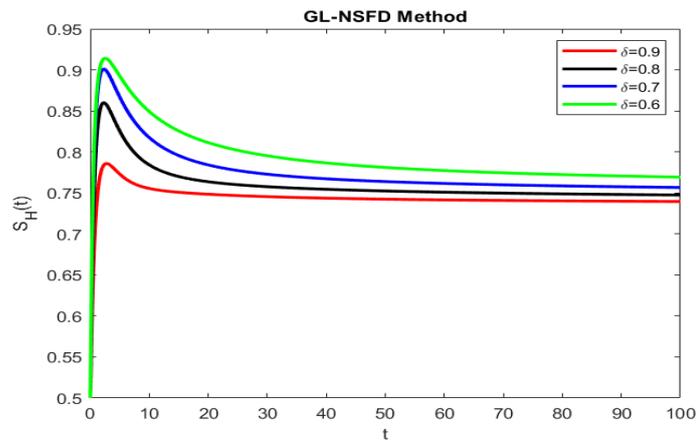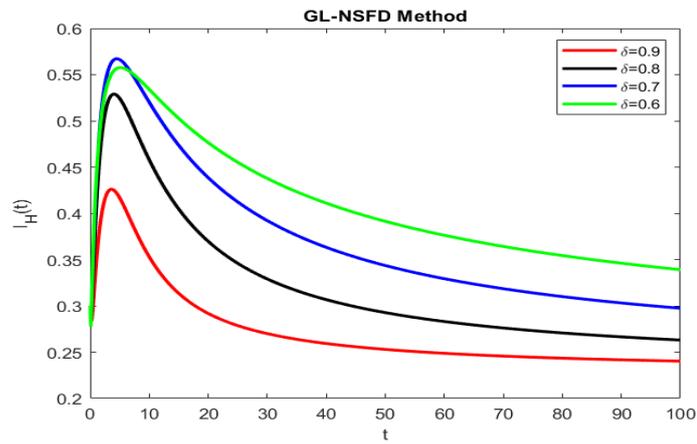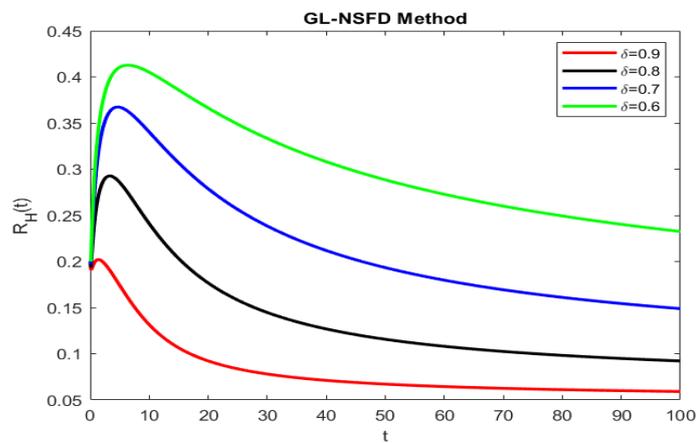**Fig. 4:** Numerical graph of susceptible vectors at DFE point.

Fig. 5: Numerical graph of infected vectors at DFE point.
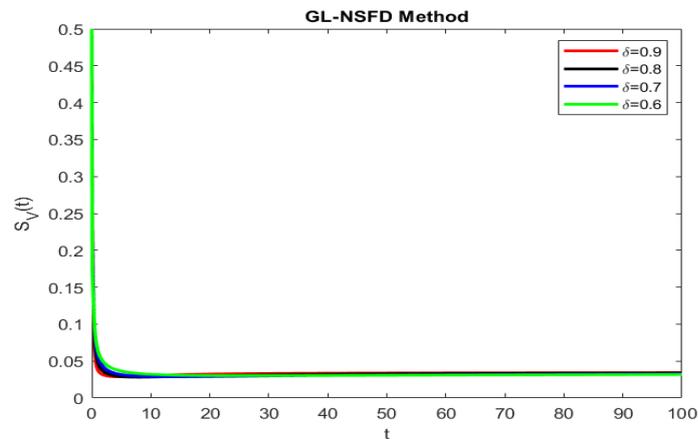
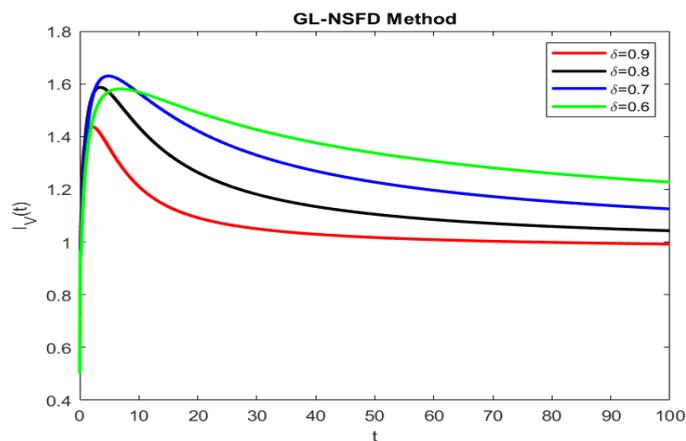**Fig. 6:** Numerical graph of susceptible individuals at endemic point.

Fig. 7: Numerical graph of infected individuals at endemic point.

Fig. 8: Numerical graph of recovered individuals at endemic point.

Fig. 9: Numerical graph of susceptible vectors at endemic point.

Fig. 10: Numerical graph of infected vectors at endemic point.

7 Conclusion

In this paper, a fractional order Buruli ulcer model is considered for the study. The mathematical model consists of S_H, I_H, R_H, S_V, I_V, S_{FF}, I_{FF}, and U_M as the state variables. Two steady equilibrium states i.e. disease free and endemic steady states are worked out for the model. The basic reproduction number is computed by the next-generation matrix technique. The stability of the model is also investigated at DFE. A time efficient structure preserving numerical scheme is proposed for the solution of the underlying Buruli ulcer fractional order mathematical model. The positivity and boundedness of the proposed scheme for the underlying model is ensured. A numerical example and simulations are also presented for the validation of the pre-results. This work may be applied on stochastic fractional epidemic models.

References

[1] Clancey, J. K. (1964). Mycobacterial skin ulcers in Uganda: description of a new mycobacterium (Mycobacterium buruli). *Journal of Pathology and Bacteriology*, 88(1), 175-187.

[2] Cook, A. (1897). The Mengo hospital notes. *Make College Medical School Library, Kampala*, 85(3), 175-77.

[3] Henry, R. (2020). Etymologia: Buruli Ulcer. *Emerging Infectious Diseases*, 26(3), 504.

[4] Van der Werf, T. S., Van der Graaf, W. T., Tappero, J. W., & Asiedu, K. (1999). Mycobacterium ulcerans infection. *The Lancet*, 354(9183), 1013-1018.

[5] Janssens, P. G., Quertinmont, M. J., Sieniawski, J. T., & Gatti, F. (1959). Necrotic tropical ulcers and mycobacterial causative agents. *Tropical and Geographical Medicine*, 11(4), 293-312.

[6] Clancey, J. K., Dodge, O. G., Lunn, H. F., & Oduori, M. L. (1961). Mycobacterial skin ulcers in Uganda. *The Lancet*, 278(7209), 951-954.

[7] Lunn, H. F., Connor, D. H., Wilks, N. E., Barnley, G. R., Kamunvi, F., Clancey, J. K., & Bee, J. D. A. (1965). Buruli (Mycobacterial) Ulceration in Uganda. (A New Focus of Buruli Ulcer in Madi District, Uganda.). *East African Medical Journal*, 42(6), 275-288.

[8] Farhan, M., Shah, Z., Jan, R., & Islam, S. (2023). A fractional modeling approach of Buruli ulcer in Possum mammals. *Physica Scripta*, 98(6), 065219.

[9] Chavda, V. P., Haritopoulou-Sinanidou, M., Bezbaruah, R., & Apostolopoulos, V. (2022). Vaccination efforts for Buruli ulcer. *Expert Review of Vaccines*, 21(10), 1419-1428.

[10] Nakanaga, K., Yotsu, R. R., Hoshino, Y., Suzuki, K., Makino, M., & Ishii, N. (2013). Buruli ulcer and mycolactone-producing mycobacteria. *Japanese journal of infectious diseases*, 66(2), 83-88.

[11] Zingue, D., Bouam, A., Tian, R. B., & Drancourt, M. (2018). Buruli ulcer, a prototype for ecosystem-related infection, caused by Mycobacterium ulcerans. *Clinical microbiology reviews*, 31(1), 10-1128.

[12] Amofah, G. K., Sagoe-Moses, C., Adjei-Acquah, C., & Frimpong, E. H. (1993). Epidemiology of buruli ulcer in Amansie west district, Ghana. *Transactions of the Royal Society of Tropical Medicine and Hygiene*, 87(6), 644-645.

[13] Duker, A. A., Carranza, E., & Hale, M. (2005). Arsenic geochemistry and health. *Environment international*, 31(5), 631-641.

[14] Merritt, R. W., Walker, E. D., Small, P. L., Wallace, J. R., Johnson, P. D., Benbow, M. E., & Boakye, D. A. (2010). Ecology and transmission of Buruli ulcer disease: a systematic review. *PLoS neglected tropical diseases*, 4(12), e911.

[15] Wansbrough-Jones, M., & Phillips, R. (2006). Buruli ulcer: emerging from obscurity. *The Lancet*, 367(9525), 1849-1858.

[16] Barogui, Y., Johnson, R. C., van der Werf, T. S., Sopoh, G., Dossou, A., Dijkstra, P. U., & Stienstra, Y. (2009). Functional limitations after surgical or antibiotic treatment for Buruli ulcer in Benin. *American Journal of Tropical Medicine and Hygiene*, 81(1), 82.

[17] Converse, P. J., Nuermberger, E. L., Almeida, D. V., & Grosset, J. H. (2011). Treating Mycobacterium ulcerans disease (Buruli ulcer): from surgery to antibiotics, is the pill mightier than the knife? *Future Microbiology*, 6(10), 1185-1198.

[18] Andreoli, A., Ruf, M. T., Sopoh, G. E., Schmid, P., & Pluschke, G. (2014). Immunohistochemical monitoring of wound healing in antibiotic treated Buruli ulcer patients. *PLoS neglected tropical diseases*, 8(4), e2809.

[19] Ahmed, S., Jahan, S., & Nisar, K. S. Hybrid Fibonacci wavelet method to solve fractional-order logistic growth model. *Mathematical Methods in the Applied Sciences*.

[20] Sontakke, B. R., & Shaikh, A. S. (2015). Properties of Caputo operator and its applications to linear fractional differential equations. *Int. J. Eng. Res. Appl*, 5(5), 22-27.

[21] Sheng, H., Li, Y., & Chen, Y. (2011). Application of numerical inverse Laplace transform algorithms in fractional calculus. *Journal of the Franklin Institute*, 348(2), 315-330.

[22] de Reyna, J. A. (1988). A generalized mean-value theorem. *Monatshefte für Mathematik*, 106, 95-97.

[23] Podlubny, I. (1998). Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. *Elsevier*.

[24] Liu, C., Liu, L., Cao, J., Abdel-Aty, M. (2023) Intermittent Event-Triggered Optimal Leader-Following Consensus for Nonlinear Multi-Agent Systems Via Actor-Critic Algorithm, IEEE Transactions on Neural Networks and Learning Systems, 34(8), pp. 3992–4006 http://dx.doi.org/10.1109/TNNLS.2021.3122458

[25] Wang, Z., Cao, J., Lu, G., Abdel-Aty, M. (2020) Fixed-Time Passification Analysis of Interconnected Memristive Reaction-Diffusion Neural Networks, IEEE Transactions on Network Science and Engineering, 7(3), pp. 1814–1824, 8906166 http://dx.doi.org/10.1109/TNSE.2019.2954463

[26] Wang, Z., Cao, J., Cai, Z., Abdel-Aty, M. (2020) A novel Lyapunov theorem on finite/fixed-time stability of discontinuous impulsive systems, Chaos, 2020, 30(1), 013139 http://dx.doi.org/10.1063/1.5121246

[27] Abdel-Aty, M., Moya-Cessa, H. (2007) Sudden death and long-lived entanglement of two trapped ions, Physics Letters A, 369(5-6), pp. 372–376 http://dx.doi.org/10.1016/j.physleta.2007.05.003

[28] Abdalla, M.S., Abdel-Aty, M., Obada, A.-S.F. (2002) Degree of entanglement for anisotropic coupled oscillators interacting with a single atom, Journal of Optics B 4(6), pp. 396–401 http://dx.doi.org/10.1088/1464-4266/4/6/305

[29] Abdel-Aty, M. (2002) General formalism of interaction of a two-level atom with cavity field in arbitrary forms of nonlinearities, Physica A 313(3-4), pp. 471–487 http://dx.doi.org/10.1016/S0378-4371(02)00999-8

[30] Abdalla, M.S., Obada, A.-S.F., Abdel-Aty, M. (2005) Von Neumann entropy and phase distribution of two mode parametric amplifier interacting with a single atom, Annals of Physics, 318(2), pp. 266–285 http://dx.doi.org/10.1016/j.aop.2005.01.002

[31] Abdel-Aty, M., Abdel-Khalek, S., Obada, A.-S.F. (2000) Pancharatnam phase of two-mode optical fields with Kerr nonlinearity, Optical Review, 7(6), pp. 499–504 http://dx.doi.org/10.1007/s10043-000-0499-6

[32] Obada, A.-S.F., Abdel-Hafez, A.M., Abdelaty, M. (1998) Phase properties of a Jaynes-Cummings model with Stark shift and Kerr medium, European Physical Journal D, 3(3), pp. 289–294 http://dx.doi.org/10.1007/s100530050176

[33] Nurul Syafiqah Mohd Ariff, Nur Sabiha Md Hussin, Khang Wen Goh, (2022), A Review of the Prevalence of Thromboembolic Complications among Pregnant Women Infected with COVID-19, *MDPI- Journal of Clinical Medicine*, Volume 11 Issue 19.