

A Quadratically Self-Modulated Exponential Distribution for Flexible Lifetime Modeling: Properties, Estimation, Simulation, and Application

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Received: 14 Apr. 2026, Revised: 08 Jul. 2026, Accepted: 30 Sep. 2026

Published online: 1 Nov. 2026

Abstract: This paper presents the Quadratically Self-Modulated Exponential (QSME) distribution, a new two-parameter lifetime model, which extends the classical exponential distribution while retaining its simplicity and the richness of interpretability. The exponential distribution is shown as an exact special case for $\alpha = 1$ but other values of α generate the range of hazard rate shapes, such as bathtub, increasing, stationary shoulder, and increase-decrease-increase. Also, some mathematical and reliability properties are obtained such as survival, cumulative hazard, quantile, moments, incomplete moments, entropy measures, residual life functions, order statistics, stress-strength reliability, and stochastic ordering. We analyze 8 estimation methods using a Monte Carlo simulation with sample sizes 20, 50, 125, and 250. The results show that percentile matching is the best for small samples and maximum likelihood is the most stable overall for moderate and large samples. An application to 128 bladder cancer remission times illustrates the practical use of the model. Among the models that achieved numerical convergence, QSME attained AIC = 832.5577, BIC = 838.2618, and KS = 0.0805 ($p = 0.3774$), with the smallest information criteria among the converged competitors.

Keywords: QSME distribution; exponential distribution; hazard rate; lifetime model; reliability; maximum likelihood; probability distribution

1 Introduction

Flexible lifetime distributions continue to play an important role in reliability analysis, survival studies, engineering, medicine, actuarial science, and risk analysis because failure mechanisms do not necessarily have to be constant but can be different in the classical exponential distribution. The exponential model is attractive as it is simple in terms of its mathematical properties and can be easily interpreted through a constant hazard rate. However, this same simple structure can limit the ability to describe data which includes early failures, aging, bathtub-shaped risks, non-monotone failure rates, or

different hazard behavior in the long term. This has led to more studies on generalized and extended lifetime distributions that need to be more flexible in shape but still have tractable distributional and inferential properties.

A major direction in this literature has been to define systematic methods of extending familiar baseline distributions. Marshall and Olkin [19] introduced an important parameter extension scheme which allows for flexibility of standard family distributions without losing useful analytical properties. Other important developments include the Kumaraswamy-generalized family of Cordeiro and de Castro [14], the T-X approach

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in Alzaatreh et al. [10], power Lindley-G family [29], inverse Weibull-G [33], new sine family [26], unit exponentiated half logistic power series class of distributions [30], Topp-Leone odd Fréchet-G [27], complementary geometric Topp-Leone-G family [32], exponentiated power generalized Weibull power series family [28], new Topp-Leone Kumaraswamy Marshall-Olkin-G [31], odd inverse power generalized Weibull-G [25], and arctangent Zubair-G [34]. For more extensions see [35–37]. These approaches established a broad principle: flexibility in distributions can often be achieved by changing or improving a simple baseline model, and not dropping its basic probabilistic nature. This has motivated a great many flexible models that are suitable for skewed, heavy-tailed, heterogeneous and non-standard lifetime data. The exponential distribution has been particularly under the focus of this research because of its importance in lifetime and reliability modeling. Chettri et al. [13] proposed a generalized power transformation of the exponential family and studied its mathematical properties and applications. Afify et al. [2] introduced the power-modified Kies-exponential distribution, obtaining additional shape flexibility together with several distributional, inferential and applied properties. Bhatti et al. [11] developed an extended quadratic hazard-rate distribution in which a polynomial-type hazard structure allows for a much wider range of behavior of failure rates than the ordinary exponential distribution. More recently, Bousseba et al. [12] studied a two-parameter quadratic exponential distribution and proved that simple extensions can provide great flexibility without the need for highly parameterized formulation.

More recent studies have shown the practical relevance of flexible exponential and lifetime models. Aljohani et al. [9] extended the exponential distribution to complex life and environmental data and also showed how breaking away from a classical exponential distribution may improve the representation of heterogeneous observations. Orji et al. [21] proposed an odd reparameterized exponential transformed-X family with applications to public health data. This work further illustrates the utility of flexible exponential-based models in modeling. Heavy-tailed statistical models have also been studied for complex real-data structures [16], and generalized flexible families have been developed to improve theoretical adaptability, estimation and risk analysis [3]. Extended lifetime distributions have likewise shown value in survival applications, including the analysis of cancer-survival data [18]. Further recent contributions include one-parameter and extended Lindley-type lifetime models with exact inference and actuarial risk measures [17, 22], and odds-based Weibull modifications for engineering failure-time data [5]. So, flexible lifetime modeling is still an active field where both analytical tractability and practical adaptability have been discussed.

Despite this progress, more flexibility is often achieved by introducing additional free parameters, nested transformations, generators and multi-stage parameterization. While such constructions can improve goodness of fit they can also complicate interpretation, numerical optimization, parameter identifiability and finite sample inference. This is especially true in reliability analysis, where the parameters should ideally retain meaningful roles related to scale and hazard shape. Hence, a parsimonious lifetime distribution can be produced to generate several important hazard patterns while leaving a small and interpretable parameter vector.

A useful two-parameter lifetime model should thus satisfy a number of requirements. It should display a familiar baseline distribution as a true nested special case. It should distinguish scale and shape effects as clearly as possible. Its cumulative distribution function should remain proper over the entire positive support. Its density and survival functions should have mathematically manageable forms. Most importantly, its hazard function should have patterns that cannot be represented in the standard exponential model, such as decreasing then increasing behavior and non-linear forms. At the same time, the model should be parsimonious enough to be able to provide reasonable estimation, simulation and practical implementation.

The Quadratically Self-Modulated Exponential (QSME) distribution proposed in this paper is designed to achieve both of these results by a self-modulation mechanism in the hazard structure. Rather than introducing an additional independent parameter to control the quadratic modification, the modulation coefficient is specified by the shape parameter as $c_\alpha = (\alpha^2 - 1)^2$ and vanishes only when $\alpha = 1$.

This is the key feature of the proposed model. When $\alpha = 1$ the self-modulation disappears automatically and the QSME distribution is exactly the same as the classical exponential one. Thus, the additional hazard flexibility does not need a third free parameter. The rate parameter $\lambda > 0$ plays the usual scale role and shape parameter $\alpha > 0$ is responsible for both the baseline power behavior and the magnitude of the quadratic modulation. Therefore, the resulting model has the parsimony and significantly richer hazard structure. Another big feature of the proposed construction is that the flexibility of the QSME model is based on a relationship between its existing parameters rather than an externally-added generator. This makes it different from many transformation-based families and parameter-added extensions. The self-modulation mechanism allows the hazard rate to change qualitatively as α varies, while the model remains truly two-parameter. Depending on the shape parameter, the hazard structure can also reproduce the constant exponential case as well as increasing, bathtub-type and more complicated increase-decrease-increase patterns. This makes the model important in the lifetime process where the

underlying risk mechanism is not constant. The theoretical contribution of this paper goes beyond defining a new density function. The QSME model is constructed by the hazard and cumulative hazard structures, and it is shown to be a proper continuous probability distribution analytically. We obtain survival, cumulative distribution, density, cumulative hazard, reversed hazard, and quantile functions.

The exponential model is obtained explicitly as an exact nested special case. We study the derivative of the standardized hazard to obtain the conditions of the different hazard rate shapes and their turning points. We also look at a large collection of distributional properties to give a complete mathematical description of the model. These include ordinary and incomplete moments, moment and characteristic generating functions, entropy measures, residual life and inactivity time functions, probability weighted moments, order statistics, inequality measures, stochastic ordering, and stress strength reliability. The building of these properties is critical because a flexible probability model should not only be able to have graphical flexibility, but also need a coherent analytical framework in which it is possible to interpret the predictions. In our inferential part of the study, we consider several complementary estimation approaches and not only the maximum likelihood. We study the parameters using maximum likelihood, least squares, weighted least squares, Cramér-von Mises, Anderson-Darling, maximum product spacings, percentile matching and quantile least squares estimation. Such a multi-method framework allows the behaviour of the estimators to be examined under different objective criteria and can provide a stronger basis for evaluating finite sample performance. A Monte Carlo experiment is therefore reserved to compare the accuracy of the estimators in terms of bias, mean squared error, root mean squared error, mean absolute error, dispersion and convergence behavior. In the real data, we will evaluate the empirical utility of the QSME distribution relative to the existing lifetime competitors.

The main contribution of the proposed QSME distribution is of three kinds: a truly two-parameter structure, a perfect recovery of the exponential distribution and very strong hazard shape flexibility via the deterministic self-modulation mechanism. It aims to bridge the gap between the very parsimonious exponential model and the more detailed life extension. It also provides a mathematically sensible approach in which other hazard flexibility can be created instead of using a modulation parameter independent of this. The rest of the paper is organized as follows. In Section 2, we introduce the QSME distribution and present its basic reliability structure, validity conditions, special cases, limiting behavior, and hazard rate classification. In Section 3, the major mathematical and distributional properties of the model are derived. Section 4 establishes the statistical inference and eight estimation procedures. In Section 5, we conduct a Monte Carlo simulation study,

while in Section 6, we perform the real data application and comparison. The final section 7 summarizes the main findings and makes some theoretical recommendations for future developments of our model.

2 The Quadratically Self-Modulated Exponential Distribution

2.1 Definition

Let X be a positive continuous random variable and let $X \sim QSME(\alpha, \lambda)$, $\alpha > 0$, $\lambda > 0$. Define the standardized variable $t = \lambda x$, and the self-modulation coefficient $c_\alpha = (\alpha^2 - 1)^2 \geq 0$.

For convenience, define

$$g_\alpha(t) = 1 + c_\alpha(t-1)^2.$$

The hazard rate of the QSME distribution is

$$h(x; \alpha, \lambda) = \lambda t^{\alpha-1} g_\alpha(t), \quad t = \lambda x.$$

$$h(x; \alpha, \lambda) = \lambda t^{\alpha-1} [1 + c_\alpha(t-1)^2], \quad t = \lambda x.$$

Substitution of t and c_α gives

$$h(x; \alpha, \lambda) = \lambda (\lambda x)^{\alpha-1} [1 + (\alpha^2 - 1)^2 (\lambda x - 1)^2]. \quad (1)$$

The hazard rate is strictly positive for every $x > 0$ because

$$1 + (\alpha^2 - 1)^2 (\lambda x - 1)^2 > 0.$$

Because $t > 0$ and $g_\alpha(t) > 0$, the hazard is strictly positive for every $x > 0$. Integrating the standardized hazard from zero to t gives

$$A_\alpha(t) = \int_0^t s^{\alpha-1} g_\alpha(s) ds,$$

so that

$$A_\alpha(t) = \frac{(1 + c_\alpha)t^\alpha}{\alpha} - \frac{2c_\alpha t^{\alpha+1}}{\alpha + 1} + \frac{c_\alpha t^{\alpha+2}}{\alpha + 2}. \quad (2)$$

The cumulative hazard function is therefore

$$H(x; \alpha, \lambda) = A_\alpha(\lambda x). \quad (3)$$

An equivalent representation is

$$A_\alpha(t) = \frac{t^\alpha}{\alpha} + c_\alpha \left[\frac{t^{\alpha+2}}{\alpha + 2} - \frac{2t^{\alpha+1}}{\alpha + 1} + \frac{t^\alpha}{\alpha} \right].$$

The survival function, cumulative distribution function, and probability density function are

$$S(x; \alpha, \lambda) = \exp\{-A_\alpha(\lambda x)\},$$

$$S(x; \alpha, \lambda) = \exp\left\{-\frac{(1+c_\alpha)(\lambda x)^\alpha}{\alpha} + \frac{2c_\alpha(\lambda x)^{\alpha+1}}{\alpha+1} - \frac{c_\alpha(\lambda x)^{\alpha+2}}{\alpha+2}\right\}, \quad (4)$$

Consequently, the cumulative distribution function is

$$F(x; \alpha, \lambda) = 1 - \exp[-A_\alpha(\lambda x)].$$

Its expanded form is

$$F(x; \alpha, \lambda) = 1 - \exp\left\{-\frac{(1+c_\alpha)(\lambda x)^\alpha}{\alpha} + \frac{2c_\alpha(\lambda x)^{\alpha+1}}{\alpha+1} - \frac{c_\alpha(\lambda x)^{\alpha+2}}{\alpha+2}\right\}. \quad (5)$$

Using the identity $f(x) = h(x)S(x)$, the probability density function becomes

$$f(x; \alpha, \lambda) = \lambda(\lambda x)^{\alpha-1} [1 + c_\alpha(\lambda x - 1)^2] \exp[-A_\alpha(\lambda x)].$$

The complete expression is

$$\begin{aligned} f(x; \alpha, \lambda) &= \lambda(\lambda x)^{\alpha-1} \left[1 + (\alpha^2 - 1)^2 (\lambda x - 1)^2\right] \\ &\times \exp\left\{-\frac{\left[1 + (\alpha^2 - 1)^2\right](\lambda x)^\alpha}{\alpha} \right. \\ &+ \frac{2(\alpha^2 - 1)^2 (\lambda x)^{\alpha+1}}{\alpha+1} \\ &\left. - \frac{(\alpha^2 - 1)^2 (\lambda x)^{\alpha+2}}{\alpha+2}\right\}. \end{aligned} \quad (6)$$

These expressions completely define the proposed model on $x > 0$.

Theorem 1 (Validity of the QSME distribution).

For every $\alpha > 0$ and $\lambda > 0$, the function in (5) is a proper cumulative distribution function on $(0, \infty)$ and the density in (6) integrates to one.

Proof. From (2),

$$A'_\alpha(t) = t^{\alpha-1} g_\alpha(t) > 0, \quad t > 0.$$

Thus $A_\alpha(t)$ is strictly increasing. Also $A_\alpha(0) = 0$. If $\alpha = 1$, then $c_1 = 0$ and $A_1(t) = t \rightarrow \infty$. If $\alpha \neq 1$, then $c_\alpha > 0$ and the leading term is

$$A_\alpha(t) \sim \frac{c_\alpha}{\alpha+2} t^{\alpha+2}, \quad t \rightarrow \infty,$$

which also tends to infinity. Consequently,

$$F(0^+) = 1 - e^{-A_\alpha(0)} = 0, \quad \lim_{x \rightarrow \infty} F(x) = 1.$$

Because $A'_\alpha(t) > 0$, differentiating (5) yields the nonnegative density in (6). Finally,

$$\int_0^\infty f(x) dx = F(\infty) - F(0^+) = 1.$$

Hence the QSME model is a proper continuous probability distribution. $\square$

2.2 Exponential Special Case and Scale Structure

Proposition 1 (Exact exponential sub-model).

When $\alpha = 1$, the QSME distribution reduces exactly to the exponential distribution with rate λ .

Proof. At $\alpha = 1$, $c_1 = (1^2 - 1)^2 = 0$, and therefore

$$A_1(t) = t.$$

Substituting into (5) and (6) gives

$$F(x; 1, \lambda) = 1 - e^{-\lambda x},$$

and

$$f(x; 1, \lambda) = \lambda e^{-\lambda x}.$$

Thus $QSME(1, \lambda) \equiv Exp(\lambda)$. $\square$

Proposition 2 (Scale representation).

If $X \sim QSME(\alpha, \lambda)$ and $T = \lambda X$, then the distribution of T depends only on α and has CDF

$$G_\alpha(t) = 1 - e^{-A_\alpha(t)}, \quad t > 0. \quad (7)$$

Consequently, λ is a pure inverse-scale parameter.

Proof. For $t > 0$,

$$P(T \leq t) = P\left(X \leq \frac{t}{\lambda}\right) = F\left(\frac{t}{\lambda}; \alpha, \lambda\right) = 1 - e^{-A_\alpha(t)}.$$

The right-hand side contains no λ . $\square$

2.3 Survival and Related Reliability Functions

The principal reliability functions follow directly. The ordinary hazard is already given by (1). The cumulative hazard is (3). The reversed hazard rate is

$$r(x) = \frac{f(x)}{F(x)} = \frac{\lambda t^{\alpha-1} g_\alpha(t)}{e^{A_\alpha(t)} - 1}, \quad t = \lambda x. \quad (8)$$

The odds function is

$$O(x) = \frac{F(x)}{S(x)} = e^{A_\alpha(t)} - 1, \quad (9)$$

and the cumulative reversed hazard is

$$R(x) = -\log F(x) = -\log [1 - e^{-A_\alpha(t)}]. \quad (10)$$

These functions are useful in reliability analysis because they describe forward failure risk, accumulated risk, backward failure risk, and failure odds under a common two-parameter structure.

2.4 Limiting Behavior

The behavior near the lower endpoint follows from $g_\alpha(t) \rightarrow 1 + c_\alpha$, $A_\alpha(t) \sim \frac{1+c_\alpha}{\alpha} t^\alpha$, $t \rightarrow 0^+$.

Hence $f(x) \sim \lambda (1 + c_\alpha) (\lambda x)^{\alpha-1}$, $x \rightarrow 0^+$.

Therefore, the density diverges at the origin when $0 < \alpha < 1$, equals λ when $\alpha = 1$, and approaches zero when $\alpha > 1$. For $\alpha \neq 1$, as $t \rightarrow \infty$,

$$g_\alpha(t) \sim c_\alpha t^2, \quad (11)$$

$$A_\alpha(t) \sim \frac{c_\alpha}{\alpha+2} t^{\alpha+2}, \quad (12)$$

and

$$f(x) \sim \lambda c_\alpha t^{\alpha+1} \exp \left\{ -\frac{c_\alpha}{\alpha+2} t^{\alpha+2} \right\}. \quad (13)$$

Thus the non-exponential QSME members have rapidly decaying upper tails.

2.5 Shapes of the Hazard Rate Function

Differentiating (1) with respect to x gives

$$\frac{d}{dx} h(x; \alpha, \lambda) = \lambda^2 t^{\alpha-2} P_\alpha(t),$$

where

$$P_\alpha(t) = c_\alpha(\alpha+1)t^2 - 2c_\alpha\alpha t + (\alpha-1)(1+c_\alpha). \quad (14)$$

Because $\lambda^2 t^{\alpha-2} > 0$, the sign of the hazard derivative is determined entirely by $P_\alpha(t)$. Its discriminant is

$$\Delta_\alpha = 4c_\alpha [c_\alpha - (\alpha^2 - 1)] = 4(\alpha^2 - 1)^3 (\alpha^2 - 2). \quad (15)$$

Theorem 2 (Exact hazard-shape classification).

The QSME hazard has the following shapes:

Range of α	Hazard-rate shape
$0 < \alpha < 1$	Bathtub-shaped
$\alpha = 1$	Constant exponential hazard
$1 < \alpha < \sqrt{2}$	Strictly increasing
$\alpha = \sqrt{2}$	Increasing with a stationary shoulder
$\alpha > \sqrt{2}$	Increase-decrease-increase (N-shaped)

Proof. For $0 < \alpha < 1$, the constant term in (14) is negative whereas the leading coefficient is positive. Hence $P_\alpha(0) < 0$ and $P_\alpha(t) \rightarrow \infty$, so there is exactly one positive zero and the hazard first decreases and then increases. At $\alpha = 1$, $c_1 = 0$ and $h(x) = \lambda$.

For $1 < \alpha < \sqrt{2}$, equation (15) gives $\Delta_\alpha < 0$. Since the leading coefficient and constant term in (14) are both positive, $P_\alpha(t) > 0$ for all $t > 0$, so the hazard is strictly increasing. At $\alpha = \sqrt{2}$, the discriminant is zero and the quadratic touches zero at a double root, giving an increasing hazard with a stationary shoulder. For $\alpha > \sqrt{2}$, the discriminant is positive and both roots are positive. The quadratic is positive before the smaller root, negative between the roots, and positive after the larger root. Therefore the hazard increases, decreases, and increases again. $\square$

For $\alpha > \sqrt{2}$, the turning points on the standardized scale are

$$t_\pm = \frac{\alpha \pm \sqrt{1 - \frac{1}{\alpha^2 - 1}}}{\alpha + 1}, \quad (16)$$

and on the original scale

$$x_\pm = \frac{t_\pm}{\lambda}. \quad (17)$$

At $\alpha = \sqrt{2}$, the two points merge into

$$t_0 = \frac{\alpha}{\alpha + 1}.$$

To illustrate the main characteristics and flexibility of the QSME distribution, Figures 1 and 2 present its density, distribution, hazard, and cumulative hazard functions under different values of α and λ .

Figure 1 shows that the QSME distribution can provide different density shapes and probability accumulation rates. The case $\alpha = 1$ leads to exponential distribution, while for large values of α the density is more flexible to unimodal and curved. In Figure 2 we show the flexibility of the QSME hazard function (from constant exponential hazard to increasing risk and increase-decrease-increase forms). The hazard is high for all combinations of parameters and the cumulative hazard is still high so the risk structure is valid.

3 Distributional Properties

3.1 Quantile Function and Random-Variate Generation

Let $0 < u < 1$. From (6), the quantile $Q(u)$ satisfies

$$A_\alpha(\lambda Q(u)) = -\log(1 - u). \quad (18)$$

Because $A'_\alpha(t) > 0$, A_α is one-to-one from $(0, \infty)$ onto $(0, \infty)$. Therefore (18) has a unique solution. If z_u is the unique positive root of

$$A_\alpha(z_u) + \log(1 - u) = 0, \quad (19)$$

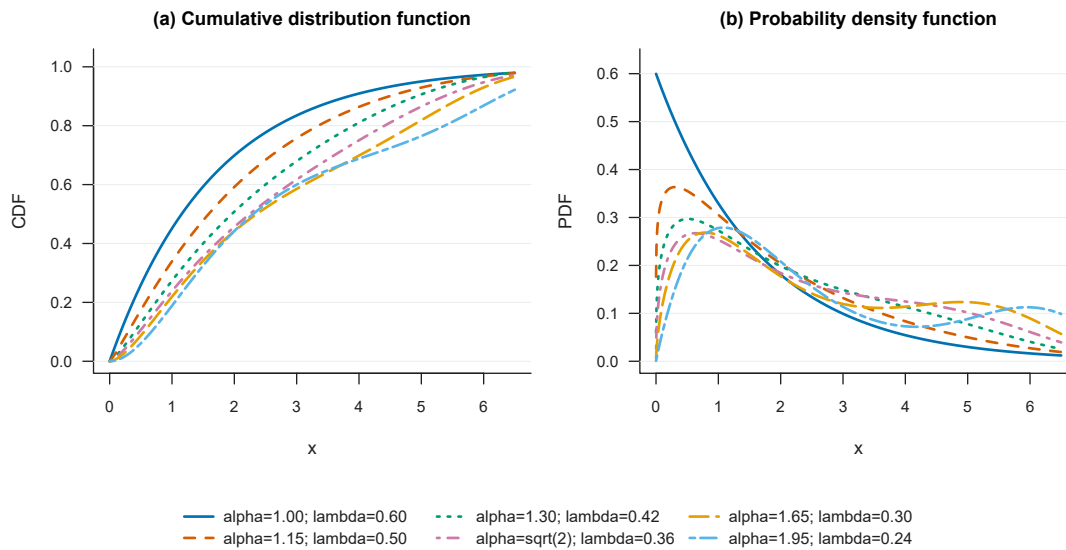


Fig. 1: QSME cumulative distribution function (a) and probability density function (b) for selected (α, λ) combinations. Distinct line types and colors identify the parameter settings; $\alpha = 1$ gives the exponential sub-model.

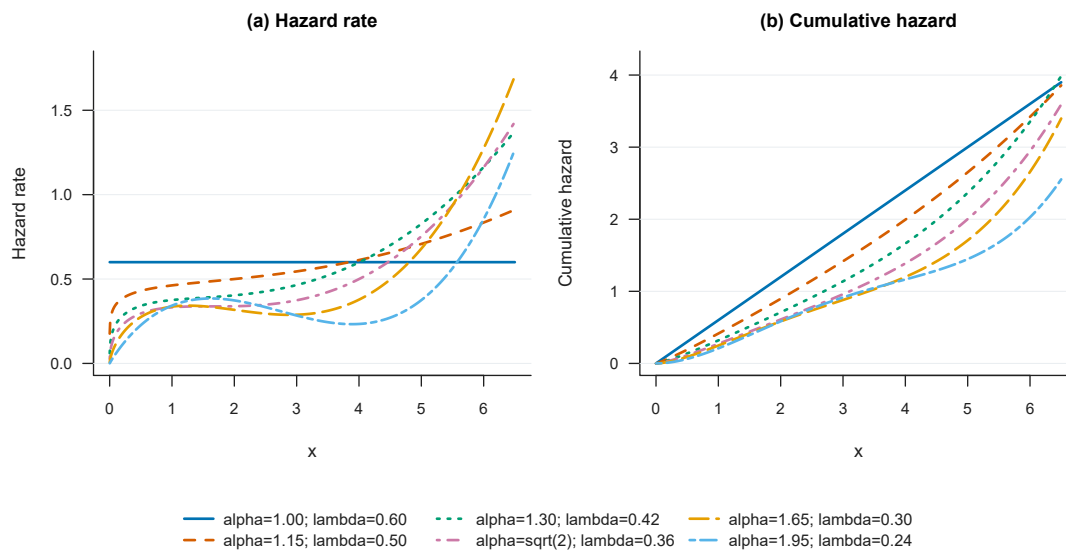


Fig. 2: QSME hazard rate (a) and cumulative hazard function (b) for selected (α, λ) combinations, illustrating the transition from the constant exponential hazard to increasing and non-monotone hazard patterns.

then

$$Q(u) = \frac{z_u}{\lambda}. \quad (20)$$

A Newton-Raphson update is

$$z_u^{(m+1)} = z_u^{(m)} - \frac{A_\alpha(z_u^{(m)}) + \log(1-u)}{\left(z_u^{(m)}\right)^{\alpha-1} g_\alpha(z_u^{(m)})}. \quad (21)$$

Since the denominator is strictly positive, numerical inversion is stable whenever a positive starting value is used. Random observations can therefore be generated by drawing $U \sim \text{Uniform}(0, 1)$ and setting $X = Q(U)$.

3.2 Raw Moments

For $r > 0$, the standard survival identity gives

$$E(X^r) = r \int_0^\infty x^{r-1} S(x) dx. \quad (22)$$

Using $t = \lambda x$ yields

$$\mu'_r = E(X^r) = \frac{r}{\lambda^r} \int_0^\infty t^{r-1} e^{-A_\alpha(t)} dt.$$

This representation makes the scale structure explicit. Define

$$I_r(\alpha) = \int_0^\infty t^{r-1} e^{-A_\alpha(t)} dt.$$

Then

$$\mu'_r = \frac{r I_r(\alpha)}{\lambda^r}.$$

In particular,

$$E(X) = \frac{1}{\lambda} \int_0^\infty e^{-A_\alpha(t)} dt,$$

$$E(X^2) = \frac{2}{\lambda^2} \int_0^\infty t e^{-A_\alpha(t)} dt,$$

and

$$\text{Var}(X) = E(X^2) - [E(X)]^2.$$

When $\alpha = 1$, $A_1(t) = t$, so

$$E(X^r) = \frac{\Gamma(r+1)}{\lambda^r},$$

which is the ordinary exponential moment formula.

Proposition 3 (Existence of all positive moments).

All positive moments of the QSME distribution exist for every $\alpha > 0$ and $\lambda > 0$.

Proof. The exponential case $\alpha = 1$ is immediate. If $\alpha \neq 1$, then by (12), $A_\alpha(t)$ grows as a positive constant times $t^{\alpha+2}$. Hence for every $r > 0$, the integrand in (22) is eventually bounded by a polynomial multiplied by $e^{-Kt^{\alpha+2}}$ for some $K > 0$. Such an integral is finite. $\square$

The coefficient of variation, skewness, and kurtosis are obtained from the first four raw moments in the usual way. Let $\mu = \mu'_1$ and $\sigma^2 = \mu'_2 - \mu^2$. Then

$$\gamma_1 = \frac{\mu'_3 - 3\mu\mu'_2 + 2\mu^3}{\sigma^3},$$

and

$$\gamma_2 = \frac{\mu'_4 - 4\mu\mu'_3 + 6\mu^2\mu'_2 - 3\mu^4}{\sigma^4} - 3.$$

3.3 Incomplete and Tail Moments

The lower incomplete moment of order $r > 0$ is

$$m_r(x) = E[X^r \mathbf{1}(X \leq x)].$$

Using integration by parts and $t = \lambda x$ gives

$$m_r(x) = \frac{1}{\lambda^r} \left[-t^r e^{-A_\alpha(t)} + r \int_0^t s^{r-1} e^{-A_\alpha(s)} ds \right]. \quad (23)$$

The upper incomplete or tail moment is

$$\begin{aligned} \bar{m}_r(x) &= E[X^r \mathbf{1}(X > x)] \\ &= \frac{1}{\lambda^r} \left[t^r e^{-A_\alpha(t)} + r \int_t^\infty s^{r-1} e^{-A_\alpha(s)} ds \right]. \end{aligned} \quad (24)$$

Equations (23) and (24) are useful for mean deviations, residual moments, inequality measures, and actuarial tail quantities.

3.4 Moment Generating and Characteristic Functions

The moment generating function can be written as

$$M_X(s) = \int_0^\infty e^{sx} f(x) dx = \int_0^\infty \exp\left(\frac{s}{\lambda} t\right) A'_\alpha(t) e^{-A_\alpha(t)} dt.$$

For $\alpha \neq 1$, the leading term of $A_\alpha(t)$ is proportional to $t^{\alpha+2}$, so $M_X(s)$ exists for every real s . When $\alpha = 1$, the QSME model is exponential and

$$M_X(s) = \frac{\lambda}{\lambda - s}, \quad s < \lambda.$$

The characteristic function is

$$\varphi_X(\omega) = \int_0^\infty \exp\left(i \frac{\omega}{\lambda} t\right) A'_\alpha(t) e^{-A_\alpha(t)} dt, \quad \omega \in \mathbb{R}.$$

3.5 Shannon and Rényi Entropies

The Shannon entropy is

$$\mathcal{H}(X) = -E[\log f(X)].$$

Let $T = \lambda X$. Since

$$P(A_\alpha(T) \leq y) = 1 - e^{-y}, \quad y > 0,$$

we have $A_\alpha(T) \sim \text{Exp}(1)$ and hence $E[A_\alpha(T)] = 1$. Since

$$\log f(X) = \log \lambda + \log A'_\alpha(T) - A_\alpha(T),$$

it follows that

$$\mathcal{H}(X) = 1 - \log \lambda - \int_0^\infty A'_\alpha(t) e^{-A_\alpha(t)} \log A'_\alpha(t) dt.$$

Because

$$A'_\alpha(t) = t^{\alpha-1} g_\alpha(t),$$

an equivalent form is

$$\mathcal{H}(X) = 1 - \log \lambda - (\alpha - 1)E(\log T) - E[\log g_\alpha(T)].$$

For order $q > 0$, $q \neq 1$, the Rényi entropy is

$$\mathcal{H}_q(X) = \frac{1}{1-q} \log \int_0^\infty f(x)^q dx.$$

After standardization,

$$\mathcal{H}_q(X) = -\log \lambda + \frac{1}{1-q} \log \left[\int_0^\infty \{A'_\alpha(t)\}^q e^{-qA_\alpha(t)} dt \right].$$

3.6 Mean Residual Life and Mean Inactivity Time

The mean residual life at age x is

$$m(x) = E(X - x | X > x) = \frac{1}{S(x)} \int_x^\infty S(u) du. \quad (25)$$

Therefore,

$$m(x) = \frac{e^{A_\alpha(t)}}{\lambda} \int_t^\infty e^{-A_\alpha(s)} ds, \quad t = \lambda x.$$

The mean inactivity time is

$$\tilde{m}(x) = E(x - X | X \leq x) = \frac{1}{F(x)} \int_0^x F(u) du,$$

and hence

$$\tilde{m}(x) = \frac{1}{\lambda [1 - e^{-A_\alpha(t)}]} \left[t - \int_0^t e^{-A_\alpha(s)} ds \right].$$

These two functions describe expected remaining lifetime and expected elapsed inactivity, respectively.

3.7 Probability-Weighted Moments

For $r \geq 0$ and $s \geq 0$, define the probability-weighted moment

$$\beta_{r,s} = E[X^r F(X)^s].$$

Using $t = \lambda x$ gives

$$\beta_{r,s} = \frac{1}{\lambda^r} \int_0^\infty t^r [1 - e^{-A_\alpha(t)}]^s A'_\alpha(t) e^{-A_\alpha(t)} dt.$$

When s is a nonnegative integer, the binomial expansion yields

$$\beta_{r,s} = \frac{1}{\lambda^r} \sum_{j=0}^s (-1)^j \binom{s}{j} \int_0^\infty t^r A'_\alpha(t) e^{-(j+1)A_\alpha(t)} dt.$$

3.8 Order Statistics

Let $X_{1:n} \leq \dots \leq X_{n:n}$ denote the order statistics from a random sample of size n . The density of the i th order statistic is

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} [F(x)]^{i-1} [1 - F(x)]^{n-i} f(x).$$

Substitution of the QSME functions gives

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} [1 - e^{-A_\alpha(t)}]^{i-1} \times e^{-(n-i+1)A_\alpha(t)} \lambda A'_\alpha(t).$$

The corresponding CDF is

$$F_{i:n}(x) = \sum_{j=i}^n \binom{n}{j} [F(x)]^j [1 - F(x)]^{n-j}.$$

The minimum and maximum follow by taking $i = 1$ and $i = n$, respectively.

3.9 Lorenz and Bonferroni Curves

Let $Q(p)$ be the quantile in (20) and let $\mu = E(X)$. The Lorenz curve is

$$L(p) = \frac{m_1(Q(p))}{\mu}, \quad 0 < p < 1,$$

where $m_1(\cdot)$ is given by (25). The Bonferroni curve is

$$B(p) = \frac{L(p)}{p}.$$

These curves can be evaluated numerically without additional integration once the incomplete first moment and quantile have been computed.

3.10 Stress-Strength Reliability

Let $X \sim QSME(\alpha_1, \lambda_1)$ represent strength and let $Y \sim QSME(\alpha_2, \lambda_2)$ represent stress, independently. The reliability probability is

$$R = P(X > Y) = \int_0^\infty S_X(y) f_Y(y) dy.$$

With $s = \lambda_2 y$ and $\rho = \lambda_1 / \lambda_2$,

$$R = \int_0^\infty \exp\{-A_{\alpha_1}(\rho s)\} A'_{\alpha_2}(s) e^{-A_{\alpha_2}(s)} ds.$$

When the two distributions are identical, continuity and symmetry imply $R = 1/2$. Stress-strength reliability has been studied extensively for other lifetime models under complete, censored, hybrid-censored and record data, with engineering and medical applications [4, 7, 15, 23].

3.11 Stochastic Ordering with Respect to the Rate Parameter

Proposition 4 (Rate ordering).

For fixed α , if $\lambda_2 > \lambda_1 > 0$, then

$$X_{\lambda_2} \leq_{st} X_{\lambda_1}.$$

Proof. By (7),

$$F(x; \alpha, \lambda) = G_\alpha(\lambda x),$$

where G_α is strictly increasing. Therefore, for every $x > 0$,

$$F(x; \alpha, \lambda_2) = G_\alpha(\lambda_2 x) \geq G_\alpha(\lambda_1 x) = F(x; \alpha, \lambda_1).$$

This is exactly the usual stochastic ordering relation.

□

4 Statistical Inference

Classical and Bayesian inference for flexible lifetime models under complete samples has been developed in [6, 8]. Here we focus on complete samples and compare eight frequentist estimators.

Let $X_1, \dots, X_n$ be a random sample from the QSME distribution and let

$$\psi = (\alpha, \lambda)^T, \quad \alpha > 0, \quad \lambda > 0.$$

For each observation define

$$t_i = \lambda x_i, \quad c = c_\alpha = (\alpha^2 - 1)^2, \quad g_i = 1 + c(t_i - 1)^2.$$

The following derivatives are used repeatedly in the estimating equations.

4.1 Derivatives Required for Estimation

The derivative of the modulation coefficient is

$$c'_\alpha = 4\alpha(\alpha^2 - 1).$$

The partial derivative of $A_\alpha(t)$ with respect to α , holding t fixed, is

$$A_{\alpha, \alpha}(t) = t^\alpha \left[\frac{c'_\alpha}{\alpha} + (1 + c_\alpha) \left(\frac{\log t}{\alpha} - \frac{1}{\alpha^2} \right) \right] - 2t^{\alpha+1} \left[\frac{c'_\alpha}{\alpha+1} + c_\alpha \left(\frac{\log t}{\alpha+1} - \frac{1}{(\alpha+1)^2} \right) \right] + t^{\alpha+2} \left[\frac{c'_\alpha}{\alpha+2} + c_\alpha \left(\frac{\log t}{\alpha+2} - \frac{1}{(\alpha+2)^2} \right) \right].$$

Since $t = \lambda x$,

$$\frac{\partial A_\alpha(t)}{\partial \lambda} = x A'_\alpha(t) = \frac{t^\alpha g_\alpha(t)}{\lambda}.$$

Because $F = 1 - e^{-A}$,

$$F_\alpha(x) = S(x) A_{\alpha, \alpha}(t),$$

and

$$F_\lambda(x) = S(x) \frac{t^\alpha g_\alpha(t)}{\lambda}.$$

These two derivatives are sufficient to write the estimating equations for the CDF-based methods.

4.2 Maximum Likelihood Estimation

The likelihood function is

$$L(\alpha, \lambda) = \prod_{i=1}^n \left[\lambda t_i^{\alpha-1} g_i e^{-A_\alpha(t_i)} \right].$$

The log-likelihood is

$$\ell(\alpha, \lambda) = n \log \lambda + (\alpha - 1) \sum_{i=1}^n \log t_i + \sum_{i=1}^n \log g_i - \sum_{i=1}^n A_\alpha(t_i).$$

The score equation for α is

$$U_\alpha = \sum_{i=1}^n \left[\log t_i + \frac{c'_\alpha (t_i - 1)^2}{g_i} - A_{\alpha, \alpha}(t_i) \right] = 0.$$

The score equation for λ is

$$U_\lambda = \frac{1}{\lambda} \sum_{i=1}^n \left[\alpha + \frac{2c_\alpha t_i (t_i - 1)}{g_i} - t_i^\alpha g_i \right] = 0.$$

Closed-form solutions are not generally available, so the maximum likelihood estimator is obtained numerically as

$$\hat{\psi}_{MLE} = \arg \min_{\alpha > 0, \lambda > 0} \{-\hat{\mathcal{J}}(\alpha, \lambda)\}.$$

Standard errors may be obtained from the inverse observed information matrix, calculated from the numerical Hessian of $-\hat{\mathcal{J}}$ at the optimum.

4.3 Least Squares Estimation

Let $X_{(1)} \leq \dots \leq X_{(n)}$ be the ordered sample and define

$$p_i = \frac{i}{n+1}.$$

The least squares criterion is

$$S_{LSE}(\psi) = \sum_{i=1}^n [F(X_{(i)}; \psi) - p_i]^2.$$

Thus

$$\hat{\psi}_{LSE} = \arg \min_{\psi} S_{LSE}(\psi).$$

For $\psi_j \in \{\alpha, \lambda\}$, the estimating equations are

$$\sum_{i=1}^n [F_i - p_i] F_{ij} = 0,$$

where $F_i = F(X_{(i)}; \psi)$ and $F_{ij} = \partial F_i / \partial \psi_j$.

4.4 Weighted Least Squares Estimation

For a standard uniform order statistic,

$$\text{Var}(U_{(i)}) = \frac{i(n-i+1)}{(n+1)^2(n+2)}.$$

The inverse-variance weight is therefore

$$w_i = \frac{(n+1)^2(n+2)}{i(n-i+1)}.$$

The weighted least squares criterion is

$$S_{WLSE}(\psi) = \sum_{i=1}^n w_i [F_i - p_i]^2,$$

with estimating equations

$$\sum_{i=1}^n w_i (F_i - p_i) F_{ij} = 0.$$

4.5 Cramér-von Mises Estimation

The Cramér-von Mises objective is

$$S_{CVM}(\psi) = \frac{1}{12n} + \sum_{i=1}^n \left[F_i - \frac{2i-1}{2n} \right]^2.$$

The corresponding estimating equations are

$$\sum_{i=1}^n \left[F_i - \frac{2i-1}{2n} \right] F_{ij} = 0.$$

4.6 Anderson-Darling Estimation

The Anderson-Darling criterion is

$$S_{AD}(\psi) = -n - \frac{1}{n} \sum_{i=1}^n (2i-1) [\log F_i + \log(1 - F_{n+1-i})].$$

For $\psi_j \in \{\alpha, \lambda\}$, the estimating equations are

$$\sum_{i=1}^n (2i-1) \left[\frac{F_{ij}}{F_i} - \frac{F_{n+1-i,j}}{1 - F_{n+1-i}} \right] = 0.$$

4.7 Maximum Product Spacings Estimation

Let

$$U_i = F(X_{(i)}; \psi), \quad U_0 = 0, \quad U_{n+1} = 1,$$

and define the spacings

$$D_i = U_i - U_{i-1}, \quad i = 1, \dots, n+1.$$

The maximum product spacings estimator minimizes

$$S_{MPS}(\psi) = - \sum_{i=1}^{n+1} \log D_i.$$

Its estimating equations are

$$\sum_{i=1}^{n+1} \frac{D_{ij}}{D_i} = 0,$$

where

$$D_{1j} = F_{1j}, \quad D_{ij} = F_{ij} - F_{i-1,j}, \quad 2 \leq i \leq n,$$

and

$$D_{n+1,j} = -F_{nj}.$$

4.8 Percentile Matching Estimation

Because the QSME distribution has two unknown parameters, two representative probabilities can be used for direct percentile matching. Let

$$\mathcal{P}_{PM} = \{0.25, 0.75\}.$$

Let $q_s(p)$ denote the sample percentile and $Q(p; \psi)$ the model percentile. Define

$$S_{PM}(\psi) = \sum_{p \in \mathcal{P}_{PM}} [Q(p; \psi) - q_s(p)]^2.$$

For a probability away from numerical boundary problems, implicit differentiation of

$$F(Q(p; \psi); \psi) = p,$$

gives

$$Q_j(p; \psi) = -\frac{F_j(Q(p; \psi); \psi)}{f(Q(p; \psi); \psi)}.$$

Hence the estimating equations are

$$\sum_{p \in \mathcal{P}_{PM}} [Q(p) - q_s(p)] Q_j(p) = 0.$$

4.9 Quantile Least Squares Estimation

Quantile least squares uses a denser probability grid,

$$\mathcal{P}_{QLS} = \{0.10, 0.20, \dots, 0.90\}.$$

The objective is

$$S_{QLS}(\psi) = \sum_{p \in \mathcal{P}_{QLS}} [Q(p; \psi) - q_s(p)]^2,$$

with estimating equations

$$\sum_{p \in \mathcal{P}_{QLS}} [Q(p) - q_s(p)] Q_j(p) = 0.$$

4.10 Numerical Optimization and Parameter Constraints

To enforce positivity automatically, define unconstrained parameters $z_1, z_2 \in \mathbb{R}$ through

$$\alpha = e^{z_1}, \quad \lambda = e^{z_2}.$$

All eight objective functions can then be minimized over $\mathbb{R}^2$. A derivative-free routine such as Nelder-Mead is suitable as a common optimizer, while BFGS or other gradient-based procedures may also be used when stable derivatives are available. Multiple starting values are useful for checking numerical stability.

4.11 Summary of the Eight Estimators

The eight estimators can be written compactly as

$$\hat{\psi}_m = \arg \min_{\psi \in \Omega} S_m(\psi),$$

where

$$m \in \{MLE, LSE, WLSE, CVM, AD, MPS, PM, QLS\}.$$

These methods provide a common basis for the Monte Carlo comparison to be completed later.

5 Monte Carlo Simulation Study

A Monte Carlo experiment was conducted to evaluate the finite-sample behavior of eight estimation procedures for the QSME model. The true parameter vector was fixed at $\alpha = 2$ and $\lambda = 0.5$. For each sample size $n \in \{20, 50, 125, 250\}$, 1000 independent simulation replications were summarized. The competing algorithms were maximum likelihood estimation (MLE), least squares estimation (LSE), weighted least squares estimation (WLSE), Cramér-von Mises estimation (CVM), Anderson-Darling estimation (AD), maximum product spacing (MPS), percentile matching (PM), and quantile least squares (QLS). The performance was measured using empirical mean, absolute bias, mean squared error (MSE), root mean squared error (RMSE), standard deviation (SD), mean absolute error (MAE), and the obtained rank scores. Lower bias and error should indicate better performance as they are more accurate. The optimization performance is 100% for all the methods in every sample size considered, so we could expect a better quality of the estimation instead of a numerical error.

Tables 1–4 show the main numerical results at the four sample sizes. Each table reports the two parameters in the same order so that accuracy can be compared within and across estimators. Absolute bias is reported. The best value in each numerical criterion is shown in bold with light-green shading, and the overall rank summarizes joint performance across both parameters.

For $n = 20$, PM has the best overall performance in small sample as well and it has an overall rank 1. It yields the smallest MSE, RMSE and MAE for both α and λ , with RMSE values 0.336460 and 0.072561, respectively. QLS is second and has the closest mean estimate of α to the true value, with $|\text{Bias}| = 0.008186$. MLE is third. The larger errors for WLSE, AD and CVM make it clear that choice of estimator is important when sample size is small.

At $n = 50$, MLE is the best method. Its α estimate has MSE = 0.053835 and RMSE = 0.232024, the smallest among the competing procedures for this parameter. PM remains effective, especially for λ where it has MSE = 0.002871 and RMSE = 0.053580, and it is second overall.

Table 1: Estimation performance for $n = 20$. Green shading identifies the best value in each numerical criterion; lower overall rank is better.

Method	Mean	Bias	α MSE	RMSE	MAE	Mean	Bias	λ MSE	RMSE	MAE	Overall rank
MLE	2.084519	0.084519	0.131363	0.362440	0.275647	0.520425	0.020425	0.008137	0.090207	0.058187	3
LSE	1.916796	0.083204	0.181250	0.425735	0.330768	0.540729	0.040729	0.011798	0.108617	0.077094	5
WLSE	1.942458	0.057542	0.187710	0.433255	0.339448	0.537007	0.037007	0.013237	0.115052	0.080676	8
CVM	1.979445	0.020555	0.203754	0.451391	0.360523	0.534904	0.034904	0.011031	0.105031	0.076619	6
AD	1.929872	0.070128	0.196429	0.443203	0.357346	0.532607	0.032607	0.011902	0.109096	0.076074	7
MPS	1.886253	0.113747	0.159271	0.399088	0.304444	0.538807	0.038807	0.013093	0.114423	0.075162	4
PM	2.015920	0.015920	0.113205	0.336460	0.274392	0.511193	0.011193	0.005265	0.072561	0.050909	1
QLS	1.991814	0.008186	0.145239	0.381102	0.314449	0.516327	0.016327	0.005637	0.075078	0.053419	2

Table 2: Estimation performance for $n = 50$. Green shading identifies the best value in each numerical criterion; lower overall rank is better.

Method	Mean	Bias	α MSE	RMSE	MAE	Mean	Bias	λ MSE	RMSE	MAE	Overall rank
MLE	2.023207	0.023207	0.053835	0.232024	0.177069	0.509970	0.009970	0.003592	0.059934	0.038569	1
LSE	1.951773	0.048227	0.081416	0.285334	0.221156	0.521818	0.021818	0.004250	0.065189	0.047789	5
WLSE	1.979019	0.020981	0.076578	0.276728	0.215225	0.515581	0.015581	0.004164	0.064526	0.046030	3
CVM	1.972797	0.027203	0.087807	0.296323	0.234416	0.519408	0.019408	0.004325	0.065762	0.048989	8
AD	1.957900	0.042100	0.084059	0.289930	0.227929	0.516797	0.016797	0.004310	0.065653	0.046005	7
MPS	1.919611	0.080389	0.073552	0.271204	0.199829	0.521870	0.021870	0.006293	0.079330	0.048636	6
PM	1.994109	0.005891	0.073780	0.271625	0.219839	0.506850	0.006850	0.002871	0.053580	0.039307	2
QLS	1.967178	0.032822	0.083343	0.288692	0.232086	0.513751	0.013751	0.003731	0.061081	0.042462	4

Table 3: Estimation performance for $n = 125$. Green shading identifies the best value in each numerical criterion; lower overall rank is better.

Method	Mean	Bias	α MSE	RMSE	MAE	Mean	Bias	λ MSE	RMSE	MAE	Overall rank
MLE	2.002645	0.002645	0.019144	0.138361	0.106577	0.504366	0.004366	0.001060	0.032560	0.022360	1
LSE	1.971769	0.028231	0.035739	0.189048	0.141194	0.510484	0.010484	0.001903	0.043619	0.030975	6
WLSE	1.979201	0.020799	0.028384	0.168477	0.131055	0.508494	0.008494	0.001427	0.037780	0.026516	2
CVM	1.979295	0.020705	0.037335	0.193222	0.145836	0.509583	0.009583	0.001955	0.044211	0.031553	8
AD	1.975122	0.024878	0.030992	0.176044	0.136115	0.507967	0.007967	0.001473	0.038379	0.026737	3.5
MPS	1.951287	0.048713	0.024870	0.157701	0.117632	0.508924	0.008924	0.001556	0.039442	0.025338	3.5
PM	1.983317	0.016683	0.036775	0.191767	0.156418	0.505356	0.005356	0.001440	0.037944	0.027353	5
QLS	1.974193	0.025807	0.039301	0.198244	0.152124	0.508302	0.008302	0.001785	0.042244	0.027296	7

WLSE is third and shows a clear improvement relative to its performance at $n = 20$.

At $n = 125$, MLE is still leading with an overall ranking of 1. Its empirical mean of 2.002645 for α and 0.504366 for λ are close to the true parameter values and the corresponding RMSE values are 0.138361 and 0.032560. WLSE is second as it improves the error for both parameters. MPS and AD are in the next position with a rank of 3.5, while CVM is still the least competitive method for this sample size.

In the case of $n = 250$, MLE ranks again first and provides the most accurate overall estimates. Its RMSE drops to 0.094717 for α and to 0.019458 for λ . MPS is the second-ranked method with an RMSE of 0.100151 and 0.020796, and WLSE is third. PM, the most popular method for $n = 20$, is at rank 8 for $n = 250$. We can see that good small-sample performance does not necessarily

lead to the strongest large-sample ranking for all estimation methods.

The rank trajectory shows clearly the change in relative performance with the sample size. MLE improves from rank 3 at $n = 20$ to rank 1 at $n = 50$ and remains first at $n = 125$ and $n = 250$. WLSE improves from rank 8 to rank 3, 2, and 3 respectively. MPS ends the experiment strong and achieves rank 2 at $n = 250$. PM, on the other hand, moves from rank 1 to rank 2, 5, and 8. As shown in Figure 8, when performance is averaged across the full simulation design, the final scores are as follows: MLE ranks first with 2.125, PM second with 3.979, WLSE is third with 4.208, and MPS is fourth with 4.396.

The $n = 20$ figures are the most demanding finite sample and thus serve as a suitable baseline for the sampling variability. The method-specific panels show larger traces and histograms, while the boxplots show much wider dispersion around the true parameter lines.

Table 4: Estimation performance for $n = 250$. Green shading identifies the best value in each numerical criterion; lower overall rank is better.

Method	Mean	Bias	α MSE	RMSE	MAE	Mean	Bias	λ MSE	RMSE	MAE	Overall rank
MLE	2.005438	0.005438	0.008971	0.094717	0.075480	0.500920	0.000920	0.000379	0.019458	0.015168	1
LSE	1.989493	0.010507	0.015690	0.125261	0.095998	0.503587	0.003587	0.000690	0.026274	0.019495	5
WLSE	1.991697	0.008303	0.013163	0.114731	0.091076	0.502986	0.002986	0.000489	0.022116	0.017218	3
CVM	1.993086	0.006914	0.016224	0.127372	0.097748	0.503143	0.003143	0.000712	0.026682	0.019788	6
AD	1.989653	0.010347	0.014142	0.118922	0.093592	0.502882	0.002882	0.000531	0.023036	0.017671	4
MPS	1.976938	0.023062	0.010030	0.100151	0.078696	0.502842	0.002842	0.000432	0.020796	0.016019	2
PM	1.984887	0.015113	0.023295	0.152626	0.121792	0.503496	0.003496	0.000958	0.030951	0.021790	8
QLS	1.987788	0.012212	0.018453	0.135844	0.103417	0.503318	0.003318	0.000747	0.027334	0.018961	7

Table 5: Evolution of method scores and ranks across the four sample sizes and the final overall ranking.

Method	Score				Rank				Final score	Final rank
	20	50	125	250	20	50	125	250		
MLE	3.250	2.083	1.583	1.583	3	1	1	1	2.125	1.000
LSE	5.417	5.250	6.000	5.250	5	5	6	5	5.479	7.000
WLSE	6.250	4.000	3.250	3.333	8	3	2	3	4.208	3.000
CVM	5.583	6.583	6.500	5.417	6	8	8	6	6.021	8.000
AD	5.750	5.833	4.000	4.000	7	7	3.5	4	4.896	5.500
MPS	5.333	5.333	4.000	2.917	4	6	3.5	2	4.396	4.000
PM	1.667	2.417	4.583	7.250	1	2	5	8	3.979	2.000
QLS	2.750	4.500	6.083	6.250	2	4	7	7	4.896	5.500

Table 6: Descriptive statistics for the bladder cancer remission-time data.

Statistic	Value
Number of observations	128
Mean	9.3656
Standard deviation	10.5083
Median	6.3950
Minimum	0.08
Maximum	79.05
Range	78.97
First quartile (Q1)	3.3475
Third quartile (Q3)	11.8375
Inter-quartile range (IQR)	8.4900
Coefficient of variation	112.20%
Skewness	3.3257
Excess kurtosis	16.1537

The comparison plots are consistent with the numerical table: PM and QLS are particularly competitive in the small sample case and several other estimators have larger error measures.

In Figures 3-6, the estimated sequences vary more significantly around reference values than in the large sample case shown later. The histograms also show more spread. In spite of this fluctuation, the convergence profiles mostly settle after the early replications, and all methods perform well. The PM and QLS panels are suited with their favorable small-sample rank in Table 5.

Figure 7 shows that the α estimates are more dispersive than the λ estimates on their respective scales. The median lines are close to the true values, but outliers are more common in the small sample. The MSE comparison confirms the numerical advantage of PM, especially for λ , and the mean comparison shows that some procedures are approximately centered even when their dispersion differs considerably.

Figure 8 shows the RMSE and MAE panels reinforce the same order in numerical results. PM has the lowest overall error profile and QLS and MLE are next to PM in the overall ranking. The rank panels show a clear separation between the leading estimators and the methods with larger sampling error. This is to be expected when $n = 20$ has little information for estimating both parameters at once. The $n = 250$ figures provide the large sample counterpart to $n = 20$. Compared with the small sample panels, the traces are more closely nested around the targets, histograms are smaller and boxplots shrink sharply. This visual contraction is in line with the sharp decline in RMSE and MAE reported in Table 5.

MLE is the best overall rank and MPS, WLSE, and AD are the other three. PM has the largest rank score at $n = 250$. With this comparison, there is a clear difference in the relative efficiency of the ranking between PM and MLE in the smallest sample size and MLE and MPS in the large sample size. The overall plots show the effect of increasing sample size without repeating the $n = 50$ and $n = 125$ figure sets. The mean curves get closer to the true parameter values as n increases, with only slight

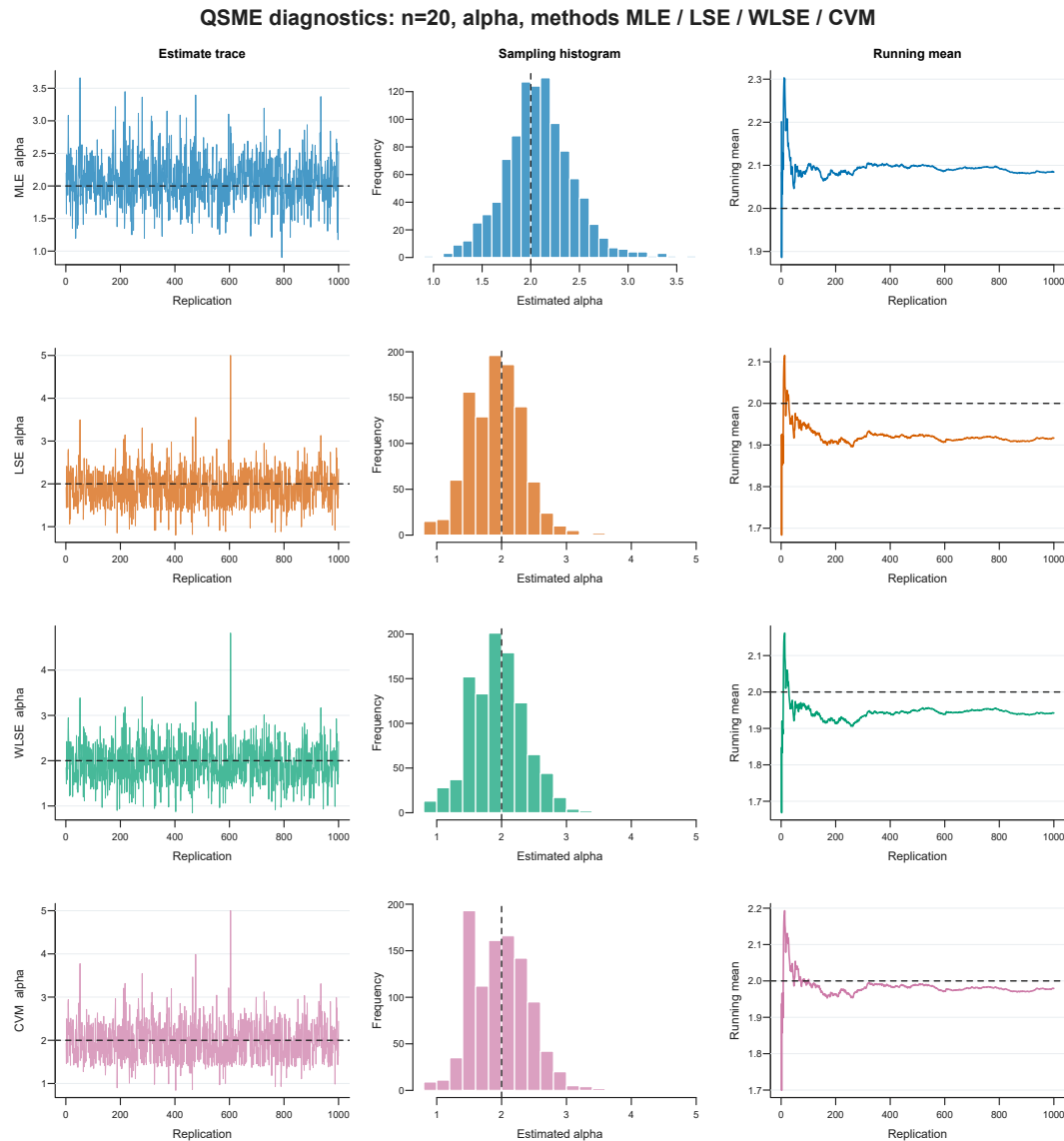


Fig. 3: Trace, histogram, and running-mean convergence diagnostics for α at $n = 20$ across MLE, LSE, WLSE, and CVM. Horizontal reference lines denote the true values $\alpha = 2$.

differences for the estimators. The RMSE decreases sharply for every method, which is the starkest evidence to show that finite sample accuracy is consistent. The rank trajectories also indicate that MLE is stable at the top position after $n = 20$, while PM loses relative rank as the sample grows. The bias panel is shown as generated. As a result, the bias values are most clearly seen in Tables 1-4.

Figure 9-11 show that the largest accuracy gains occur when moving from the smallest samples toward moderate and large samples. RMSE falls steadily for both parameters, while the mean curves move toward $\alpha = 2$ and $\lambda = 0.5$. MLE changes from third at $n = 20$ to first

from $n = 50$ onward, whereas PM follows the reverse ranking pattern.

Figure 12 shows the overall ranking places MLE first with an average score of 2.125. PM is still second overall due to its excellent performance at $n = 20$ and $n = 50$, but it is less effective at the larger sample. WLSE and MPS occupy the third and fourth positions, respectively. AD and QLS share the next position, followed by LSE and CVM. The numerical and graphical evidence clearly indicate that MLE is the most stable estimator across the full range of sample sizes, while PM performs very well

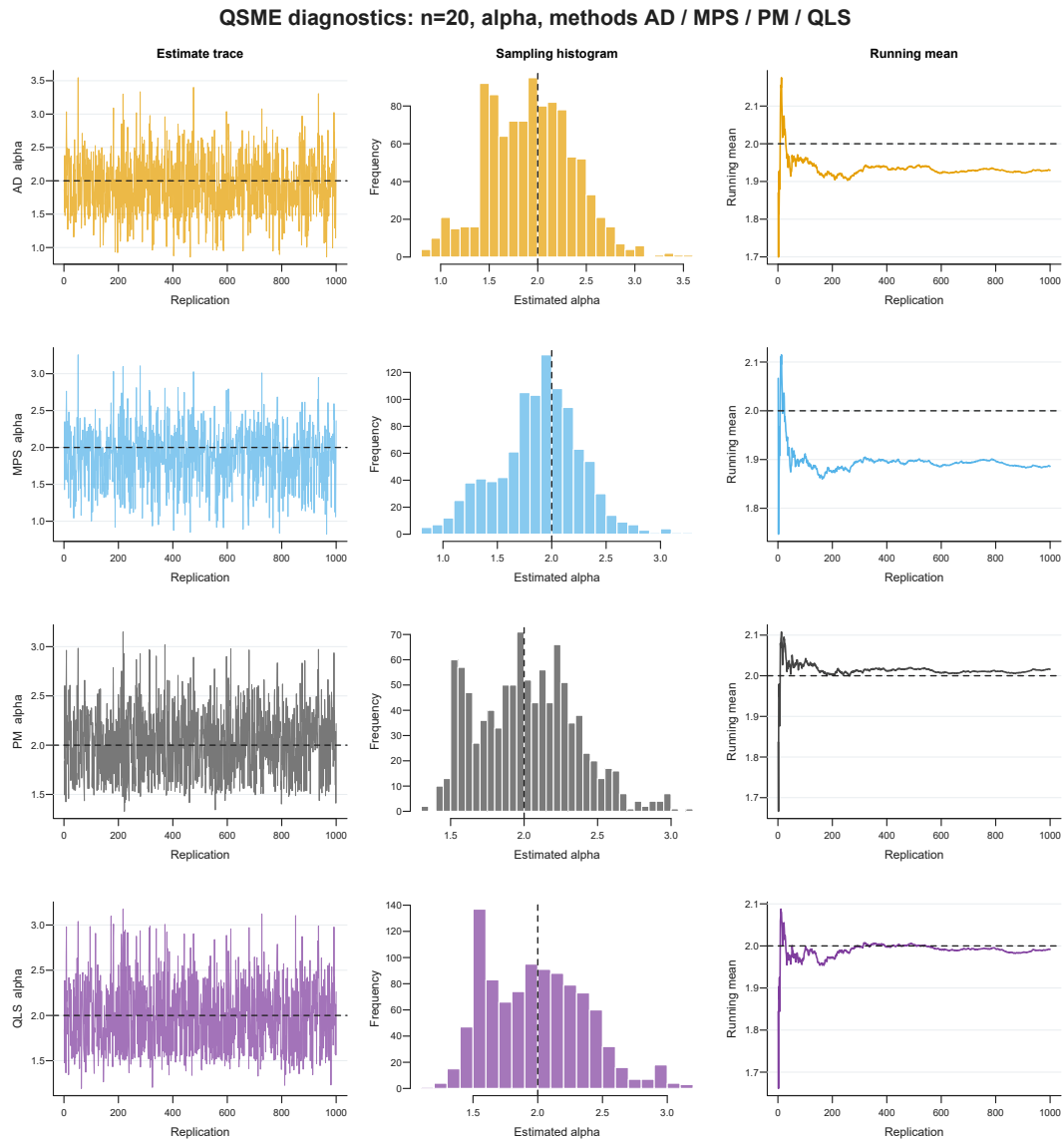


Fig. 4: Trace, histogram, and running-mean convergence diagnostics for α at $n = 20$ across AD, MPS, PM, and QLS. Horizontal reference lines denote the true values $\alpha = 2$.

in small samples and MPS is very competitive in the large-sample setting.

The simulation study shows a strong effect of sample size on estimation accuracy. Increasing n from 20 to 250 reduces RMSE and MAE for every estimation method and concentrates the empirical distributions on the true parameter values. MLE shows the best overall behavior and dominates from $n = 50$ onward. PM has the best small sample performance but decreases relative efficiency as n grows. MPS and WLSE improve with sample size and are strong alternatives at $n = 250$. These results support the choice of estimators in a sample-size-aware manner rather

than relying on a single ranking obtained under one experimental setting.

6 Application to Real Data: Bladder Cancer Remission Times

The practical performance of the proposed QSME distribution has been evaluated using the remission times (months) of 128 bladder cancer patients. The data range from 0.08 to 79.05 months and were analyzed in Klakattawi (2022) [18]. The data have a pronounced right

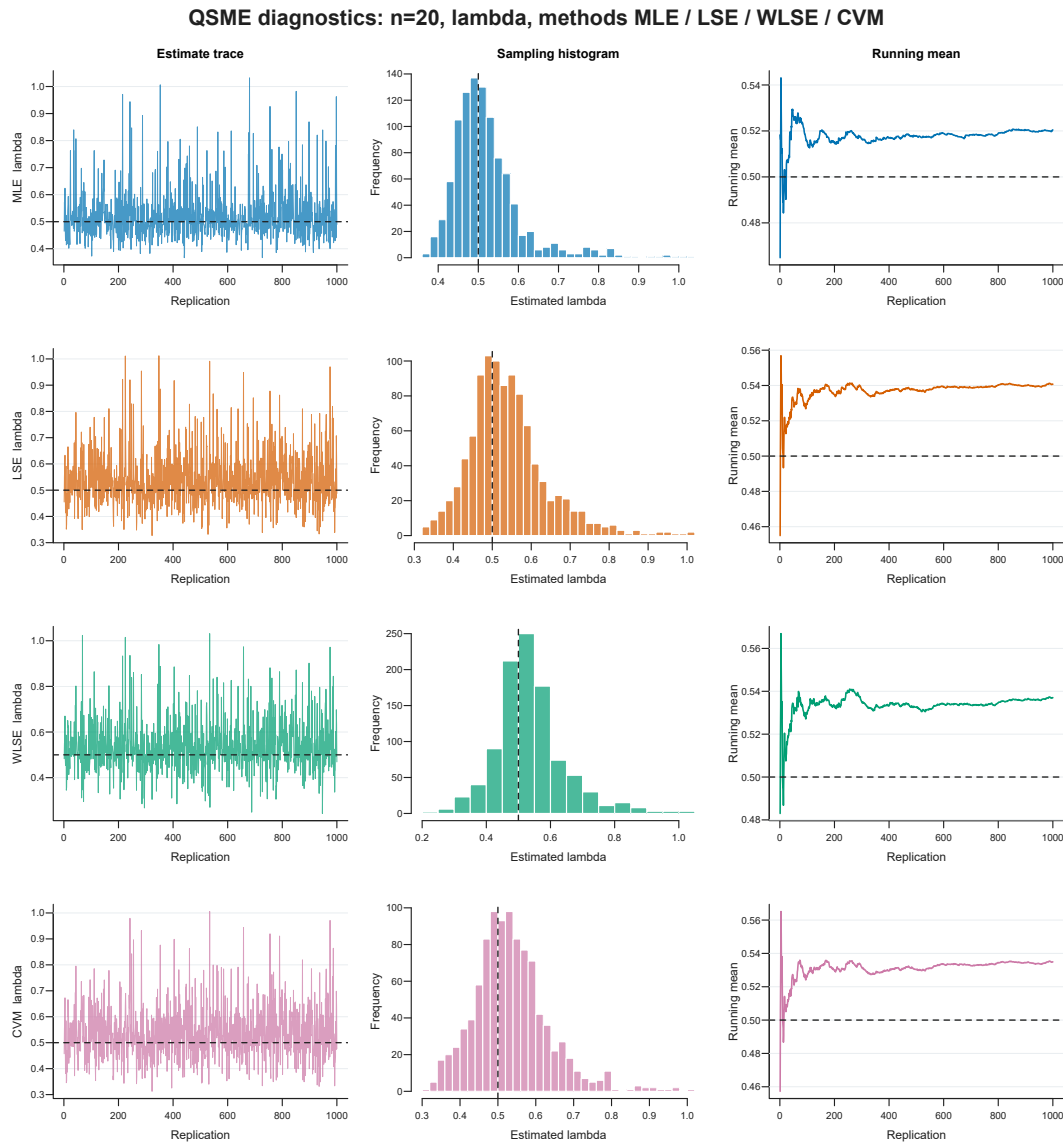


Fig. 5: Trace, histogram, and running-mean convergence diagnostics for λ at $n = 20$ across MLE, LSE, WLSE, and CVM. Horizontal reference lines denote the true values $\lambda = 0.5$.

tail so they are suitable for testing lifetime models with flexible distributional and hazard structures. The mean of the sample is 9.3656 months and the median is 6.3950 months. The standard deviation is 10.5083 and the observations range from 0.08 to 79.05 months. The positive skewness (3.3257) and high excess kurtosis (16.1537) indicate a strongly right-skewed sample with a long upper tail and a strong departure from symmetry. Table 6 shows some descriptive statistics for the bladder cancer remission-time data.

The QSME model converged well under maximum likelihood estimation. The estimated shape parameter is

$\alpha = 1.0113$ and the estimated rate parameter is $\lambda = 0.1073$. The 95% confidence interval for α includes 1, so the fitted model is close to the exponential special case but still has the additional flexibility of the QSME model. Table 7 shows the maximum likelihood estimates, standard deviations, and 95% confidence interval for the QSME model and the competing distributions.

The QSME model was compared to the six competing models of Fr, We, EG, IGo, NH, and BX. Smaller values of the negative log-likelihood, AIC, CAIC, BIC, HQIC, Cramér-von Mises statistic (W), Anderson-Darling statistic (A) and Kolmogorov-Smirnov statistic are better

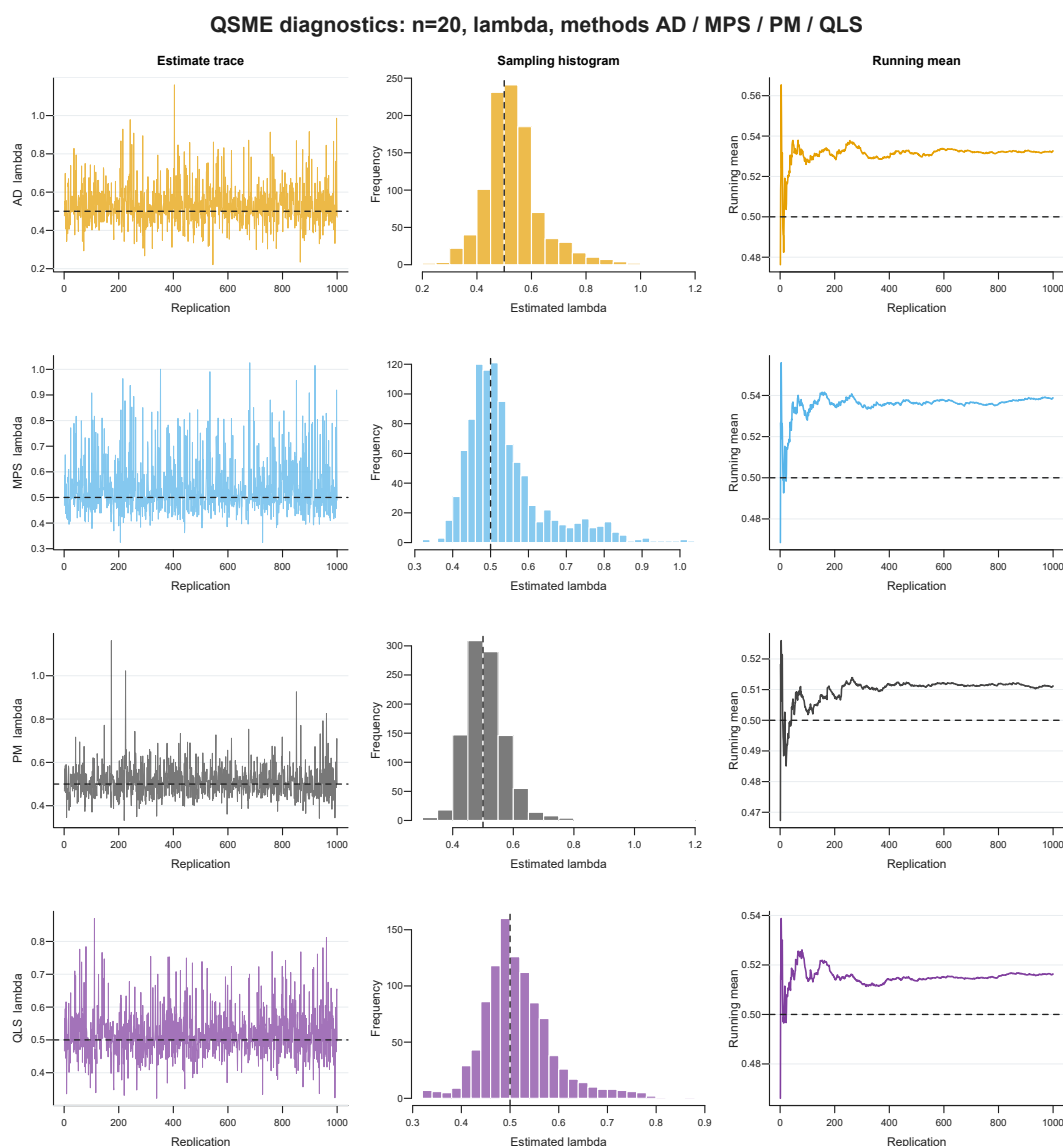


Fig. 6: Trace, histogram, and running-mean convergence diagnostics for λ at $n = 20$ across AD, MPS, PM, and QLS. Horizontal reference lines denote the true values $\lambda = 0.5$.

fit [1, 20, 24], and larger K-S p-values show better agreement with the empirical distribution.

Among the models that reached algorithmic convergence, QSME has the smallest negative log-likelihood (414.2788), AIC (832.5577), CAIC (832.6537), BIC (838.2618), and HQIC (834.8753). Its K-S statistic is 0.0805 with $p = 0.3774$, so the fitted QSME distribution is not rejected by the K-S test at the 5% significance level. The comparison of information criteria is therefore restricted to the models that satisfied the numerical convergence criterion.

EG and NH have competitive numerical values, and NH has the smallest K-S statistic (0.0746) with the largest p-value (0.4753). However, we report both optimization procedures as not converged. Their fit statistics should therefore be interpreted cautiously because the parameters estimated may not be the most stable optimum solutions.

Fr and We are clearly less competitive. Their AIC values are much larger than those of QSME, and their K-S p-values are 0.0125 and 0.0042, respectively, indicating large differences from the empirical distribution. BX converges and has a non-significant K-S result ($p =$

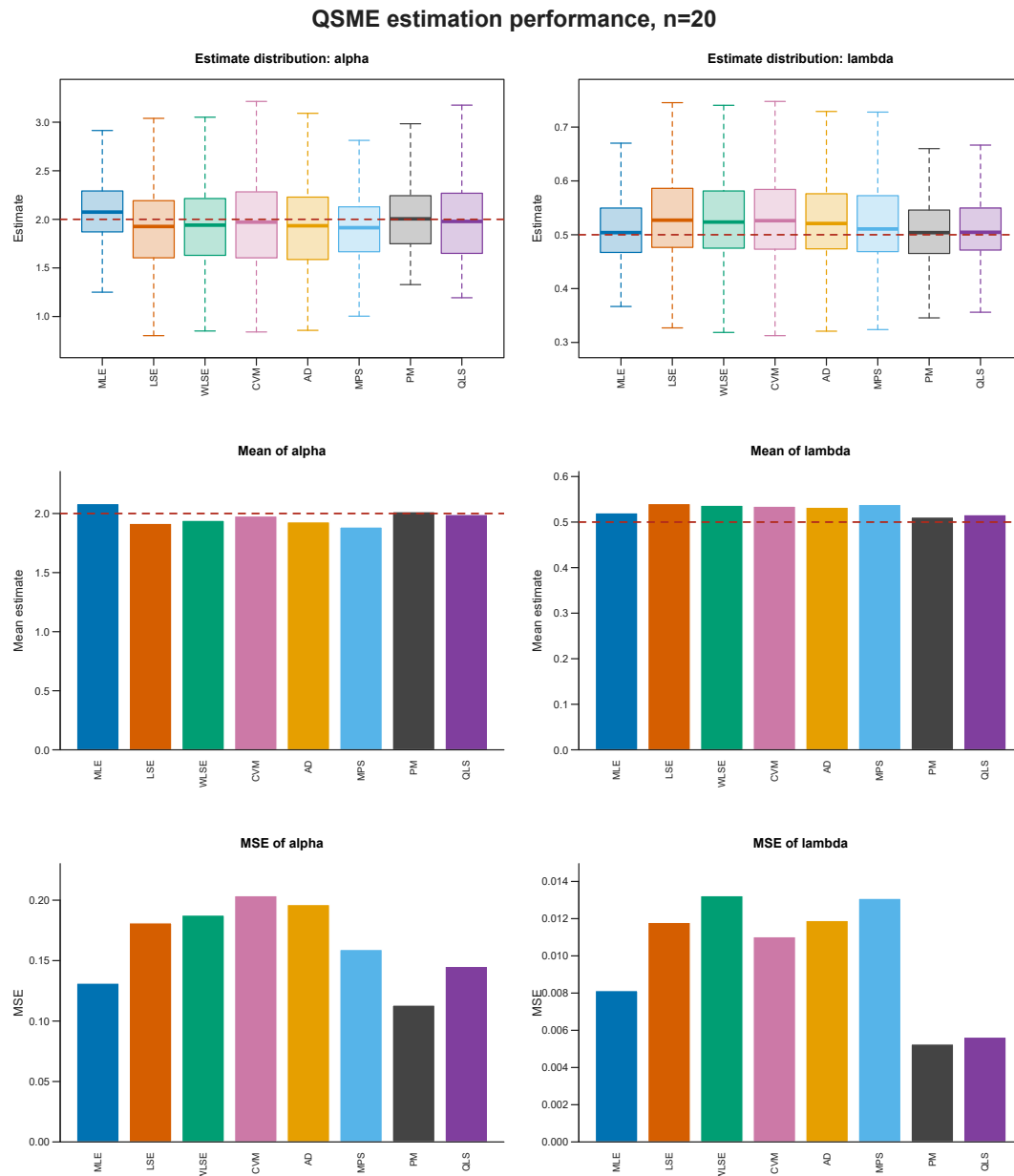


Fig. 7: Distributional spread and mean/MSE comparisons across all estimators at $n = 20$. Panels (a) and (b) show the α and λ boxplots; panels (c) and (d) show the mean and MSE comparisons. Horizontal reference lines denote the true parameter values where applicable.

0.1413), but its information criteria and empirical distribution statistics are clearly inferior to that of QSME.

The IGo result is reported only for transparency because its optimization routine did not satisfy the convergence criterion. Although the reported W and A statistics are small, the fitted CDF is inconsistent with the empirical accumulation pattern. Accordingly, the IGo fit

is excluded from comparative interpretation and from the final model-selection conclusion.

Figure 13 compares the empirical cumulative distribution function with the fitted CDF of each model. The QSME curve follows the empirical step function closely over most of the observed range. EG and NH also show close visual agreement, but their optimization routines did not reach the convergence criterion;

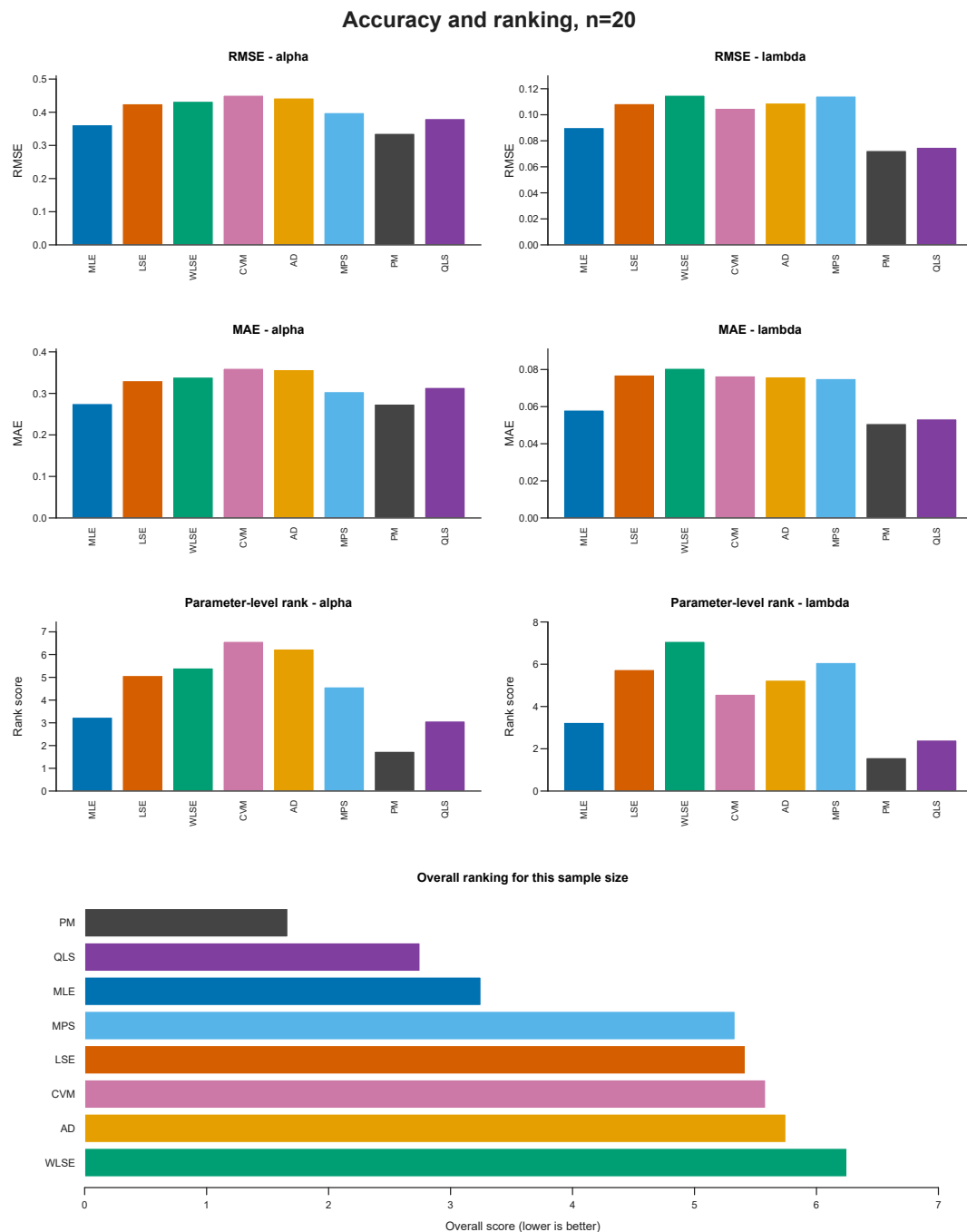


Fig. 8: RMSE, MAE, parameter-level rank scores, and overall method rank score across all estimators at $n = 20$; lower error and rank values indicate better performance.

therefore, their curves are displayed only as diagnostics. Fr, We, and BX show larger visible deviations, while the non-converged IGo curve does not reproduce the empirical accumulation pattern.

Considering only models that achieved numerical convergence, QSME provides the smallest

likelihood-based information criteria and a non-significant K-S result, and its fitted CDF shows close agreement with the empirical distribution. Non-converged models are retained in Tables 8-9 and Figure 13 for transparency but are not ranked against the converged fits. The estimated α value being close to 1 indicates behavior

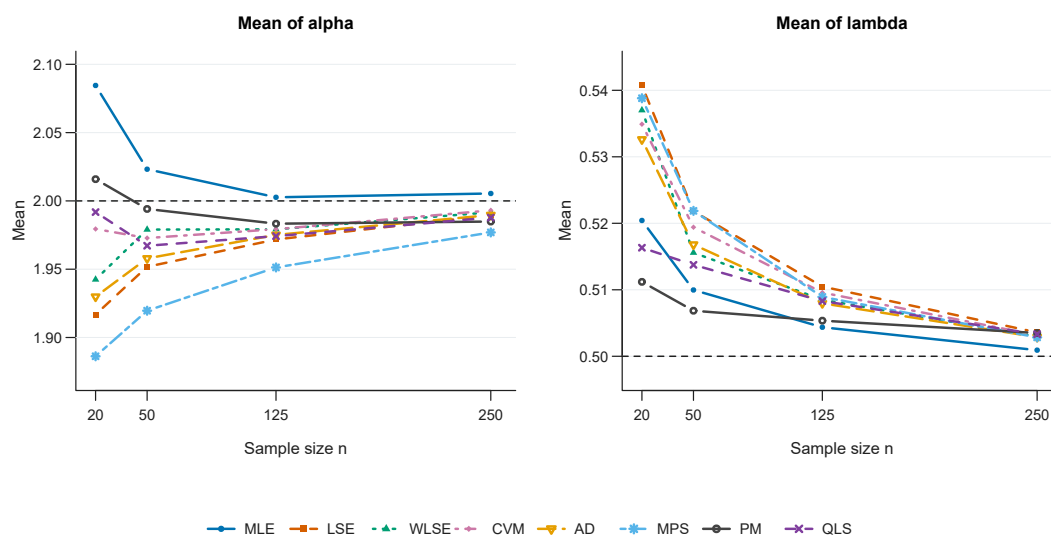


Fig. 9: Mean versus sample size

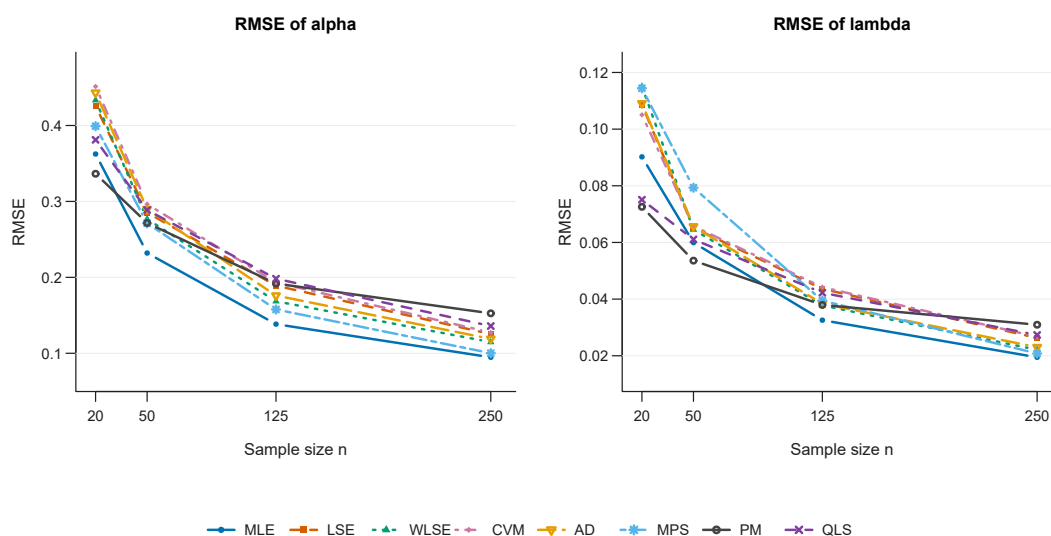


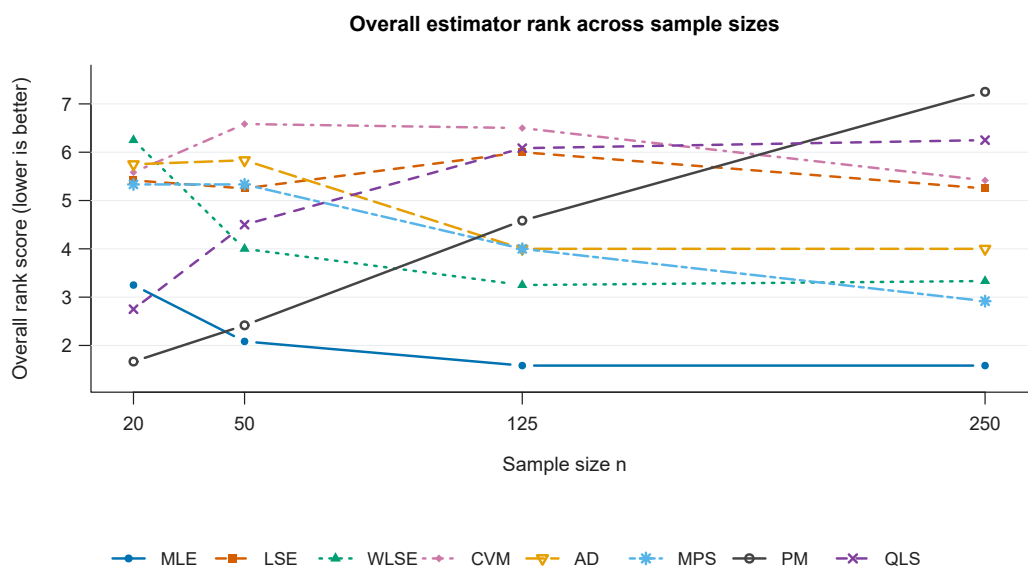
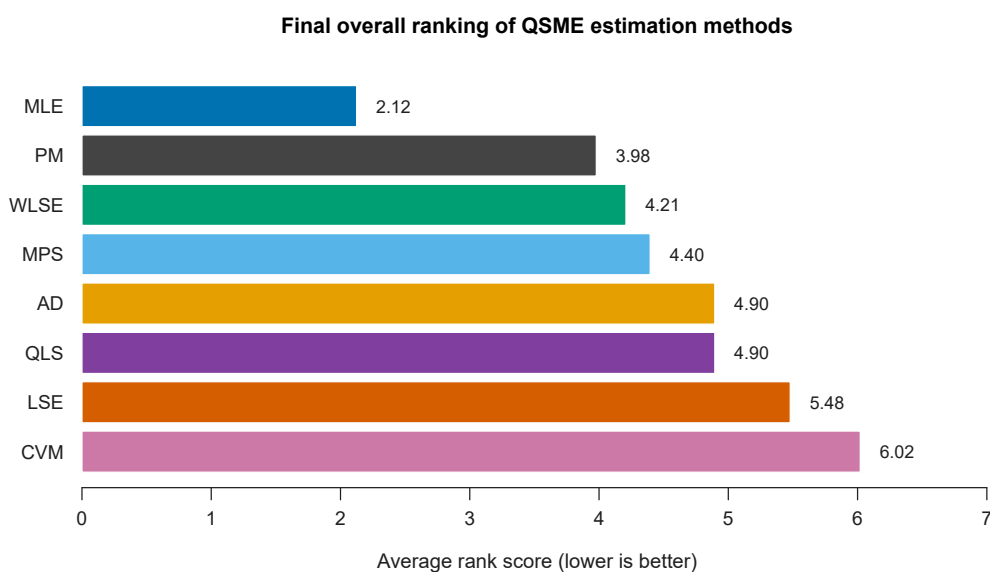
Fig. 10: RMSE versus sample size

near the exponential special case while preserving the additional QSME flexibility when departures from that case occur.

7 Conclusion

The QSME distribution provides a flexible and parsimonious 2-parameter extension of the exponential

model. Its main advantage is that it generates many important hazard rate shapes without introducing any other independent parameter, and it keeps the exponential case exactly at $\alpha = 1$. The theoretical results validate the distribution and provide a wide range of mathematical and reliability properties for lifetime analysis. The simulation study shows that the accuracy of our estimates increases as the sample size increases. Percentile matching is well-suited for small samples, and maximum

**Fig. 11:** Overall rank versus sample size**Fig. 12:** Final overall ranking of the eight QSME estimation methods using the average rank score across the four sample sizes; lower scores indicate better overall performance.

likelihood is the best overall estimator from $n=50$ to keep the first final rank for our whole simulation design. The real-data application to the remission times in bladder cancer also confirms the effectiveness of the model. QSME met the convergence criteria, the lowest information criteria among the converged competitors,

and a non-significant Kolmogorov-Smirnov test. These results show that QSME is a practical and interpretable model for positive lifetime and reliability data.

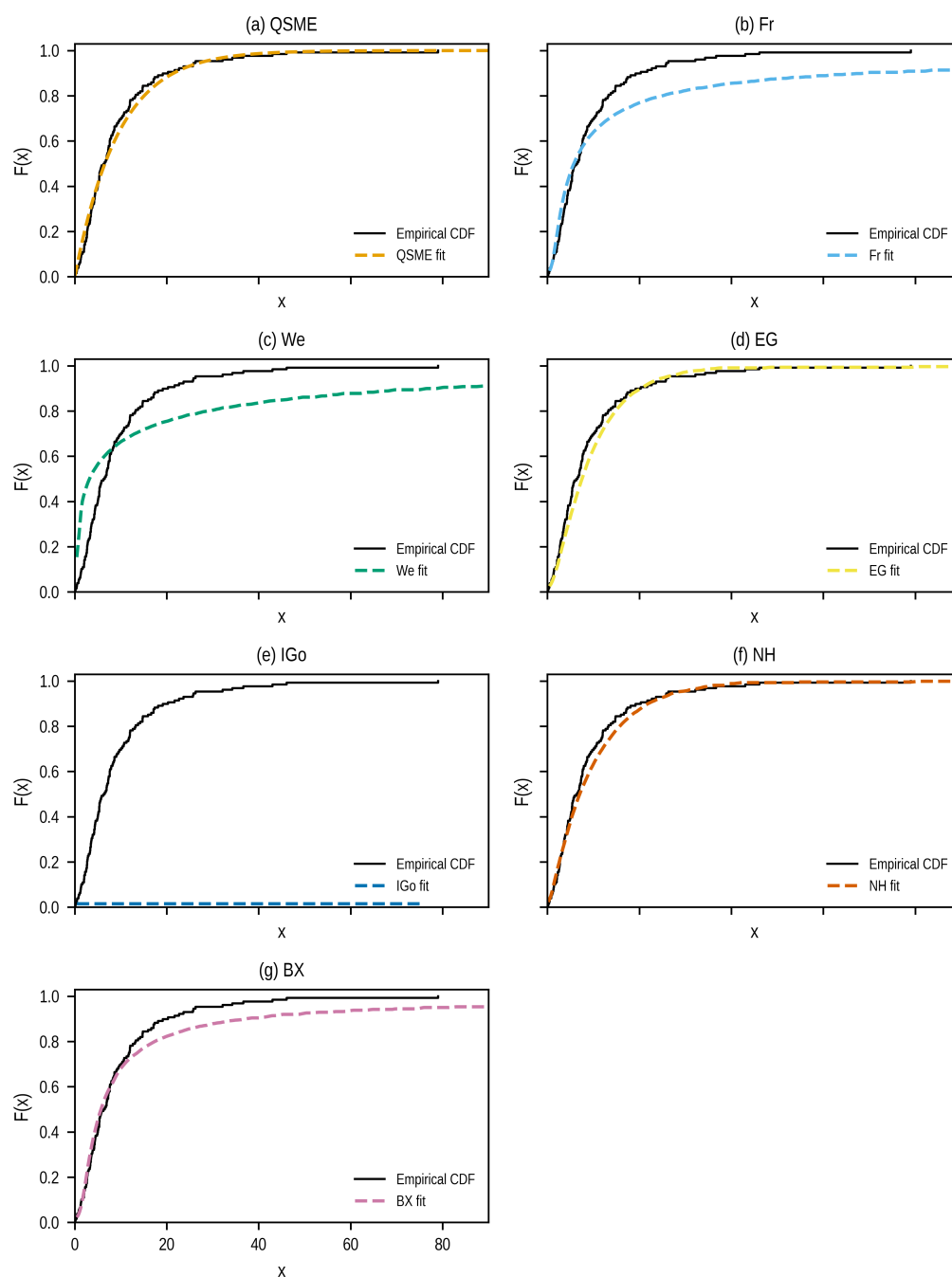


Fig. 13: Empirical CDF of competing models for the bladder cancer remission.

Acknowledgment

The authors extend their appreciation to Umm Al-Qura University, Saudi Arabia for funding this research work through grant number: 26UQU4300206GSSR05.

Funding Statement

This research work was funded by Umm Al-Qura University, Saudi Arabia under grant number: 26UQU4300206GSSR05.

Table 7: Maximum-likelihood estimates, standard deviations, and 95% confidence intervals for QSME and competing models.

Models	Parameter	MLE	Std. Dev.	Lower 95% CI	Upper 95% CI
QSME	α	1.0113414	0.030881245	0.95081530	1.0718676
	λ	0.1072519	0.009373342	0.08888047	0.1256233
Fr	α	3.2580877	0.40741298	2.4595730	4.0566025
	λ	0.7520845	0.04242387	0.6689353	0.8352338
We	α	-0.04763393	0.003906354	-0.05529024	-0.03997762
	λ	0.36412526	0.037320288	0.29097884	0.43727168
EG	α	0.1375019	0.01500547	0.1080917	0.1669120
	λ	1.4793362	0.19149887	1.1040053	1.8546670
IGo	α	1.9657167	0.37477659	1.23116810	2.7002653
	λ	0.0745271	0.01897826	0.03733039	0.1117238
NH	α	1.1207167	0.23184181	0.66631515	1.5751183
	λ	0.0888666	0.03016837	0.02973768	0.1479955
BX	α	1.033294	0.06044343	0.9148267	1.151761
	λ	4.206964	0.40536973	3.4124540	5.001474

Table 8: Likelihood-based model-selection criteria for the fitted distributions.

Model	-log L	AIC	CAIC	BIC	HQIC	Convergence
QSME	414.2788	832.5577	832.6537	838.2618	834.8753	Converged
Fr	444.0008	892.0015	892.0975	897.7056	894.3191	Converged
We	429.2248	862.4497	862.5457	868.1537	864.7673	Converged
EG	414.3123	832.7653	832.8613	838.4694	835.0829	Not converged
IGo	420.9485	845.9078	846.0038	851.6118	848.2253	Not converged
NH	414.8718	833.7537	833.8497	839.4577	836.0712	Not converged
BX	426.6864	857.3729	857.4689	863.0769	859.6905	Converged

Table 9: Empirical-distribution goodness-of-fit measures and convergence status.

Model	W	A	K-S	K-S p-value	Convergence
QSME	0.123054	0.738159	0.080545	0.377367	Converged
Fr	0.744321	4.546423	0.140792	0.012508	Converged
We	0.474723	2.786499	0.155112	0.004228	Converged
EG	0.103998	0.625790	0.079438	0.394477	Not converged
IGo	0.029920	0.214772	0.131556	0.023814	Not converged
NH	0.146667	0.876254	0.074554	0.475269	Not converged
BX	0.385596	2.454297	0.101737	0.141295	Converged

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