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Analysis of Stochastic Breaking Soliton Equation with Multiplicative Noise by Mapping Method

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Abstract: This article deals with the stochastic (2+1)-dimensional breaking soliton equation (SBSE) imposed by multiplicative noise. To discover new rational, hyperbolic, elliptic, and trigonometric functions, we employ the mapping approach. Moreover, we generalise some earlier results. Because of the importance of the stochastic breaking soliton equation in domains like hydrodynamics, plasma physics, nonlinear optics and some other disciplines, the solutions that have been found can be used to explain a variety of fascinating physical phenomena. Utilizing MATLAB tools to model our observations and create several 3D graphs, we demonstrate how the multiplicative Brownian motion affects the analytical SBSE solutions. Ultimately, we show how multiplicative Brownian motion affects the symmetry as well as stability of the SBSE solutions that are produced.

Keywords: stochastic breaking soliton equation; Wiener process; the mapping method.

1 Introduction

Japanese mathematician Ito initially presented the stochastic differential equation (SDE) including multiplicative noise controlled by Brownian motion around 1951 [1]. The growth of SDE has been quite rapid in recent years, and many professionals and academics have directed their research strategy toward this subject as a result of the widespread application of stochastic differential equation in the disciplines of nonlinear optics, communication and finance [2–4]. Among major areas of study for SDE are the stability of solutions, analytical solutions, existence of solution, numerical solution, the theory of martingale representations, stochastic control, variational inequality and others [5]. In contrast to deterministic differential equations, financiers, engineers and physicists, are attempting to investigate the dynamic behaviour of SDE and resolve the analytical solutions to these equations in any of these research directions.
Many specialists and academics have modified their study in the last two years about the analysis of travelling wave solutions of SDE, particularly for the popular stochastic partial differential equation (SPDE). It is impossible to directly create analytical solutions to deterministic partial differential equations because of the complexity of unpredictability when attempting to construct analytical solutions to stochastic partial differential equations. SPDE can be converted into a deterministic ordinary differential equation by employing a few specific transformations and travelling wave transformations. Consequently, it is now possible to design an analytical SPDE solution. In recent years, a large number of specialists and researchers have used several well-known techniques to build analytical SPDE solutions.
The investigation of stochastic nonlinear evolution equations and their analytical solutions connected to broader studies of nonlinear interactions and quantum dynamical phenomena. Previous research has extensively examined the effects of nonlinear media, field entropy, quantum entanglement, and entropy squeezing in atom–field interaction systems [6]- [9]. In particular,

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these studies have highlighted the influence of nonlinear interactions and external physical effects on the evolution and dynamical properties of coupled quantum systems. Furthermore, the investigation of entanglement dynamics in dissipative cavities provides additional insight into the role of environmental influences in modifying the stability and evolution of physical systems [10]. These research directions offer complementary perspectives for the present investigation of the stochastic (2+1)-dimensional breaking soliton equation, where multiplicative Brownian motion introduces stochastic perturbations that influence the analytical wave structures, symmetry properties, and stability of the resulting solutions. Although the physical models differ, these works share a common interest in understanding how nonlinear interactions and external influences govern the dynamical behavior of complex physical systems.

Riccati–Bernoulli sub-ODE [11], perturbation [12–15], tanh-function method [16], Jacobi elliptic function [17], inverse scattering transform [18], tanh-sech [19, 20], The Hirota's [21], $\exp(-\Omega(\xi))$-expansion [22], variational iteration method [23, 24], $\left(\frac{G'}{G}\right)$-expansion [25, 26] , Backlund transformation [27],sine-cosine [28, 29], etc are a few illustrations of analytical techniques. While certain numerical techniques have been developed to solve fractional stochastic partial differential equations, these techniques include the meshless method [30, 31], implicit Euler method [32, 33], Galerkin finite element method [34], modified decomposition technique [35], finite element method [36] etc.

Numerous disciplines, including physics, geophysics, climatic dynamics, biology, chemistry, and others have recently come to agree that it is appropriate to include random influences in the analysis, forecasting, and simulation of complex processes. Partially Differential Equations are the ideal mathematical problems to depict complicated systems where there are noise as well as random effects. It seems more important to look at the NLEEs under some random force. As a result, we will consider the SBSE in this manner [37]:

$$du_x - [4u_x u_{xy} + 2u_{xx}u_y - u_{xxxy}]dt = \sigma u_x d\xi, \qquad (1.1)$$

Whereas u means a real stochastic function, σ indicates noise level, ξ denotes Brownian motion in a single variable t, and $u_x d\xi$ represents multiplicative noise inside the ito sense.

Noise can lead to statistical characteristics and essential phenomena, thus it must not be disregarded. In comparison to classical PDEs, stochastic PDEs are generally more challenging to find exact solutions to. We see that the interpretation of the stochastic integral $\int_0^t \gamma d\xi$ can be done in a variety of methods, including the Itô and Stratonovich calculus. In contrast to a Stratonovich stochastic integral (shown by $\int_0^t \gamma o d\xi$), which is calculated in the centre, the stochastic integral is Itô (marked by $\int_0^t \gamma d\xi$) when it is evaluated at the left-end [38]. The Itô integral and Stratonovich integral are connected by the equation shown below:

$$\int_0^t \gamma(Z_\tau)\, d\xi(\tau) = \int_0^t \gamma(Z_\tau)\, o\, d\xi(\tau) + \frac{1}{2}\int_0^t \gamma(Z_\tau)\frac{\partial \gamma(Z_\tau)}{\partial z} d\tau, \qquad (1.2)$$

When it is assumed that γ is appropriately regular as well as $Z_t, t \geq 0$ is a stochastic procedure. The solutions of the Stochastic breaking soliton equation. when $\sigma = 0$ have been employed in a variety of domains, including plasma physics, nonlinear optics, hydrodynamics and many other disciplines and thus the solution found help to explain a variety of remarkable physical phenomena.

We are driven to complete this task in order to obtain the analytical stochastic solutions for the SBSE (1.1). With the use of the mapping approach, we are able to find many different stochastic solutions, including elliptic, hyperbolic functions, rational as well as trigonometric. This equation's usage in describing the hydrodynamic wave model involving shallow-water waves, the leading flow of fluid and plasma physics is relevant because it describes these three fields. Such analytical stochastic solutions are much more comprehensive and helpful in describing many incredibly complicated physical processes. Additionally, the SBSE (1.1) gained analytical solutions in this study enhance a number of previously acquired results, including the result described in [39, 40]. Furthermore, to investigate how Brownian motion affects the symmetry and stability of such SBSE solutions that are produced and with the help of MATLAB tools, we created 3D graphics for some of the solutions that were created.

Here is how this article is organised: In the upcoming sections 2, we employ an appropriate wave transformation to derive the SBSE (1.1)'s wave equation. We use the mapping method approach in Section 3 to obtain the analytical stochastic solution of the SBSE (1.1). We show how the Wiener process impacts the analytical solutions of the SBSE in Section 4. Lastly, we offer the article's conclusion.

2 The Wave Equation of the SBSE

Using the following wave transformation, we can derive the wave equation for such SBSE

$$u(x,y,t) = U(\eta)\, e^{\sigma\xi(t) - \frac{\sigma^2}{2}t}, \quad \eta = \eta_1 x + \eta_1 y + \eta_3 t, \qquad (2.1)$$

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when real function U is used, η_i is a constant for all $i = 1, 2, and\, 3$. We see that

$$\begin{aligned} u_x &= \eta_1 U' e^{\sigma\xi(t)-\frac{\sigma^2}{2}t}, \\ \frac{du}{dt} &= \left(\eta_1\eta_3 U'' + \frac{1}{2}\sigma^2\eta_1 U' - \frac{1}{2}\sigma^2\eta_1 U'\right) e^{\sigma\xi(t)-\frac{\sigma^2}{2}t} dt \\ &\quad + \left(\sigma\eta_1 U' d\beta\right) e^{\sigma\xi(t)-\frac{\sigma^2}{2}t}, \\ \frac{\partial u}{\partial y} &= \eta_2 U' e^{\sigma\xi(t)-\frac{\sigma^2}{2}t}, \quad \frac{d^2u}{dxdy} = \eta_1\eta_2 U'' e^{\sigma\xi(t)-\frac{\sigma^2}{2}t}, \\ \frac{d^2u}{dx^2} &= \eta_1^2 U'' e^{\sigma\xi(t)-\frac{\sigma^2}{2}t}, \quad \frac{d^4u}{dx^3dy} = \eta_1^3\eta_2 U'''' e^{\sigma\xi(t)-\frac{\sigma^2}{2}t}, \end{aligned} \tag{2.2}$$

where Ito correction term $\frac{\sigma^2}{2}dt$ is being used. Eq.(2.1) is then substituted into Eq.(1.1), and by applying Eq.(2.2), the following ordinary differential equation is produced:

$$\eta_3 U'' - 6\eta_1\eta_2 U' U'' e^{\sigma\xi(t)-\frac{\sigma^2}{2}t} + \eta_1^2\eta_2 U'''' = 0. \tag{2.3}$$

Remembering that U is just the deterministic function, we may obtain the following by assuming expectation on both sides of Eq.(2.3)

$$\eta_3 U'' - 6\eta_1\eta_2 U' U'' e^{-\frac{\sigma^2}{2}t} E\left(e^{\sigma\xi(t)}\right) + \eta_1^2\eta_2 U'''' = 0 \tag{2.4}$$

For each standard Gaussian process Z, we observe that:

$$E\left(e^{pZ}\right) = e^{\frac{p^2}{2}t}, \tag{2.5}$$

wherein p denotes a real number. Eq.(2.5) arises from the reality $\sigma\xi(t)$ is dispersed in a manner $\sigma\sqrt{t}Z$. Consequently, Equation Eq.(2.4) takes the following shape

$$\eta_3 U'' - 6\eta_1\eta_2 U' U'' + \eta_1^2\eta_2 U'''' = 0. \tag{2.6}$$

Integrate Eq.(2.6) in terms of η, We locate

$$\eta_3 U' - 3\eta_1\eta_2\left[U'\right]^2 + \eta_1^2\eta_2 U''' = r, \tag{2.7}$$

here r is a constant used in integration. To make things simple, we set $r = 0$ to get

$$\eta_1^2\eta_2 U''' + \eta_3 U' - 3\eta_1\eta_2\left[U'\right]^2 = 0. \tag{2.8}$$

Placing

$$U' = \psi, \tag{2.9}$$

In Eq.(2.8) yields the following result:

$$\psi'' + k_1\psi - k_2\psi^2 = 0, \tag{2.10}$$

Therefore

$$k_1 = \frac{\eta_3}{\eta_1^2\eta_2}, and\ \ k_2 = \frac{3}{\eta_1}. \tag{2.11}$$

3 The Mapping Method

Below, we employ the mapping approach [41] to determine the solutions to Eq.(2.10). As a result, we are able to determine the analytical solutions for SBSEs Eq.(1.1).

$$\psi(\eta) = \sum_{i=0}^{n} a_i W^i, \tag{3.1}$$

when W is solved

$$W' = \sqrt{\frac{1}{2}p_1 W^4 + p_2 W^2 + p_3}, \tag{3.2}$$

wherein p_1, p_2, p_3 are actual parameters. We observe that there are various solutions to Eq.(3.2), based on $p_1, p_2\, and\, p_3$, as shown in the following. Every answer to Eq.(3.2) depend upon for a variation of $p_1, p_2\, and\, p_3$ values.

Type 1: If $p_1 = 2m^2, p_2 = -(1+m^2), p_3 = 1$ then $W(\eta) = sn(\eta)$,

Type 2: If $p_1 = 2, p_2 = 2m^2 - 1, p_3 = -m^2\left(1-m^2\right)$ then $W(\eta) = ds(\eta)$,

Type 3: If $p_1 = 2, p_2 = 2 - m^2, p_3 = 1 - m^2$ then $W(\eta) = cs(\eta)$,

Type 4: If $p_1 = -2m^2, p_2 = 2m^2 - 1, p_3 = 1 - m^2$ then $W(\eta) = cn(\eta)$,

Type 5: If $p_1 = -2, p_2 = 2 - m^2, p_3 = m^2 - 1$ then $W(\eta) = dn(\eta)$,

Type 6: If $p_1 = \frac{m^2}{2}, p_2 = \frac{m^2-2}{2}, p_3 = \frac{1}{4}$ then $W(\eta) = \frac{sn(\eta)}{1\pm dn(\eta)}$,

Type 7: If $p_1 = \frac{m^2}{2}, p_2 = \frac{m^2-2}{2}, p_3 = \frac{m^2}{4}$ then $W(\eta) = \frac{sn(\eta)}{1\pm dn(\eta)}$,

Type 8: If $p_1 = \frac{-1}{2}, p_2 = \frac{m^2+1}{2}, p_3 = \frac{-\left(1-m^2\right)^2}{4}$ then $W(\eta) = mcn(\eta) + dn(\eta)$,

Type 9: If $p_1 = \frac{m^2-1}{2}, p_2 = \frac{m^2+1}{2}, p_3 = \frac{m^2-1}{4}$ then $W(\eta) = \frac{dn(\eta)}{1\pm sn(\eta)}$,

Type 10: If $p_1 = \frac{1-m^2}{2}, p_2 = \frac{1-m^2}{2}, p_3 = \frac{1-m^2}{4}$ then $W(\eta) = \frac{cn(\eta)}{1\pm sn(\eta)}$,

Type 11: If $p_1 = \frac{\left(1-m^2\right)^2}{2}, p_2 = \frac{\left(1-m^2\right)^2}{2}, p_3 = \frac{1}{4}$ then $W(\eta) = \frac{sn(\eta)}{dn\pm cn(\eta)}$,

Type 12: If $p_1 = 2, p_2 = 0, p_3 = 0$ then $W(\eta) = \frac{c}{\eta}$,

Type 13: If $p_1 = 0, p_2 = 1, p_3 = 0$ then $W(\eta) = ce^{\eta}$.

such that $dn(\eta) = dn(\eta, m), sn(\eta) = sn(\eta, m), cn(\eta) = cn(\eta, m)$ and for $0 < m < 1$ seem to be the Jacobi elliptic functions (JEFs). The subsequent hyperbolic functions are generated from JEFs whenever $m \to 1$

$$cs(\eta) \to csch(\eta), sn(\eta) \to tanh(\eta), cn(\eta) \to sech(\eta),$$

$$dn(\eta) \to sech(\eta), ds(\eta) \to csch(\eta).$$

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The e triangular functions that result from $m \to 0$ are just as follows:

$$sn(\eta) \to sin(\eta), cn(\eta) \to cos(\eta), dn(\eta) \to 1,$$
$$cs(\eta) \to cot(\eta), ds(\eta) \to csc(\eta).$$

4 Application of the Mapping Method

Consider random(nonlinear) stochastic equation

$$du_x - [4u_x u_{xy} + 2u_{xx}u_y - u_{xxxy}]dt = \sigma u_x d\xi \tag{4.1}$$

where $u(x,y,t)$ is complex valued, and $\xi(t)$ is a real valued white noise and $\xi_t = \frac{d\xi}{dt}$ Now we split the wave profile into amplitude and phase components, respectively, as

$$u(x,y,t) = U(\eta) e^{\sigma\xi(t) - \frac{1}{2}\sigma^2 t}, \tag{4.2}$$

where $\eta = (\eta_1 x + \eta_2 y + \eta_3 t)$. Substituting Eq. (4.2) into Eq. (4.1)

$$\psi'' + k_1\psi + k_2\psi^2 = 0. \tag{4.3}$$

Balancing ψ'' and ψ^2 Eq.(4.3) gives $N = 2$ Thus, (4.3) has the formal solution:

$$\psi(\eta) = a_0 + a_1 W^1 + a_2 W^2, \tag{4.4}$$

where a_0, a_1 and a_2 are constants to be determined, such that $a_2 \neq 0$. Substituting Eq. (4.4) along with Eq.(3.2) into Eq. (4.3) and setting the coefficients of all powers of $W^i, i = 0,1,...,6$ to zero, we yield the following system of algebraic equations

$$\begin{cases} 3a_2p_1 - k_2{a_2}^2 = 0. \\ a_1p_1 - 2k_2a_1a_2 = 0, \\ 4a_2p_2 - 2k_2a_0a_2 + k_1a_2 - k_2{a_1}^2 = 0, \\ k_1a_1 + a_1p_2 - 2k_2a_0a_1 = 0, \\ 2a_2p_3 + k_1a_0 - k_2{a_0}^2 = 0. \end{cases} \tag{4.5}$$

Solving the system of algebraic equations(4.5) with the aid of Maple, we have the following set of results:

$$a_0 = \frac{p_2 + k_1}{2k_2}, a_1 = \pm\frac{3\sqrt{p_1p_2}}{k_2}, a_2 = \frac{3p_1}{k_2}, \tag{4.6}$$

Substituting Eq.(4.6) along with Eq.(3.2) into Eq.(4.4), we get the following exact solution of Eq.(4.3). Now, Eq.(4.3) has the following many new exact solutions:
If $m \to 1$, then we obtain the solitary wave solution
(1) If $p_1 = 2m^2$, $p_2 = -(1+m^2)$, $p_3 = 1$ then $W(\eta) = sn(\eta, k)$ and the solution of the Eq.(2.10) are

$$\psi(\eta) = \frac{p_2 + k_1}{2k_2} + \frac{3\sqrt{p_1p_2}}{k_2} sn\eta + \frac{3p_1}{k_2} sn^2\eta \tag{4.7}$$

$$\psi(\eta) = \frac{p_2 + k_1}{2k_2} + \frac{3\sqrt{p_1p_2}}{k_2} \tanh\eta + \frac{3p_1}{k_2} \tanh^2\eta \tag{4.8}$$

Integrate above eq

$$U = \frac{p_2 + k_1}{2k_2}\eta + \frac{3\sqrt{p_1p_2}}{k_2} ln\cosh\eta + \frac{3p_1}{k_2}(\eta - \tanh\eta) \tag{4.9}$$

$$u_1(x,y,t) = \left\{ \frac{p_2 + k_1}{2k_2}\eta + \frac{3\sqrt{p_1p_2}}{k_2} ln\cosh\eta + \frac{3p_1}{k_2}(\eta - \tanh\eta) \right\} e^{\sigma\xi(t) - \frac{1}{2}\sigma^2 t} \tag{4.10}$$

$$u_1(\eta) = \left\{ \frac{1}{6}\left(-1 - m^2 + \frac{\eta_3}{{\eta_1}^2\eta_2}\right)\eta\,\eta_1 + \sqrt{2}\sqrt{m^2(-1-m^2)} ln\cosh\eta\,\eta_1 + 2m^2(\eta - \tanh\eta)\,\eta_1 \right\} e^{\sigma\xi(t) - \frac{1}{2}\sigma^2 t} \tag{4.11}$$

(2) If $p_1 = 2, p_2 = 2m^2 - 1, p_3 = -m^2(1-m^2)$ and the solution of the Eq.(2.10) are

$$\psi(\eta) = \frac{p_2 + k_1}{2k_2} + \frac{3\sqrt{p_1p_2}}{k_2} ds\eta + \frac{3p_1}{k_2} ds^2\eta \tag{4.12}$$

$$\psi(\eta) = \frac{p_2 + k_1}{2k_2} + \frac{3\sqrt{p_1p_2}}{k_2} \operatorname{csch}\eta + \frac{3p_1}{k_2} \operatorname{csch}^2\eta \tag{4.13}$$

Integrate above eq

$$U = \frac{p_2 + k_1}{2k_2}\eta + \frac{3\sqrt{p_1p_2}}{k_2} \ln\left|\tanh\frac{\eta}{2}\right| - \frac{3p_1}{k_2}\coth\eta \tag{4.14}$$

$$u_1(x,y,t) = \left\{ \frac{p_2 + k_1}{2k_2}\eta + \frac{3\sqrt{p_1p_2}}{k_2} \ln\left|\tanh\frac{\eta}{2}\right| - \frac{3p_1}{k_2}\coth\eta \right\} e^{\sigma\xi(t) - \frac{1}{2}\sigma^2 t} \tag{4.15}$$

$$u_2(\eta) = \left\{ \frac{1}{6}\left(2m^2 - 1 + \frac{\eta_3}{{\eta_1}^2\eta_2}\right)\eta\,\eta_1 + \sqrt{4m^2 - 2}\ln\left|\tanh\frac{\eta}{2}\right|\eta_1 - 2coth\eta\,\eta_1 \right\} e^{\sigma\xi(t) - \frac{1}{2}\sigma^2 t} \tag{4.16}$$

(3) If $p_1 = 2, p_2 = 2 - m^2, p_3 = 1 - m^2$ and the solution of the Eq.(2.10) are

$$\psi(\eta) = \frac{p_2 + k_1}{2k_2} + \frac{3\sqrt{p_1p_2}}{k_2} cs\eta + \frac{3p_1}{k_2} cs^2\eta \tag{4.17}$$

$$\psi(\eta) = \frac{p_2 + k_1}{2k_2} + \frac{3\sqrt{p_1p_2}}{k_2} \operatorname{csch}\eta + \frac{3p_1}{k_2} \operatorname{csch}^2\eta \tag{4.18}$$

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Integrate above eq

$$U=\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{k_2}\ln\left|\tanh\frac{\eta}{2}\right|-\frac{3p_1}{k_2}\coth\eta \quad (4.19)$$

$$u_1(x,y,t)=\left\{\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{k_2}\ln\left|\tanh\frac{\eta}{2}\right|-\frac{3p_1}{k_2}\coth\eta\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \quad (4.20)$$

$$u_3(\eta)=\left\{\frac{1}{6}\left(2-m^2+\frac{\eta_3}{{\eta_1}^2\eta_2}\right)\eta\,\eta_1+\sqrt{4-2m^2}\ln\left(\tanh\frac{\eta}{2}\right)\eta_1-2coth\eta\,\eta_1\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \quad (4.21)$$

(4) If $p_1=-2m^2, p_2=2m^2-1, p_3=1-m^2$ and the solution of the Eq.(2.10) are

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}cn\eta+\frac{3p_1}{k_2}cn^2\eta \quad (4.22)$$

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}\operatorname{sech}\eta+\frac{3p_1}{k_2}\operatorname{sech}^2\eta \quad (4.23)$$

Integrate above eq

$$U=\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{k_2}\arctan(sinh\eta)+\frac{3p_1}{k_2}\tanh\eta \quad (4.24)$$

$$u_1(x,y,t)=\left\{\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{k_2}\arctan(sinh\eta)+\frac{3p_1}{k_2}\tanh\eta\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \quad (4.25)$$

$$u_4(\eta)=\left\{\frac{1}{6}\left(2m^2-1+\frac{\eta_3}{{\eta_1}^2\eta_2}\right)\eta\,\eta_1+\sqrt{-2m^2(2m^2-1)}\arctan(sinh\eta)\,\eta_1-2m^2tanh\eta\,\eta_1\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \quad (4.26)$$

(5) If $p_1=-2, p_2=2-m^2, p_3=m^2-1$ and the solution of the Eq.(2.10) are

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}dn\eta+\frac{3p_1}{k_2}dn^2\eta \quad (4.27)$$

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}\operatorname{sech}\eta+\frac{3p_1}{k_2}\operatorname{sech}^2\eta \quad (4.28)$$

Integrate above eq

$$U=\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{k_2}\arctan(sinh\eta)+\frac{3p_1}{k_2}\tanh\eta \quad (4.29)$$

$$u_1(x,y,t)=\left\{\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{k_2}\arctan(sinh\eta)+\frac{3p_1}{k_2}\tanh\eta\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \quad (4.30)$$

$$u_5(\eta)=\left\{\frac{1}{6}\left(2-m^2+\frac{\eta_3}{{\eta_1}^2\eta_2}\right)\eta\,\eta_1+\sqrt{-4+2m^2}\arctan(sinh\eta)\,\eta_1-2tanh\eta\,\eta_1\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \quad (4.31)$$

(6) If $p_1=\frac{m^2}{2}, p_2=\frac{m^2-2}{2}, p_3=\frac{1}{4}$ and the solution of the Eq.(2.10) are

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}\frac{\operatorname{sn}\eta}{1\pm\operatorname{dn}\eta}+\frac{3p_1}{k_2}\frac{(\operatorname{sn}\eta)^2}{(1\pm\operatorname{dn}\eta)^2}. \quad (4.32)$$

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}\frac{tanh\eta}{1\pm\operatorname{sech}\eta}+\frac{3p_1}{k_2}\frac{(tanh\eta)^2}{(1\pm senh\eta)^2} \quad (4.33)$$

Integrate above eq

$$U=\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{2k_2}\left(\ln(sinh\eta)-\ln\left(\tanh\frac{\eta}{2}\right)\right)+\frac{3p_1}{4k_2}(\eta-2coth\eta+2cosech\eta) \quad (4.34)$$

$$u_1(x,y,t)=\left\{\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{2k_2}\left(\ln(sinh\eta)-\ln\left(\tanh\frac{\eta}{2}\right)\right)+\frac{3p_1}{4k_2}(\eta-2coth\eta+2cosech\eta)\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \quad (4.35)$$

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$$u_6(\eta)=\left\{\frac{1}{6}\left(\frac{1}{2}m^2-1+\frac{\eta_3}{{\eta_1}^2\eta_2}\right)\eta_1 + \frac{1}{2}\sqrt{2}\sqrt{m^2\left(\frac{1}{2}m^2-1\right)}\left(\ln(sinh\eta)-\ln\left(\tanh\frac{\eta}{2}\right)\right)\eta_1 + \frac{1}{2}m^2(\eta-2coth\eta+2cosech\eta)\eta_1\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \quad (4.37)$$

(7) If $p_1=\frac{m^2}{2}, p_2=\frac{m^2-2}{2}, p_3=\frac{m^2}{4}$ and the solution of the Eq.(2.10) are

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}\frac{sn\eta}{1\pm dn\eta}+\frac{3p_1}{k_2}\frac{(sn\eta)^2}{(1\pm dn\eta)^2} \quad (4.38)$$

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}\frac{tanh\eta}{1\pm\operatorname{sech}\eta}+\frac{3p_1}{k_2}\frac{(tanh\eta)^2}{(1\pm senh\eta)^2} \quad (4.39)$$

Integrate above eq

$$U=\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{2k_2}\left(\ln(sinh\eta)-\ln\left(\tanh\frac{\eta}{2}\right)\right)+\frac{3p_1}{4k_2}(\eta-2coth\eta+2cosech\eta) \quad (4.40)$$

$$u_1(x,y,t)=\left\{\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{2k_2}\left(\ln(sinh\eta)-\ln\left(\tanh\frac{\eta}{2}\right)\right)+\frac{3p_1}{4k_2}(\eta-2coth\eta+2cosech\eta)\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \quad (4.41)$$

$$u_7(\eta)=\left\{\frac{1}{6}\left(\frac{1}{2}m^2-1+\frac{\eta_3}{{\eta_1}^2\eta_2}\right)\eta_1 + \frac{1}{2}\sqrt{2}\sqrt{m^2\left(\frac{1}{2}m^2-1\right)}\left(\ln(sinh\eta)-\ln\left(\tanh\frac{\eta}{2}\right)\right)\eta_1 + \frac{1}{2}m^2(\eta-2coth\eta+2cosech\eta)\eta_1\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \quad (4.42)$$

(8) If $p_1=\frac{-1}{2}, p_2=\frac{m^2+1}{2}, p_3=\frac{-(1-m^2)^2}{4}$ and the solution of the Eq.(2.10) are

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}(mcn\eta\pm dn\eta)+\frac{3p_1}{k_2}(mcn\eta\pm dn\eta)^2 \quad (4.43)$$

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}(msech\eta\pm sech\eta)+\frac{3p_1}{k_2}(msech\eta\pm sech\eta)^2 \quad (4.44)$$

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}(msech\eta+sech\eta)+\frac{3p_1}{k_2}\left(m^2sech^2\eta+sech^2\eta+2msech^2\eta\right) \quad (4.45)$$

Integrate above eq

$$U=\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{k_2}(m\arctan(sinh\eta)+\arctan(sinh\eta))+\frac{3p_1}{k_2}\left(m^2tanh\eta+tanh\eta+2mtanh\eta\right) \quad (4.46)$$

$$u_1(x,y,t)=\left\{\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{k_2}(m\arctan(sinh\eta)+\arctan(sinh\eta))+\frac{3p_1}{k_2}\left(m^2tanh\eta+tanh\eta+2mtanh\eta\right)\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \quad (4.47)$$

$$u_8(\eta)=\left\{\frac{1}{6}\left(\frac{1}{2}+\frac{1}{2}m^2+\frac{\eta_3}{{\eta_1}^2\eta_2}\right)\eta_1 + \frac{1}{2}\sqrt{-1-m^2}(m\arctan(sinh\eta)+\arctan(sinh\eta))\eta_1 - \frac{1}{2}\left(m^2tanh\eta+tanh\eta+2mtanh\eta\right)\eta_1\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \quad (4.48)$$

(9) If $p_1=\frac{m^2-1}{2}, p_2=\frac{m^2+1}{2}, p_3=\frac{m^2-1}{4}$ and the solution of the Eq.(2.10) are

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}\frac{dn\eta}{1\pm sn\eta}+\frac{3p_1}{k_2}\frac{(dn\eta)^2}{(1\pm sn\eta)^2} \quad (4.49)$$

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}\frac{sech\eta}{1\pm tanh\eta}+\frac{3p_1}{k_2}\frac{(sech\eta)^2}{(1\pm tanh\eta)^2} \quad (4.50)$$

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Integrate above eq

$$U=\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{k_2}\left(sinh\eta-cosh\eta\right)+\frac{3p_1}{4k_2}\left(\frac{1}{2}\sinh\left(2\eta\right)-\frac{1}{2}\cosh\left(2\eta\right)\right) \quad (4.51)$$

$$u_1(x,y,t)=\left\{\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{k_2}\left(sinh\eta-cosh\eta\right)+\frac{3p_1}{4k_2}\left(\frac{1}{2}\sinh\left(2\eta\right)-\frac{1}{2}\cosh\left(2\eta\right)\right)\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \quad (4.52)$$

$$u_9\left(\eta\right)=\left\{\frac{1}{6}\left(\frac{1}{2}+\frac{1}{2}m^2+\frac{\eta_3}{{\eta_1}^2\eta_2}\right)\eta_1+\frac{1}{2}\sqrt{2m\left(\frac{1}{2}+\frac{1}{2}m^2\right)^2-2}\left(sinh\eta-cosh\eta\right)\eta_1+\left(\frac{1}{2}m^2-\frac{1}{2}\right)\left(\frac{1}{2}\sinh\left(2\eta\right)-\frac{1}{2}\cosh\left(2\eta\right)\right)\eta_1\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \quad (4.53)$$

(10) If $p_1=\frac{1-m^2}{2},p_2=\frac{1-m^2}{2},p_3=\frac{1-m^2}{4}$ and the solution of the Eq.(2.10) are

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}\frac{cn\eta}{1\pm sn\eta}+\frac{3p_1}{k_2}\frac{(cn\eta)^2}{(1\pm sn\eta)^2} \quad (4.54)$$

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}\frac{sech\eta}{1\pm tanh\eta}+\frac{3p_1}{k_2}\frac{(sech\eta)^2}{(1\pm tanh\eta)^2} \quad (4.55)$$

Integrate above eq

$$U=\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{k_2}\left(sinh\eta-cosh\eta\right)+\frac{3p_1}{4k_2}\left(\frac{1}{2}\sinh\left(2\eta\right)-\frac{1}{2}\cosh\left(2\eta\right)\right) \quad (4.56)$$

$$u_1(x,y,t)=\left\{\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{k_2}\left(sinh\eta-cosh\eta\right)+\frac{3p_1}{4k_2}\left(\frac{1}{2}\sinh\left(2\eta\right)-\frac{1}{2}\cosh\left(2\eta\right)\right)\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \quad (4.57)$$

$$u_{10}\left(\eta\right)=\left\{\frac{1}{6}\left(\frac{1}{2}-\frac{1}{2}m^2+\frac{\eta_3}{{\eta_1}^2\eta_2}\right)\eta_1+\frac{1}{2}\sqrt{2}\sqrt{\left(1-m^2\right)\left(\frac{1}{2}-\frac{1}{2}m^2\right)}\left(sinh\eta-cosh\eta\right)\eta_1+\left(\frac{1}{2}-\frac{1}{2}m^2\right)\left(\frac{1}{2}\sinh\left(2\eta\right)-\frac{1}{2}\cosh\left(2\eta\right)\right)\eta_1\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \quad (4.58)$$

(11) If $p_1=\frac{\left(1-m^2\right)^2}{2},p_2=\frac{\left(1-m^2\right)^2}{2},p_3=\frac{1}{4}$ and the solution of the Eq.(2.10) are

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}\frac{sn\eta}{dn\pm cn\eta}+\frac{3p_1}{k_2}\frac{(sn\eta)^2}{(dn\pm cn\eta)^2} \quad (4.59)$$

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}\frac{\tanh\eta}{(\mathrm{sech}\,\eta+\mathrm{sech}\,\eta)}+\frac{3p_1}{k_2}\frac{(\tanh\eta)^2}{(\mathrm{sech}\,\eta+\mathrm{sech}\,\eta)^2} \quad (4.60)$$

Integrate above eq

$$U=\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{2k_2}\left(\cosh\eta\right)+\frac{3p_1}{4k_2}\left(\frac{1}{4}\sinh\left(2\eta\right)-\frac{\eta}{2}\right) \quad (4.61)$$

$$u_1(x,y,t)=\left\{\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{2k_2}\left(\cosh\eta\right)+\frac{3p_1}{4k_2}\left(\frac{1}{4}\sinh\left(2\eta\right)-\frac{\eta}{2}\right)\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \quad (4.62)$$

$$u_{11}\left(\eta\right)=\left\{\frac{1}{6}\left(\frac{1}{2}\left(1-m^2\right)^2+\frac{\eta_3}{{\eta_1}^2\eta_2}\right)\eta_1+\frac{1}{4}\left(1-m^2\right)^2cosh\eta\,\eta_1+\frac{1}{16}\left(1-m^2\right)^2\left(\frac{1}{2}\sinh\left(2\eta\right)-\eta\right)\eta_1\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \quad (4.63)$$

(12) If $p_1=2,p_2=0,p_3=0$ and the solution of the Eq.(2.10) are

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}\frac{c}{\eta}+\frac{3p_1}{k_2}\left(\frac{c}{\eta}\right)^2 \quad (4.64)$$

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Integrate above eq

$$U=\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{k_2}cln\eta-\frac{3p_1}{k_2}\frac{c^2}{\eta} \tag{4.65}$$

$$u_1(x,y,t)=\left\{\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{k_2}cln\eta-\frac{3p_1}{k_2}\frac{c^2}{\eta}\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \tag{4.66}$$

$$u_{12}(\eta)=\left\{\frac{1}{6}\frac{\eta_3}{\eta_1\eta_2}-\frac{2c^2\eta_1}{\eta}\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \tag{4.67}$$

(13) If $p_1=2,p_2=0,p_3=0$ and the solution of the Eq.(2.10) are

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}ce^{\eta}+\frac{3p_1}{k_2}(ce^{\eta})^2 \tag{4.68}$$

Integrate above eq

$$U=\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{k_2}ce^{\eta}+\frac{3p_1}{k_2}\frac{1}{2}\frac{c^2e^{\eta2}}{\eta} \tag{4.69}$$

$$u_1(x,y,t)=\left\{\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{k_2}ce^{\eta}+\frac{3p_1}{2k_2}\frac{c^2e^{\eta2}}{\eta}\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \tag{4.70}$$

$$u_{13}(\eta)=\left\{\frac{1}{6}\left(1+\frac{\eta_3}{\eta_1{}^2\eta_2}\right)\eta_1\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \tag{4.71}$$

If $m\to 0$, then we obtain the solitary wave solution

(1) If $p_1=2m^2,p_2=-(1+m^2),p_3=1$ then $W(\eta)=sn(\eta,k)$ and the solution of the Eq.(2.10) are

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}sn\eta+\frac{3p_1}{k_2}sn^2\eta \tag{4.72}$$

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}\sin\eta+\frac{3p_1}{k_2}\sin^2\eta \tag{4.73}$$

Integrate above eq

$$U=\frac{p_2+k_1}{2k_2}\eta-\frac{3\sqrt{p_1p_2}}{k_2}\cos\eta+\frac{3p_1}{k_2}\left(\frac{\eta}{2}-\frac{sin2\eta}{4}\right) \tag{4.74}$$

$$u_1(x,y,t)=\left\{\frac{p_2+k_1}{2k_2}\eta-\frac{3\sqrt{p_1p_2}}{k_2}\cos\eta+\frac{3p_1}{k_2}\left(\frac{\eta}{2}-\frac{sin2\eta}{4}\right)\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \tag{4.75}$$

$$u_1(\eta)=\left\{\frac{1}{6}\left(-1-m^2+\frac{\eta_3}{\eta_1{}^2\eta_2}\right)\eta\,\eta_1-\sqrt{2}\sqrt{m^2(-1-m^2)}\cos(\eta)\,\eta_1+\left(m^2\left(\eta-\frac{\sin2\eta}{2}\right)\right)\eta_1\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \tag{4.76}$$

(2) If $p_1=2,p_2=2m^2-1,p_3=-m^2\left(1-m^2\right)$ and the solution of the Eq.(2.10) are

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}ds\eta+\frac{3p_1}{k_2}ds^2\eta \tag{4.77}$$

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}\csc\eta+\frac{3p_1}{k_2}\csc^2\eta \tag{4.78}$$

Integrate above eq

$$U=\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{k_2}ln\left|csc\eta-cot\eta\right|-\frac{3p_1}{k_2}\cot\eta \tag{4.79}$$

$$u_1(x,y,t)=\left\{\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{k_2}ln\left|csc\eta-cot\eta\right|-\frac{3p_1}{k_2}\cot\eta\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \tag{4.80}$$

$$u_2(\eta)=\left\{\frac{1}{6}\left(2m^2-1+\frac{\eta_3}{\eta_1{}^2\eta_2}\right)\eta\,\eta_1+\sqrt{4m^2-2}ln\left|csc\eta-cot\eta\right|\eta_1-2cot\eta\,\eta_1\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \tag{4.81}$$

(3) If $p_1=2,p_2=2-m^2,p_3=1-m^2$ and the solution of the Eq.(2.10) are

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}cs\eta+\frac{3p_1}{k_2}cs^2\eta \tag{4.82}$$

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}\cot\eta+\frac{3p_1}{k_2}\cot^2\eta \tag{4.83}$$

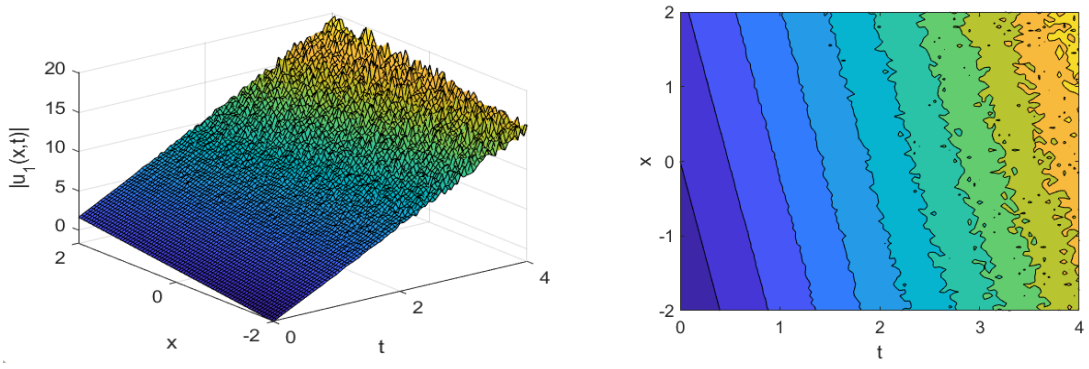
Integrate above eq

$$U=\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{k_2}ln\left|csc\eta-cot\eta\right|-\frac{3p_1}{k_2}\cot\eta \tag{4.84}$$

$$u_1(x,y,t)=\left\{\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{k_2}ln\left|sin\eta\right|+\frac{3p_1}{k_2}(-\cot\eta-\eta)\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \tag{4.85}$$

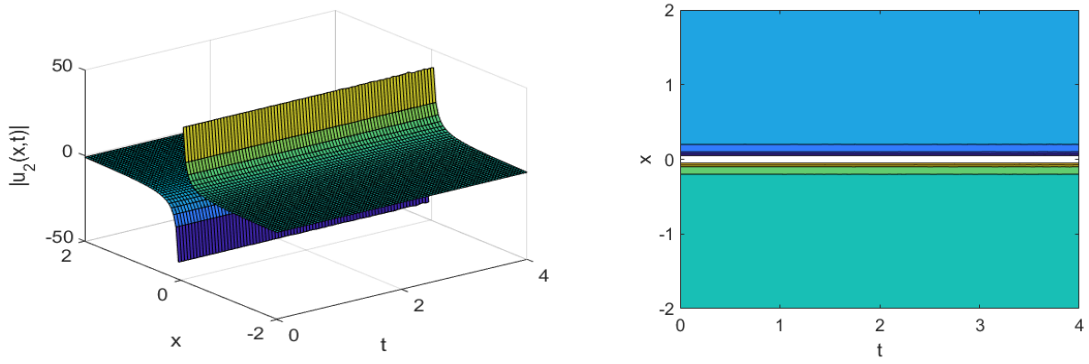
$$u_3(\eta)=\left\{\frac{1}{6}\left(2-m^2+\frac{\eta_3}{\eta_1{}^2\eta_2}\right)\eta\,\eta_1+\sqrt{4-2m^2}\,ln\sin\eta\,\eta_1+2(-\cot\eta-\eta)\,\eta_1\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \tag{4.86}$$

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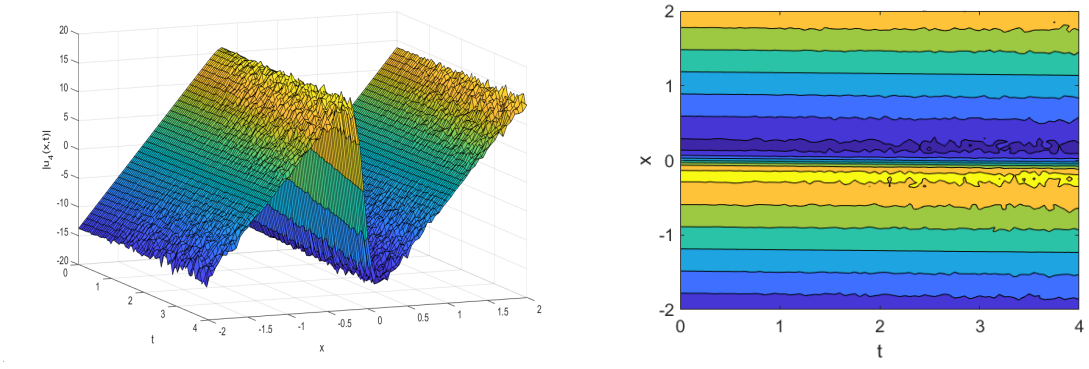
(a) **(b)**

Fig. 1: The values are $\eta_1 = 0.2, \eta_2 = 0.2, \eta_3 = 1.0012, \sigma = 0.1, \xi = randn$. The 3-D graph is on the left and the contour graph is on the right, for the solution $u_1(x,y,t)$.



(a) **(b)**

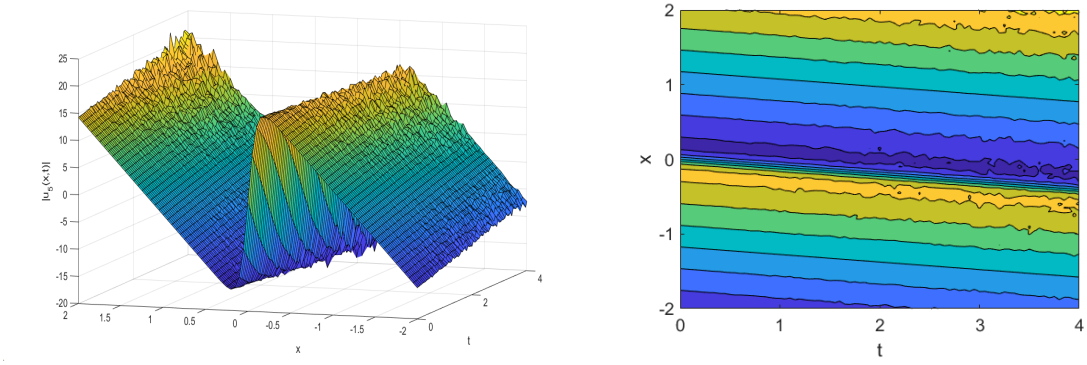
Fig. 2: The values are $\eta_1 = 0.2, \eta_2 = 0.1, \eta_3 = 0, \sigma = 0.009, \xi = randn$. The 3-D graph is on the left and the contour graph is on the right, for the solution $u_2(x,y,t)$.



(a) **(b)**

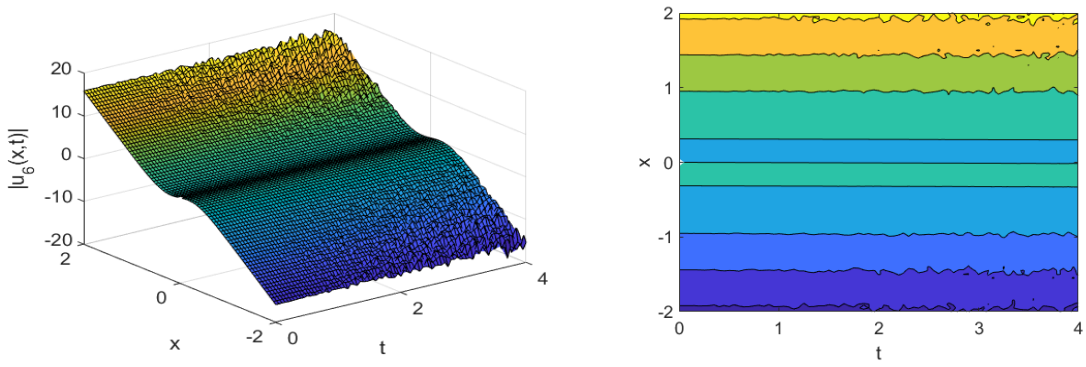
Fig. 3: The values are $\eta_1 = 10, \eta_2 = 0.1, \eta_3 = 0.12, \sigma = 0.1, \xi = randn$. The 3-D graph is on the left and the contour graph is on the right, for the solution $u_4(x,y,t)$.

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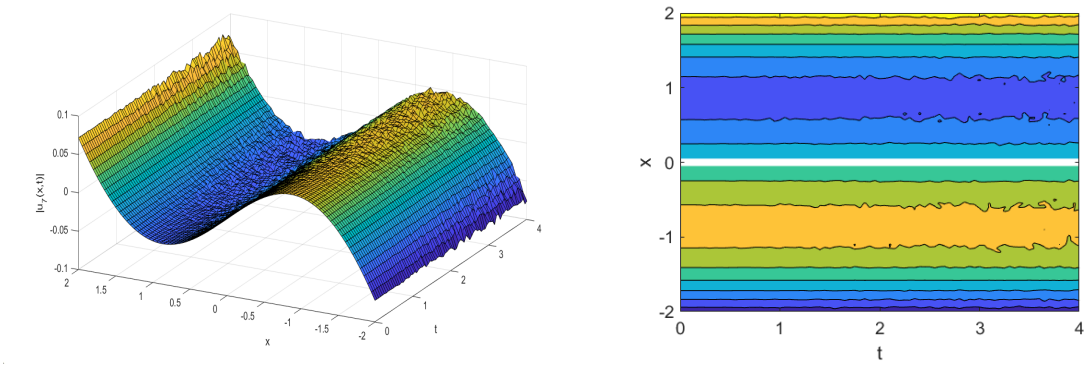
(a) **(b)**

Fig. 4: The values are $\eta_1 = 10.12, \eta_2 = 1, \eta_3 = 1.02, \sigma = 0.1, \xi = randn$. The 3-D graph is on the left and the contour graph is on the right, for the solution $u_5(x,y,t)$.



(a) **(b)**

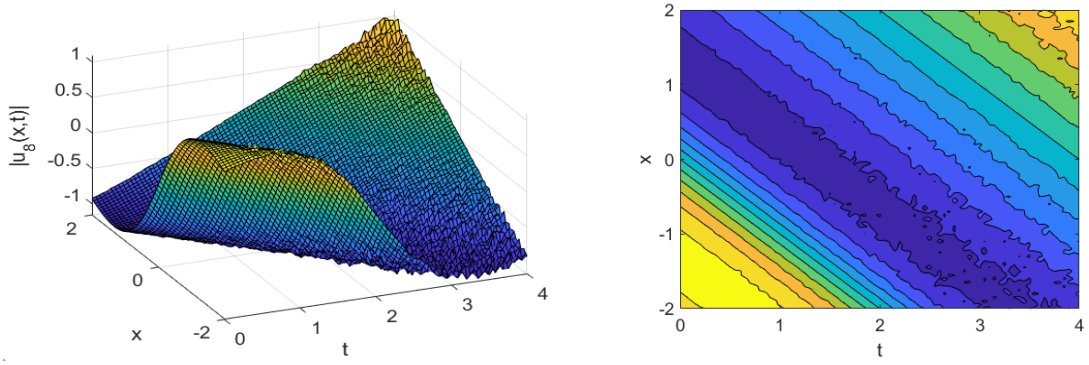
Fig. 5: The values are $\eta_1 = 5, \eta_2 = 1.5, \eta_3 = 0.02, \sigma = 0.1, \xi = randn$. The 3-D graph is on the left and the contour graph is on the right, for the solution $u_6(x,y,t)$.



(a) **(b)**

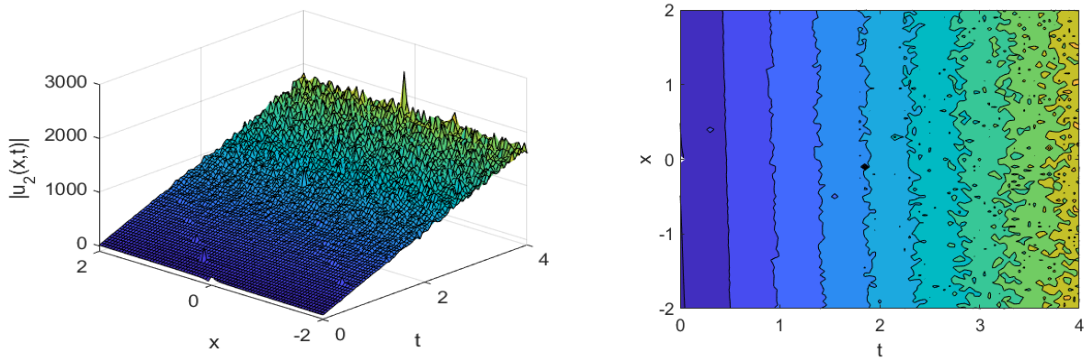
Fig. 6: The values are $\eta_1 = 1, \eta_2 = 0.5, \eta_3 = 0, \sigma = 0.1, \xi = randn$. The 3-D graph is on the left and the contour graph is on the right, for the solution $u_7(x,y,t)$.

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(a) **(b)**

Fig. 7: The values are $\eta_1 = 1, \eta_2 = 0.5, \eta_3 = 1, \sigma = 0.1, \xi = randn$. The 3-D graph is on the left and the contour graph is on the right, for the solution $u_8(x,y,t)$.



(a) **(b)**

Fig. 8: The values are $\eta_1 = 0.2, \eta_2 = 0.2, \eta_3 = 10.2, \sigma = 0.1, \xi = randn$. The 3-D graph is on the left and the contour graph is on the right, for the solution $u_2(x,y,t)$.

(4) If $p_1 = -2m^2, p_2 = 2m^2 - 1, p_3 = 1 - m^2$ and the solution of the Eq.(2.10) are

$$\psi(\eta) = \frac{p_2 + k_1}{2k_2} + \frac{3\sqrt{p_1 p_2}}{k_2} cn\eta + \frac{3p_1}{k_2} cn^2\eta \tag{4.87}$$

$$\psi(\eta) = \frac{p_2 + k_1}{2k_2} + \frac{3\sqrt{p_1 p_2}}{k_2} \cos\eta + \frac{3p_1}{k_2} \cos^2\eta \tag{4.88}$$

Integrate above eq

$$U = \frac{p_2 + k_1}{2k_2}\eta + \frac{3\sqrt{p_1 p_2}}{k_2} \sin\eta + \frac{3p_1}{k_2}\left(\frac{\eta}{2} + \frac{\sin 2\eta}{4}\right) \tag{4.89}$$

$$u_1(x,y,t) = \left\{ \frac{p_2 + k_1}{2k_2}\eta + \frac{3\sqrt{p_1 p_2}}{k_2} \sin\eta + \frac{3p_1}{k_2}\left(\frac{\eta}{2} + \frac{\sin 2\eta}{4}\right) \right\} e^{\sigma\xi(t) - \frac{1}{2}\sigma^2 t} \tag{4.90}$$

$$u_4(\eta) = \left\{ \frac{1}{6}\left(2m^2 - 1 + \frac{\eta_3}{{\eta_1}^2 \eta_2}\right)\eta\,\eta_1 + \sqrt{-2m^2(2m^2-1)}\, sin\eta\,\eta_1 - m^2\left(\eta + \frac{\sin 2\eta}{2}\right)\eta_1 \right\} e^{\sigma\xi(t) - \frac{1}{2}\sigma^2 t} \tag{4.91}$$

(5) If $p_1 = -2, p_2 = 2 - m^2, p_3 = m^2 - 1$ and the solution of the Eq.(2.10) are

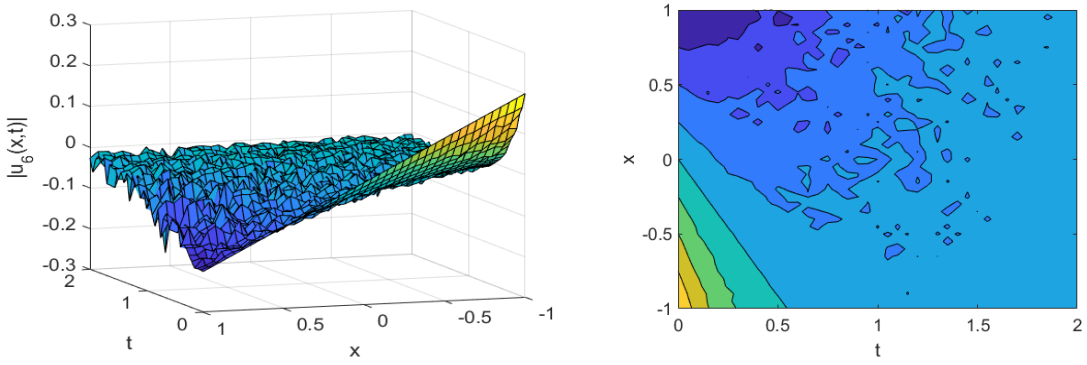
$$\psi(\eta) = \frac{p_2 + k_1}{2k_2} + \frac{3\sqrt{p_1 p_2}}{k_2} dn\eta + \frac{3p_1}{k_2} dn^2\eta \tag{4.92}$$

$$\psi(\eta) = \frac{p_2 + k_1}{2k_2} + \frac{3\sqrt{p_1 p_2}}{k_2}(1) + \frac{3p_1}{k_2}(1)^2 \tag{4.93}$$

Integrate above eq

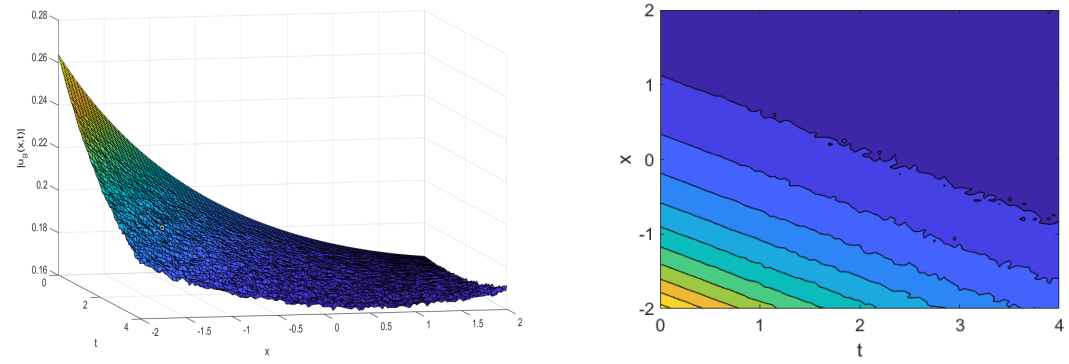
$$U = \frac{p_2 + k_1}{2k_2}\eta + \frac{3\sqrt{p_1 p_2}}{k_2}\eta + \frac{3p_1}{k_2}\eta \tag{4.94}$$

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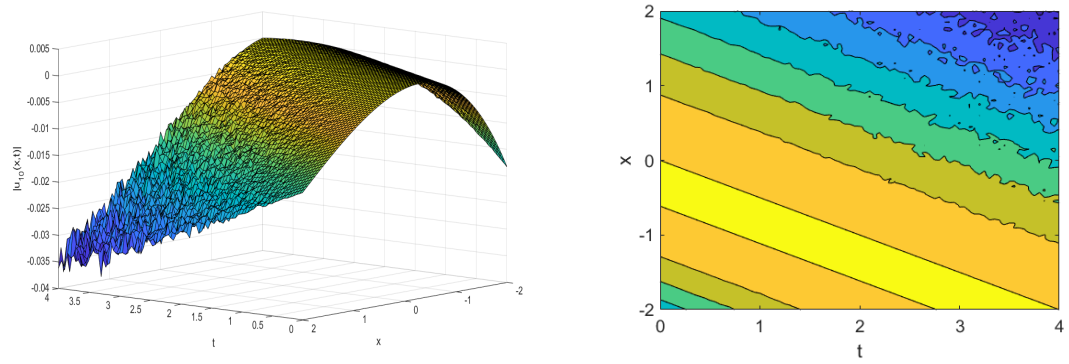
(a) **(b)**

Fig. 9: The values are $\eta_1 = 1.2, \eta_2 = 9.2, \eta_3 = 2.2, \sigma = 2, \xi = randn$. The 3-D graph is on the left and the contour graph is on the right, for the solution $u_6(x,y,t)$.



(a) **(b)**

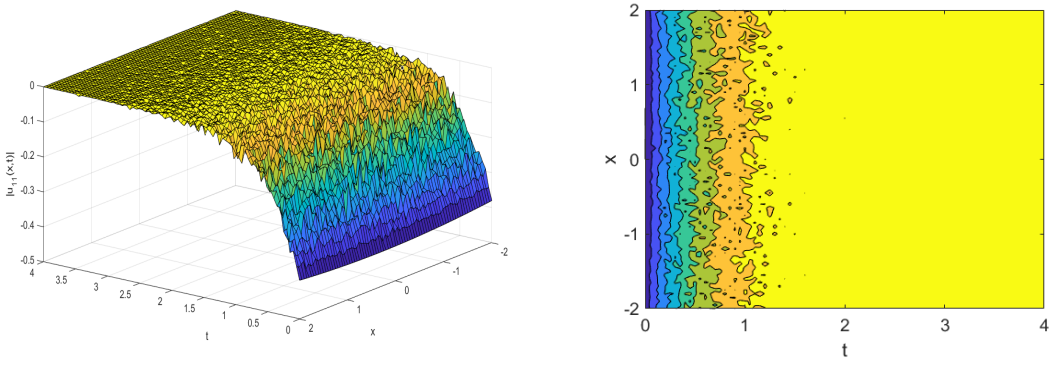
Fig. 10: The values are $\eta_1 = 0.2, \eta_2 = 1.2, \eta_3 = 0.1, \sigma = 0.01, \xi = randn$. The 3-D graph is on the left and the contour graph is on the right, for the solution $u_9(x,y,t)$.



(a) **(b)**

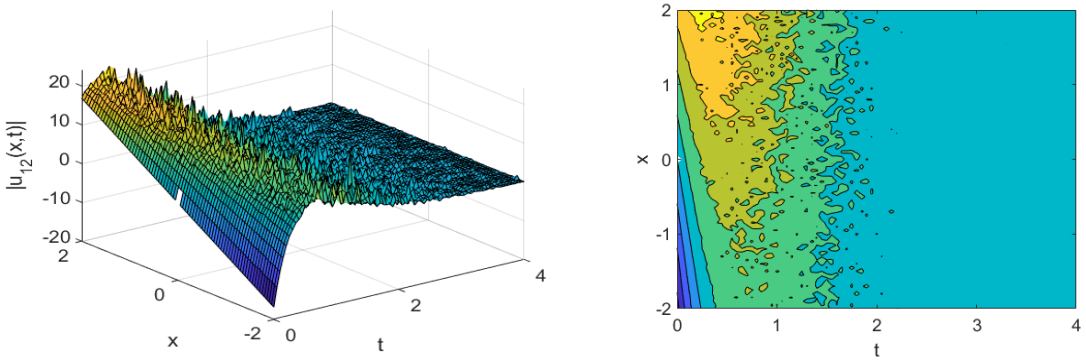
Fig. 11: The values are $\eta_1 = 0.2, \eta_2 = 1.2, \eta_3 = 0.1, \sigma = 0.1, \xi = randn$. The 3-D graph is on the left and the contour graph is on the right, for the solution $u_{10}(x,y,t)$.

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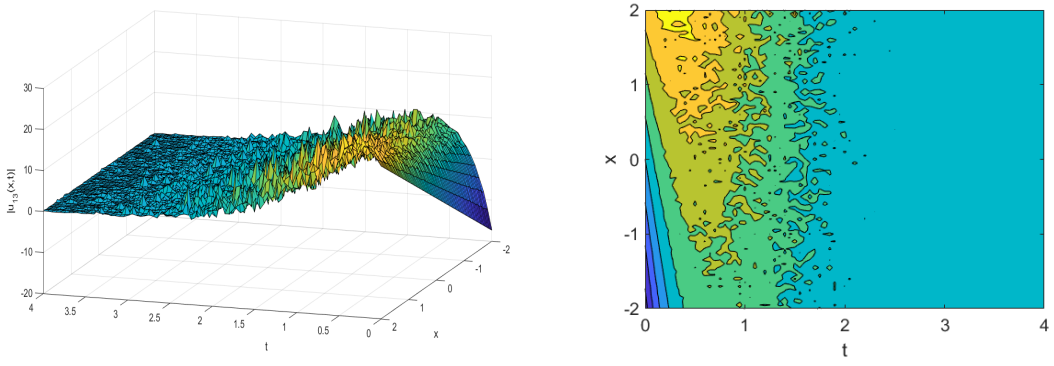
(a) **(b)**

Fig. 12: The values are $\eta_1 = 0.2, \eta_2 = 1.2, \eta_3 = 0.1, , \sigma = 2, \xi = randn$. The 3-D graph is on the left and the contour graph is on the right, for the solution $u_{11}(x,y,t)$.



(a) **(b)**

Fig. 13: The values are $\eta_1 = 1.2, \eta_2 = 0.2, \eta_3 = 10.1, c = 0, \sigma = 2, \xi = randn$. The 3-D graph is on the left and the contour graph is on the right, for the solution $u_{12}(x,y,t)$.



(a) **(b)**

Fig. 14: The values are $\eta_1 = 1.2, \eta_2 = 0.2, \eta_3 = 10.1, c = 0, \sigma = 2, \xi = randn$. The 3-D graph is on the left and the contour graph is on the right, for the solution $u_{13}(x,y,t)$.

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$$u_1(x,y,t)=\left\{\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{k_2}\eta+\frac{3p_1}{k_2}\eta\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \quad (4.95)$$

$$u_5(\eta)=\left\{\frac{1}{6}\left(2-m^2+\frac{\eta_3}{{\eta_1}^2\eta_2}\right)\eta\,\eta_1+\sqrt{-4+2m^2}\eta\,\eta_1-2\,\eta\,\eta_1\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \quad (4.96)$$

(6) If $p_1=\frac{m^2}{2},p_2=\frac{m^2-2}{2},p_3=\frac{1}{4}$ and the solution of the Eq.(2.10) are

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}\frac{sn\eta}{1\pm(dn\eta)}+\frac{3p_1}{k_2}\frac{(sn\eta)^2}{(1\pm dn\eta)^2} \quad (4.97)$$

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}\frac{sin\eta}{1\pm1}+\frac{3p_1}{k_2}\frac{(sin\eta)^2}{(1\pm1)^2} \quad (4.98)$$

Integrate above eq

$$U=\frac{p_2+k_1}{2k_2}\eta-\frac{3\sqrt{p_1p_2}}{2k_2}\cos\eta+\frac{3p_1}{4k_2}\left(\frac{\eta}{2}-\frac{sin2\eta}{4}\right) \quad (4.99)$$

$$u_1(x,y,t)=\left\{\frac{p_2+k_1}{2k_2}\eta-\frac{3\sqrt{p_1p_2}}{2k_2}\cos\eta+\frac{3p_1}{4k_2}\left(\frac{\eta}{2}-\frac{sin2\eta}{4}\right)\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \quad (4.100)$$

$$u_6(\eta)=\left\{\frac{1}{6}\left(\frac{1}{2}m^2-1+\frac{\eta_3}{{\eta_1}^2\eta_2}\right)\eta\,\eta_1-\frac{1}{4}\sqrt{2}\sqrt{m^2\left(\frac{1}{2}m^2-1\right)}\cos\eta\,\eta_1+\frac{1}{16}m^2\left(\eta-\frac{\sin2\eta}{2}\right)\eta_1\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \quad (4.101)$$

(7) If $p_1=\frac{m^2}{2},p_2=\frac{m^2-2}{2},p_3=\frac{m^2}{4}$ and the solution of the Eq.(2.10) are

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}\frac{sn\eta}{1\pm(dn\eta)}+\frac{3p_1}{k_2}\frac{(sn\eta)^2}{(1\pm dn\eta)^2} \quad (4.102)$$

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}\frac{sin\eta}{1\pm1}+\frac{3p_1}{k_2}\frac{(sin\eta)^2}{(1\pm1)^2} \quad (4.103)$$

Integrate above eq

$$U=\frac{p_2+k_1}{2k_2}\eta-\frac{3\sqrt{p_1p_2}}{2k_2}\cos\eta+\frac{3p_1}{4k_2}\left(\frac{\eta}{2}-\frac{sin2\eta}{4}\right) \quad (4.104)$$

$$u_1(x,y,t)=\left\{\frac{p_2+k_1}{2k_2}\eta-\frac{3\sqrt{p_1p_2}}{2k_2}\cos\eta+\frac{3p_1}{4k_2}\left(\frac{\eta}{2}-\frac{sin2\eta}{4}\right)\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \quad (4.105)$$

$$u_7(\eta)=\left\{\frac{1}{6}\left(\frac{1}{2}m^2-1+\frac{\eta_3}{{\eta_1}^2\eta_2}\right)\eta\,\eta_1-\frac{1}{4}\sqrt{2}\sqrt{m^2\left(\frac{1}{2}m^2-1\right)}\cos\eta\,\eta_1+\frac{1}{16}m^2\left(\eta-\frac{\sin2\eta}{2}\right)\eta_1\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \quad (4.106)$$

(8) If $p_1=\frac{-1}{2},p_2=\frac{m^2+1}{2},p_3=\frac{-\left(1-m^2\right)^2}{4}$ and the solution of the Eq.(2.10) are

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}(mcn\eta\pm dn\eta)+\frac{3p_1}{k_2}(mcn\eta\pm dn\eta)^2 \quad (4.107)$$

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}(mcos\eta\pm1)+\frac{3p_1}{k_2}(mcos\eta\pm1)^2 \quad (4.108)$$

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}(mcos\eta\pm1)+\frac{3p_1}{k_2}\left(m^2con^2\eta+1\pm2m\cos\eta\right) \quad (4.109)$$

Integrate above eq

$$U=\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{k_2}(msin\eta\pm\eta)+\frac{3p_1}{k_2}\left(m^2\left(\frac{\eta}{2}+\frac{\sin2\eta}{4}\right)+\eta+2m\sin\eta\right) \quad (4.110)$$

$$u_1(x,y,t)=\left\{\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{k_2}(msin\eta\pm\eta)+\frac{3p_1}{k_2}\left(m^2\left(\frac{\eta}{2}+\frac{\sin2\eta}{4}\right)+\eta+2m\sin\eta\right)\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \quad (4.111)$$

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$$u_8(\eta)=\left\{\frac{1}{6}\left(\frac{1}{2}m^2+\frac{1}{2}+\frac{\eta_3}{{\eta_1}^2\eta_2}\right)\eta\,\eta_1 + \frac{1}{2}\sqrt{-m^2-1}\,(msin\eta\pm\eta)\,\eta_1 - \frac{1}{2}\left(m^2\left(\frac{\eta}{2}+\frac{sin2\eta}{4}\right)+\eta\pm 2msin\eta\right)\eta_1\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2 t} \quad (4.112)$$

(9) If $p_1=\frac{m^2-1}{2}, p_2=\frac{m^2+1}{2}, p_3=\frac{m^2-1}{4}$ and the solution of the Eq.(2.10) are

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}\frac{dn\eta}{1\pm(sn\eta)}+\frac{3p_1}{k_2}\frac{(dn\eta)^2}{(1\pm sn\eta)^2} \quad (4.113)$$

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}\frac{1}{1\pm sin\eta}+\frac{3p_1}{k_2}\frac{(1)^2}{(1\pm sin\eta)^2} \quad (4.114)$$

Integrate above eq

$$U=\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{k_2}(tan\eta-sec\eta)+\frac{3p_1}{4k_2}\left(-\frac{1}{cos\eta+sin\eta\,cos\eta}\right) \quad (4.115)$$

$$u_1(x,y,t)=\left\{\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{k_2}(tan\eta-sec\eta)+\frac{3p_1}{4k_2}\left(-\frac{1}{cos\eta+sin\eta\,cos\eta}\right)\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2 t} \quad (4.116)$$

$$u_9(\eta)=\left\{\frac{1}{6}\left(\frac{1}{2}m^2+\frac{1}{2}+\frac{\eta_3}{{\eta_1}^2\eta_2}\right)\eta_1 + \frac{1}{2}\sqrt{2m\left(\frac{1}{2}m^2+\frac{1}{2}\right)^2-2}\,(tan\eta-sec\eta)\,\eta_1 - \frac{(m^2-1)\,\eta_1}{2(\cos\eta+\sin\eta\cos\eta)}\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2 t} \quad (4.117)$$

(10) If $p_1=\frac{1-m^2}{2}, p_2=\frac{1-m^2}{2}, p_3=\frac{1-m^2}{4}$ and the solution of the Eq.(2.10) are

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}\frac{cn\eta}{1\pm(sn\eta)}+\frac{3p_1}{k_2}\frac{(cn\eta)^2}{(1\pm sn\eta)^2} \quad (4.118)$$

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}\frac{con\eta}{1+(sin\eta)}+\frac{3p_1}{k_2}\frac{(con\eta)^2}{(1+sin\eta)^2} \quad (4.119)$$

Integrate above eq

$$U=\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{k_2}ln(1-\sin\eta)+\frac{3p_1}{k_2}(2\tan\eta-2\sec\eta-\eta) \quad (4.120)$$

$$u_1(x,y,t)=\left\{\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{k_2}ln(1-\sin\eta)+\frac{3p_1}{k_2}(2\tan\eta-2\sec\eta-\eta)\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2 t} \quad (4.121)$$

$$u_{10}(\eta)=\left\{\frac{1}{6}\left(\frac{1}{2}-\frac{1}{2}m^2+\frac{\eta_3}{{\eta_1}^2\eta_2}\right)\eta_1 + \frac{1}{2}\left(1-m^2\right)\ln(1-sin\eta)\,\eta_1 + \frac{1}{2}\left(1-m^2\right)(2tan\eta-2sec\eta-\eta)\,\eta_1\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2 t} \quad (4.122)$$

(11) If $p_1=\frac{\left(1-m^2\right)^2}{2}, p_2=\frac{\left(1-m^2\right)^2}{2}, p_3=\frac{1}{4}$ and the solution of the Eq.(2.10) are

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}\frac{sn\eta}{dn\pm(cn\eta)}+\frac{3p_1}{k_2}\frac{(sn\eta)^2}{(dn\pm cn\eta)^2} \quad (4.123)$$

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}\frac{\sin\eta}{(1+\cos\eta)}+\frac{3p_1}{k_2}\frac{\sin\eta^2}{(1+\cos\eta)^2} \quad (4.124)$$

Integrate above eq

$$U=\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{k_2}(-\ln(1+cos\eta))+\frac{3p_1}{k_2}\left(2\tan\frac{\eta}{2}-\eta\right) \quad (4.125)$$

$$u_1(x,y,t)=\left\{\frac{p_2+k_1}{2k_2}\eta-\frac{3\sqrt{p_1p_2}}{k_2}\ln(1+cos\eta)+\frac{3p_1}{k_2}\left(2\tan\frac{\eta}{2}-\eta\right)\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2 t} \quad (4.126)$$

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$$u_{11}(\eta)=\left\{\frac{1}{6}\left(\frac{1}{2}\left(1-m^2\right)^2+\frac{\eta_3}{{\eta_1}^2\eta_2}\right)\eta_1 -\frac{1}{2}\left(1-m^2\right)^2\ln\left(1+cos\eta\right)\eta_1 +\left(1-m^2\right)^2\left(\tan\frac{\eta}{2}-\eta\right)\eta_1\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \quad (4.127)$$

(12) If $p_1=2,p_2=0,p_3=0$ and the solution of the Eq.(2.10) are

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}\frac{c}{\eta}+\frac{3p_1}{k_2}\left(\frac{c}{\eta}\right)^2 \quad (4.128)$$

Integrate above eq

$$U=\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{k_2}cln\eta-\frac{3p_1}{k_2}\frac{c^2}{\eta} \quad (4.129)$$

$$u_1(x,y,t)=\left\{\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{k_2}cln\eta -\frac{3p_1}{k_2}\frac{c^2}{\eta}\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \quad (4.130)$$

$$u_{12}(\eta)=\left\{\frac{1}{6}\frac{\eta_3}{\eta_1\eta_2}-\frac{2c^2\eta_1}{\eta}\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \quad (4.131)$$

(13) If $p_1=2,p_2=0,p_3=0$ and the solution of the Eq.(2.10) are

$$\psi(\eta)=\frac{p_2+k_1}{2k_2}+\frac{3\sqrt{p_1p_2}}{k_2}ce^{\eta}+\frac{3p_1}{k_2}\left(ce^{\eta}\right)^2 \quad (4.132)$$

Integrate above eq

$$U=\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{k_2}ce^{\eta}+\frac{3p_1}{k_2}\frac{1}{2}\frac{c^2e^{\eta2}}{\eta} \quad (4.133)$$

$$u_1(x,y,t)=\left\{\frac{p_2+k_1}{2k_2}\eta+\frac{3\sqrt{p_1p_2}}{k_2}ce^{\eta} +\frac{3p_1}{2k_2}\frac{c^2e^{\eta2}}{\eta}\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \quad (4.134)$$

$$u_{13}(\eta)=\left\{\frac{1}{6}\left(1+\frac{\eta_3}{{\eta_1}^2\eta_2}\right)\eta_1\right\}e^{\sigma\xi(t)-\frac{1}{2}\sigma^2t} \quad (4.135)$$

5 Discussion and Conclusion

In the present research, we used the maping method to get analytical stochastic solutions for the SBSE, which included elliptic, rational, trigonometric and hyperbolic solutions. Several already obtained results are given in [39, 40]. Because of the significance of this equation, It is used to represent the plasma physics, shallow-water wave hydrodynamic equation and the wave of leading fluid flow. These analytically stochastic solutions have become extensive and important for describing a wide range of very complex physical processes. Finally, we used Matlab tools to visualise the impact of the Wiener process on the acquired SBSE solutions, and we ultimately concluded that the multiplicative Wiener process, within the itô sense, stabilises the SBSE solutions.

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