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Oscillation of Second-Order Neutral Dynamic Equations with Mixed Arguments

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Abstract: We present some new oscillation criteria for a class of half-linear second-order neutral dynamic equations with mixed arguments. An example is given to illustrate the main results.

Keywords: Oscillation, neutral dynamic equation with mixed arguments, time scale

1 Introduction

This paper is concerned with oscillation of a second-order half-linear neutral functional dynamic equation

$$\left(r(t)\left((x(t)+p_1(t)x(\eta_1(t))+p_2(t)x(\eta_2(t)))^{\Delta}\right)^{\gamma}\right)^{\Delta} \\ +q_1(t)x^{\gamma}(\tau_1(t))+q_2(t)x^{\gamma}(\tau_2(t))=0 \quad (1)$$

on an arbitrary time scale $\mathbb{T}$ unbounded above, where γ is a quotient of odd positive integers, r, p_1, p_2, q_1, and q_2 are real-valued positive rd-continuous functions on $\mathbb{T}$. Also, we assume that $\eta_1,\eta_2,\tau_1,\tau_2 : \mathbb{T}\to\mathbb{T}$ are rd-continuous, $\eta_1(t)\le t$, $\eta_2(t)\ge t$, $\tau_1(t)\le t$, $\tau_2(t)\ge t$, and $\lim_{t\to\infty}\eta_1(t)=\lim_{t\to\infty}\tau_1(t)=\infty$. Since we are interested in oscillatory behavior, we assume $t_0\in\mathbb{T}$ and it is convenient to assume $t_0>0$, and define the time scale interval of the form $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}}:=[t_0,\infty)\cap\mathbb{T}$.

Set $z(t):=x(t)+p_1(t)x(\eta_1(t))+p_2(t)x(\eta_2(t))$. By a solution of (1) we mean a non-trivial real-valued function $x\in C^1_{\mathrm{rd}}[T_x,\infty)_{\mathbb{T}}$, $T_x\in[t_0,\infty)_{\mathbb{T}}$ which has the properties that z and $r(z^{\Delta})^{\gamma}$ are defined and Δ-differentiable for $t\in\mathbb{T}$, and satisfies (1) for $t\in[T_x,\infty)_{\mathbb{T}}$. The solutions vanishing in some neighbourhood of infinity will be excluded from our consideration. A solution x of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is termed nonoscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory.

The study of dynamic equations on time scales, which goes back to its founder Hilger [1], is an area of mathematics that has recently received a lot of attention. Several authors have expounded on various aspects of this new theory, see the survey paper by Agarwal et al. [2] and the references cited therein. The books on the subject of time scales, i.e., measure chain, by Bohner and Peterson [3,4], summarize and organize much of time scale calculus. In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions to different classes of dynamic equations on time scales, we refer the reader to the papers [5–25] and the references cited therein. Therein, Agarwal et al. [6], Candan [7], Erbe et al. [8], Şahiner [10], Saker [11], Saker et al. [12,13], Saker and O'Regan [14], Tripathy [15], Chen [16], Zhang and Wang [17], Wu et al. [18], and Thandapani et al. [20] studied a class of half-linear dynamic equations

$$\left(r(t)\left((x(t)+p_1(t)x(\eta_1(t)))^{\Delta}\right)^{\gamma}\right)^{\Delta}+q_1(t)x^{\gamma}(\tau_1(t))=0,$$

where $\eta_1(t)\le t$, and obtained some oscillation results under the assumption that

$$\int_{t_0}^{\infty}\frac{\Delta t}{r^{1/\gamma}(t)}=\infty. \quad (2)$$

The purpose of this paper is to derive new oscillation criteria for (1) in the cases where (2) holds or

$$\int_{t_0}^{\infty}\frac{\Delta t}{r^{1/\gamma}(t)}<\infty. \quad (3)$$

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In what follows, all functional inequalities are assumed to hold eventually, that is, they are satisfied for all t large enough.

2 Main results

In this section, using the Riccati transformation technique, we establish some oscillation criteria for (1). In the sequel, we let

$$d_+(t) := \max\{0, d(t)\}, \quad \text{and} \quad d_-(t) := \max\{0, -d(t)\}.$$

Theorem 2.1. Let (2) hold. Assume that there exists a positive real-valued Δ-differentiable function δ such that for all sufficiently large t_1 and for $\tau_1(T) > t_1$,

$$\limsup_{t\to\infty} \int_T^t \left[\delta(s)f(s) - \frac{r(s)(\delta_+^\Delta(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^\gamma(s)}\right]\Delta s = \infty, \quad (4)$$

where

$$\begin{aligned} f(s) : = q_1(s)&\left(\frac{m(\tau_1(s))}{m(s)}\right)^\gamma \\ &\cdot\left(1 - p_1(\tau_1(s)) - p_2(\tau_1(s))\frac{m(\eta_2(\tau_1(s)))}{m(\tau_1(s))}\right)^\gamma \\ &+q_2(s) \\ &\cdot\left(1 - p_1(\tau_2(s)) - p_2(\tau_2(s))\frac{m(\eta_2(\tau_2(s)))}{m(\tau_2(s))}\right)^\gamma, \end{aligned} \quad (5)$$

and m is a positive real-valued Δ-differentiable function such that

$$\frac{m(t)}{r^{1/\gamma}(t)\int_{t_1}^t \frac{1}{r^{1/\gamma}(s)}\Delta s} - m^\Delta(t) \le 0 \quad (6)$$

and

$$1 - p_1(t) - p_2(t)\frac{m(\eta_2(t))}{m(t)} > 0. \quad (7)$$

Then (1) is oscillatory.

Proof. Let x be a nonoscillatory solution of (1). Without loss of generality, we assume $x(t) > 0$, $x(\eta_i(t)) > 0$, and $x(\tau_i(t)) > 0$ for $i = 1, 2$ and $t \in [t_0, \infty)_{\mathbb{T}}$. In view of (1), we obtain

$$\begin{aligned} (r(t)(z^\Delta(t))^\gamma)^\Delta &= -q_1(t)x^\gamma(\tau_1(t)) \\ &\quad -q_2(t)x^\gamma(\tau_2(t)) < 0, \quad t \in [t_0, \infty)_{\mathbb{T}}. \end{aligned} \quad (8)$$

It follows from (2) that there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $z^\Delta(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. By virtue of (8), we see that

$$\begin{aligned} z(t) &= z(t_1) + \int_{t_1}^t \frac{(r(s)(z^\Delta(s))^\gamma)^{1/\gamma}}{r^{1/\gamma}(s)}\Delta s \\ &\ge r^{1/\gamma}(t)z^\Delta(t)\int_{t_1}^t \frac{\Delta s}{r^{1/\gamma}(s)}. \end{aligned}$$

Since

$$\begin{aligned} \left(\frac{z(t)}{m(t)}\right)^\Delta &= \frac{z^\Delta(t)m(t) - z(t)m^\Delta(t)}{m(t)m^\sigma(t)} \\ &\le \frac{z(t)}{m(t)m^\sigma(t)}\left[\frac{m(t)}{r^{1/\gamma}(t)\int_{t_1}^t \frac{1}{r^{1/\gamma}(s)}\Delta s} - m^\Delta(t)\right] \le 0, \end{aligned}$$

we find that z/m is nonincreasing. Hence we have

$$\begin{aligned} x(t) &= z(t) - p_1(t)x(\eta_1(t)) - p_2(t)x(\eta_2(t)) \\ &\ge z(t) - p_1(t)z(\eta_1(t)) - p_2(t)z(\eta_2(t)) \\ &\ge \left(1 - p_1(t) - p_2(t)\frac{m(\eta_2(t))}{m(t)}\right)z(t). \end{aligned}$$

By virtue of (8), we have

$$\begin{aligned} &(r(t)(z^\Delta(t))^\gamma)^\Delta \le -q_1(t) \\ &\cdot\left(1 - p_1(\tau_1(t)) - p_2(\tau_1(t))\frac{m(\eta_2(\tau_1(t)))}{m(\tau_1(t))}\right)^\gamma z^\gamma(\tau_1(t)) \\ &- q_2(t)\left(1 - p_1(\tau_2(t)) - p_2(\tau_2(t))\frac{m(\eta_2(\tau_2(t)))}{m(\tau_2(t))}\right)^\gamma \\ &\cdot z^\gamma(\tau_2(t)). \end{aligned} \quad (9)$$

Define the function ω by

$$\omega(t) := \delta(t)\frac{r(t)(z^\Delta(t))^\gamma}{z^\gamma(t)}, \quad t \in [t_1, \infty)_{\mathbb{T}}. \quad (10)$$

Then $\omega(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. The rest of the proof is similar to that of [8, Theorem 2.1], and so is omitted. The proof is complete. □

Remark 2.1. The function m is existent, see, e.g., $m(t) = \int_{t_1}^t \frac{1}{r^{1/\gamma}(s)}\Delta s$.

Theorem 2.2. Let (2) hold. Assume that there exist functions H, h such that for each fixed t, $H(t,s)$ and $h(t,s)$ are rd-continuous with respect to s on $\mathbb{D} = \{(t,s) : t \ge s \ge t_0\}$ and

$$H(t,t) = 0, \quad t \ge t_0, \quad H(t,s) > 0, \quad t > s \ge t_0, \quad (11)$$

and H has a nonpositive continuous Δ-partial derivative $H^{\Delta_s}(t,s)$ with respect to the second variable and satisfies

$$\begin{aligned} &- H^{\Delta_s}(t,s) - H(t,s)\frac{\delta^\Delta(s)}{\delta^\sigma(s)} \\ &= \frac{h(t,s)}{\delta^\sigma(s)}(H(t,s))^{\gamma/(\gamma+1)}, \end{aligned} \quad (12)$$

and for all sufficiently large t_1 and for $\tau_1(T) > t_1$,

$$\begin{aligned} \limsup_{t\to\infty} \frac{1}{H(t,T)}\int_T^t &[\delta(s)H(t,s)f(s) \\ &- \frac{r(s)(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^\gamma(s)}\bigg]\Delta s = \infty, \end{aligned} \quad (13)$$

where f is defined as in (5), m is a positive real-valued Δ-differentiable function such that (6) and (7) hold, and δ is a positive real-valued Δ-differentiable function. Then (1) is oscillatory.

Proof. From Theorem 2.1, the proof is similar to that of [8, Theorem 2.2], and hence is omitted. This completes the proof. □

Theorem 2.3. Let (3) hold and $0 \le p_1(t) + p_2(t) \le p < 1$. Assume that all assumptions of Theorem 2.1 or Theorem 2.2 hold except (2). If for all sufficiently large t_1,

$$\int_{t_1}^{\infty} \left[\frac{1}{r(s)} \int_{t_1}^{s} [q_1(\tau) + q_2(\tau)] \Delta \tau \right]^{1/\gamma} \Delta s = \infty, \tag{14}$$

then every solution of (1) is oscillatory or converges to zero as $t \to \infty$.

Proof. Let x be a nonoscillatory solution of (1). Without loss of generality, we assume $x(t) > 0$, $x(\eta_i(t)) > 0$, and $x(\tau_i(t)) > 0$ for $i = 1,2$ and $t \in [t_0,\infty)_{\mathbb{T}}$. In view of (1), we obtain (8). Then there exists a $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $z^\Delta(t) > 0$, or $z^\Delta(t) < 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$.

Assume first that $z^\Delta(t) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. As in Theorem 2.1 or Theorem 2.2, we can obtain the corresponding contradictions. Assume now that $z^\Delta(t) < 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. We assert that $\lim_{t\to\infty} z(t) = 0$. If not, similar to the proof of [26, Lemma 2], there exist two constants $k > 0$ and $l > 0$ such that

$$x(t) \ge kz(t) \ge kl.$$

By virtue of (8), we have

$$\begin{aligned}(r(t)(z^\Delta(t))^\gamma)^\Delta &= -q_1(t)x^\gamma(\tau_1(t)) - q_2(t)x^\gamma(\tau_2(t)) \\ &\le -(kl)^\gamma[q_1(t) + q_2(t)].\end{aligned}$$

Integrating the latter inequality from t_1 to t, we get

$$r(t)(z^\Delta(t))^\gamma \le -(kl)^\gamma \int_{t_1}^{t} [q_1(s) + q_2(s)] \Delta s.$$

Integrating again from t_1 to t, we obtain

$$z(t) \le z(t_1) - kl \int_{t_1}^{t} \left[\frac{1}{r(s)} \int_{t_1}^{s} [q_1(\tau) + q_2(\tau)] \Delta \tau \right]^{1/\gamma} \Delta s.$$

Hence by (14), $\lim_{t\to\infty} z(t) = -\infty$, which is a contradiction. Therefore, $\lim_{t\to\infty} z(t) = 0$. Then, $\lim_{t\to\infty} x(t) = 0$ due to $0 \le x(t) \le z(t)$. This completes the proof. □

For an application of our results, we give the following example.

Example 2.1. We consider the second-order neutral functional dynamic equation

$$\begin{aligned}&\left(x(t) + \frac{1}{2}x(\eta_1(t)) + \frac{t^2}{3(\eta_2(t))^2}x(\eta_2(t)) \right)^{\Delta\Delta} \\ &+ \frac{k_1}{t}x(\tau_1(t)) + \frac{k_2}{t}x(\tau_2(t)) = 0,\end{aligned} \tag{15}$$

where $t \in [1,\infty)_{\mathbb{T}}$, $\tau_1(t) \le t$, $\tau_2(t) \ge t$, $\eta_1(t) \le t$, $\eta_2(t) \ge t$, $k_1 > 0$, and $k_2 > 0$. Let $m(t) = t^2$ and $\delta(t) = 1$. Then, every solution of (15) is oscillatory when using Theorem 2.1.

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