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Theoretical Analysis of HBV Infection Under Mittag-Leffler Derivative

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Abstract: The theoretical study of fractional calculus has grown significantly during the last few years. For the theoretical analysis of fractional differential equations, primarily two methods were employed. One is the fixed point approach, which determines whether a solution exists, and the other is the functional analysis approach, which determines whether a solution is stable. This study investigates the theoretical features of HBV infection under a fractional operator with a nonsingular and nonlocal kernel. We examine the existence and uniqueness of the model's results using the fixed point theorems of Banach and Krasnoselskii. According to the Hyres-Ulam stability studies, the HBV model's solution is stable under the Atangana-Baleanu derivative.

Keywords: HBV infection, fractional-order model, Hyres-Ulam stability, Krasnoselskii fixed point theorem.

1 Introduction, motivation and preliminaries

The hepatitis B virus, which causes hepatitis B, is a dangerous and chronic disease that affects the liver. According to the WHO, 257 million people had chronic HBV infection in 2015 [1]. HBV is now one of the world's most critical public health issues. It can lead to chronic infection and poses a higher risk of dying from cirrhosis and liver cancer in humans. To research this severe viral infection, many mathematical models [2,3] were created, starting with the innovative research of "Nowak and Bangham" [4].

Only three classes were taken into account in that first model, namely, the suseptable cells class, the infected class, and the virus-free cell's class. It is very well recognized that upon entering the uninfected cell, the virus performs a certain process, decapitation, replication, and encapsidation being the three major steps [5,6]. The significant-viral-mechanism can be added to extend the classical viral dynamic model [7]. To lessen the severity of viral infection, the humoral immune response serves a role, and B-cell-produced is the combating and neutralising antibodies responsible for the immune system [8,9]. Numerous mathematical models were developed by using ordinary differential equations (ODE) to determine the effects of the antibodies on HBV dynamics. A detailed analysis of HBV modeling by ODEs was discussed in [10]. From the other end, the ordinary derivative can be extended by a arbitrary order derivative, and it was considered an effective method for modeling various physical phenomena, this situation happened in a large number of biological systems [11,12].

A non-local operator counter distinguishes the fractional derivative from that of the integer derivative. Modeling some physical phenomena through arbitrary order derivatives has thus caught considerable interest of several authors in various fields [13,14,15] . For example, we make sure that fractional derivatives can be useful in biology for modeling and analyzing cell rheological properties [16]. Besides, a fractional-order form is modeled mostly the HIV dynamics with

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different strategies [17]. On the other side, the FDEs results showed their success in investigating the dynamic behavior of many physical systems in the context of the mechanics [18,19,20]. We recognize that fractional order differential equations have been used to investigate other models such as the “Drinfeld Sokolov Wilson” model, Kawahara equations, and KdV equations [21,22]. The behavior of several viral diseases was investigated in various articles using fractional order differential equations (FDEs) [23,24]. Recently, the use of FDEs has examined optimal control problems [25]. All these techniques have been applied to investigate how the immune system and tumor cells interact [26].

Optimum control of coexistence with diabetes and tuberculosis was constructed through arbitrary order derivatives [27]. Recently, Ahmad et. al investigated the transmission dynamics of the COVID-19 [28] by FDEs, it had been founded that the results obtained to fit the real data with the simulated data. The main aim of FDEs is thus to enhance and generalizing the system of ODE. Integer-order models are less effective at describing the retention and heritable qualities of various materials and system of DE’s than fractional calculus. As a result, different perspectives on fractional calculus were investigated, including existence theory, stability analysis, computational analysis, etc. [29,30,31]. However, Traditional fractional derivatives have a singular kernel that makes it difficult to describe some properties. Caputo and Fabrizio (CF) constructed [32], a new fractional operator that includes exponential kernels instead of singular ones, as a solution to this issue. Several real-world problems have been implemented using mathematical models using these operators, and they have shown to be successful [33,34]. Different kinds of fractional order or non-local derivatives have been presented in contemporary literature to handle the limitation of a standard derivative. For instance, Riemann-Liouville developed the idea of a fractional derivative based on a kernel of power law.

After that, M. Caputo and M. Fabrizio used the exponential-decay kernel [32] to propose a new nonsingular fractional operator. The locality of the kernel is a problem for this derivative as well. In order to fix the issue of CF, Atangana and Baleanu (AB) recently developed a new modified version of a fractional derivative in [35] using the well-known Mittag-Leffler function (MLF) as a non-singular and non-local kernel. In [36,37] we list few applications of AB derivative uses. Recently, a fractional model using the following system of nonlinear equations was presented by Danane et al. [38].

$$\begin{cases} {}^{\mathbb{C}}\mathscr{D}_0^{\lambda}\mathscr{H}(t) &= c - k\mathscr{H}\mathscr{V} - \lambda\mathscr{H}, \\ {}^{\mathbb{C}}\mathscr{D}_0^{\lambda}\mathscr{I}(t) &= k\mathscr{H}\mathscr{V} - \eta\mathscr{I}, \\ {}^{\mathbb{C}}\mathscr{D}_0^{\lambda}\mathscr{D}(t) &= a\mathscr{I} - (\beta+\delta)\mathscr{D}, \\ {}^{\mathbb{C}}\mathscr{D}_0^{\lambda}\mathscr{V}(t) &= \beta\mathscr{D} - u\mathscr{V} - q\mathscr{V}\mathscr{W}, \\ {}^{\mathbb{C}}\mathscr{D}_0^{\lambda}\mathscr{W}(t) &= g\mathscr{V}\mathscr{W} - h\mathscr{W}, \end{cases} \tag{1}$$

along with initial values $\mathscr{H}(0) = \omega_1$, $\mathscr{I}(0) = \omega_2$, $\mathscr{D}(0) = \omega_3$, $\mathscr{V}(0) = \omega_4$, $\mathscr{W}(0) = \omega_5$. Now, we use AB-fractional operator to study the above model. Consider

$$\begin{cases} {}^{\mathbb{ABC}}\mathscr{D}_0^{\lambda}\mathscr{H}(t) &= c - k\mathscr{H}\mathscr{V} - \lambda\mathscr{H}, \\ {}^{\mathbb{ABC}}\mathscr{D}_0^{\lambda}\mathscr{I}(t) &= k\mathscr{H}\mathscr{V} - \eta\mathscr{I}, \\ {}^{\mathbb{ABC}}\mathscr{D}_0^{\lambda}\mathscr{D}(t) &= a\mathscr{I} - (\beta+\delta)\mathscr{D}, \\ {}^{\mathbb{ABC}}\mathscr{D}_0^{\lambda}\mathscr{V}(t) &= \beta\mathscr{D} - u\mathscr{V} - q\mathscr{V}\mathscr{W}, \\ {}^{\mathbb{ABC}}\mathscr{D}_0^{\lambda}\mathscr{W}(t) &= g\mathscr{V}\mathscr{W} - h\mathscr{W}. \end{cases} \tag{2}$$

Applying fixed point analysis, this paper will examine the existence and uniqueness of the model (2) solution. Additionally, we analyze the suggested model’s HU-stability. For the rest of paper, FD will use for fractional derivative and FI will be use fractional integral.

Definition 1.*[36] Let $\Phi \in C[0,T]$. The Caputo FD is given as:*

$$D^{\lambda}\Phi(t) = \frac{1}{\Gamma(g-\lambda)}\left[\int_0^t (t-\varpi)^{g-\lambda-1}\frac{d^g}{d\varpi^g}\Phi(t)(\varpi)\,d\varpi\right],$$

where $g = [\lambda]+1$ and $[\lambda]$ is the greatest integer function of λ.

Definition 2.*[36] Let $\Psi \in \mathscr{H}^1(0,T)$. The $\mathbb{ABC}$ FD is:*

$${}^{\mathbb{ABC}}\mathscr{D}_0^{\lambda}\Psi(t) = \frac{\mathscr{G}(\lambda)}{(1-\lambda)}\int_0^t \mathbb{E}_{\lambda}\left(\frac{-\lambda}{\lambda-1}(t-\varpi)^{\lambda}\right)\Psi'(\varpi)d\varpi, \quad t>0,$$

where $0<\lambda\leq 1$ and $\mathscr{G}(\lambda)$ is the normalization function such that $\mathscr{G}(\lambda) = 1$ for $\lambda = 0,1$ and $\mathbb{E}_{\lambda}$ represent the Mittag-Leffler function which is:

$$\mathbb{E}_{\lambda}(\varpi) = \sum_{b=0}^{\infty}\frac{\varpi^b}{\Gamma(\lambda b+1)}.$$

Definition 3.[36] *Let $\Theta \in \mathbb{L}^1(0,T)$. Then the $\mathbb{ABC}$ FI is given by:*

$$ {}^{\mathbb{ABC}}\mathscr{I}_0^{\lambda}\Theta(t) = \frac{(1-\lambda)}{\mathscr{G}(\lambda)}\Theta(t) + \frac{\lambda}{\mathscr{G}(\lambda)}\frac{1}{\Gamma(\lambda)}\int_0^t (t-\varpi)^{\lambda-1}\Theta(\varpi)d\varpi, \quad t>0. $$

2 Main work

2.1 Existence and uniqueness results

We shall examine the system's existence and uniqueness outcomes in this portion (2). The suggested model (2) may be expressed as

$$ \begin{cases} {}^{\mathbb{ABC}}\mathscr{D}_0^{\lambda}\mathscr{H}(t) & = \mathscr{U}_1(t,\mathscr{H}), \\ {}^{\mathbb{ABC}}\mathscr{D}_0^{\lambda}\mathscr{I}(t) & = \mathscr{U}_2(t,\mathscr{I}), \\ {}^{\mathbb{ABC}}\mathscr{D}_0^{\lambda}\mathscr{D}(t) & = \mathscr{U}_3(t,\mathscr{D}), \\ {}^{\mathbb{ABC}}\mathscr{D}_0^{\lambda}\mathscr{V}(t) & = \mathscr{U}_4(t,\mathscr{V}), \\ {}^{\mathbb{ABC}}\mathscr{D}_0^{\lambda}\mathscr{W}(t) & = \mathscr{U}_5(t,\mathscr{W}), \end{cases} \tag{3} $$

where

$$ \begin{cases} \mathscr{U}_1(t,\mathscr{H}) & = c - k\mathscr{H}\mathscr{V} - \lambda\mathscr{H}, \\ \mathscr{U}_2(t,\mathscr{I}) & = k\mathscr{H}\mathscr{V} - \eta\mathscr{I}, \\ \mathscr{U}_3(t,\mathscr{D}) & = a\mathscr{I} - (\beta+\delta)\mathscr{D}, \\ \mathscr{U}_4(t,\mathscr{V}) & = \beta\mathscr{D} - u\mathbb{V} - q\mathscr{V}\mathscr{W}, \\ \mathscr{U}_5(t,\mathscr{W}) & = g\mathscr{V}\mathscr{W} - h\mathscr{W}. \end{cases} \tag{4} $$

The suggested model may be expressed concisely as:

$$ \begin{cases} {}^{\mathbb{ABC}}\mathscr{D}_0^{\lambda}\mathrm{V}(t) & = \mathfrak{D}(t,\mathrm{V}(t)) \\ \mathrm{V}(0) & = \mathrm{V}_0 \geq 0, \end{cases} \tag{5} $$

where

$$ \begin{cases} \mathrm{V}(t) & = (\mathscr{H},\mathscr{I},\mathscr{D},\mathscr{V},\mathscr{W})^T \\ \mathrm{V}_0 & = (\omega_1,\omega_2,\omega_3,\omega_4,\omega_5)^T \\ \mathfrak{D}(t,\mathrm{V}(t)) & = (\mathscr{U}_1,\mathscr{U}_2,\mathscr{U}_3,\mathscr{U}_4,\mathscr{U}_5)^T. \end{cases} \tag{6} $$

With initial values and the $\mathbb{ABC}$ FI, we can construct the equivalent representation of (5) as

$$ \mathrm{V}(t) = \mathrm{V}_0 + \frac{(1-\lambda)}{\mathscr{G}(\lambda)}\mathfrak{D}(t,\mathrm{V}(t)) + \frac{\lambda}{\mathscr{G}(\lambda)}\frac{1}{\Gamma(\lambda)}\int_0^t (t-\varpi)^{\lambda-1}\mathfrak{D}(\varpi,\mathrm{V}(\varpi))\,d\varpi. \tag{7} $$

Let Banach space $\mathscr{B} = \mathfrak{C}\left(\mathbb{X},R^5\right)$ be a Banach space on $\mathbb{X} = [0,\mathbb{T}]$ under the following norm

$$ \|\mathrm{V}\| = \sup_{t\in\mathbb{X}}\{\mathrm{V}(t) : \mathrm{V} \in \mathscr{B}\} $$

Assume that $\forall\ \mathrm{V} \in \mathscr{B}$, and $\mathfrak{D}(t,\mathrm{V}(t))$ satisfies the assumptions provided below.

–$\exists\ \lambda_{\mathfrak{D}} > 0$ and $\rho_{\mathfrak{D}} > 0$ such that

$$ |\mathfrak{D}(t,\mathrm{V}(t))| \leq \lambda_{\mathfrak{D}}|\mathrm{V}| + \rho_{\mathfrak{D}}. \tag{8} $$

–$\exists\ L_{\mathfrak{D}} > 0$ such that

$$|\mathfrak{D}(t,\mathrm{V}_1(t)) - \mathfrak{D}(t,\mathrm{V}_2(t))| \le L_{\mathfrak{D}} |\mathrm{V}_1 - \mathrm{V}_2|. \tag{9}$$

Next two operators $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are defined as:

$$\mathfrak{P}_1\mathrm{V}(t) = \mathrm{V}_0 + \frac{(1-\lambda)}{\mathscr{G}(\lambda)}\mathfrak{D}(t,\mathrm{V}(t)),$$
$$\mathfrak{P}_2\mathrm{V}(t) = \frac{\lambda}{\mathscr{G}(\lambda)}\frac{1}{\Gamma(\lambda)}\int_0^t (t-\varpi)^{\lambda-1}\mathfrak{D}(\varpi,\mathrm{V}(\varpi))\,d\varpi,$$

where $\mathfrak{P}_1 + \mathfrak{P}_2 = \mathscr{B}$.

Theorem 1.*If the equations (8), and (9) holds along with the following conditions:*

1. $\frac{(1-\lambda)}{\mathscr{G}(\lambda)}L_{\mathfrak{D}} < 1.$
2. $\nabla_1 = \left[\frac{(1-\lambda)}{\mathscr{G}(\lambda)} + \frac{T^{\lambda}}{\mathscr{G}(\lambda)\Gamma(\lambda)}\right]\rho_{\mathfrak{D}} < 1.$
3. $\nabla_2 = \left\{\frac{(1-\lambda)}{\mathscr{G}(\lambda)} + \frac{T^{\lambda}}{\mathscr{G}(\lambda)\Gamma(\lambda)}\right\}\lambda_{\mathfrak{D}} < 1.$

Then the integral equation (7) has at least one solution.

*Proof.*Let $\mathfrak{B}_\tau = \{\mathrm{V} \in \mathscr{B} : \|\mathrm{V}\| \le \tau\}$ be a convex and closed set. Our first task is show that $\mathfrak{P}_1\mathrm{V}_1 + \mathfrak{P}_2\mathrm{V}_2 \in \mathfrak{B}_\tau$, for $\mathrm{V}_1, \mathrm{V}_2 \in \mathfrak{B}_\tau$. For this use Eq. (8), we have

$$\begin{aligned}
\|\mathfrak{P}_1\mathrm{V}_1 + \mathfrak{P}_2\mathrm{V}_2\| &\le \sup_{t\in[0,\mathbb{T}]}\left\{|\mathrm{V}_0| + \frac{(1-\lambda)}{\mathscr{G}(\lambda)}|\mathfrak{D}(t,\mathrm{V}(t))| + \frac{\lambda}{\mathscr{G}(\lambda)}\frac{1}{\Gamma(\lambda)}\int_0^t (t-\varpi)^{\lambda-1}|\mathfrak{D}(\varpi,\mathrm{V}(\varpi))|\,d\varpi\right\}\\
&\le \left\{|\mathrm{V}_0| + \frac{(1-\lambda)}{\mathscr{G}(\lambda)}(\lambda_{\mathfrak{D}}\|\mathrm{V}\| + \rho_{\mathfrak{D}}) + \frac{\lambda}{\mathscr{G}(\lambda)}\frac{1}{\Gamma(\lambda)}\int_0^t (t-\varpi)^{\lambda-1}(\lambda_{\mathfrak{D}}\|\mathrm{V}\| + \rho_{\mathfrak{D}})\,d\varpi\right\}\\
&= |\mathrm{V}_0| + \left\{\frac{(1-\lambda)}{\mathscr{G}(\lambda)} + \frac{T^{\lambda}}{\mathscr{G}(\lambda)\Gamma(\lambda)}\right\}\rho_{\mathfrak{D}} + \left\{\frac{(1-\lambda)}{\mathscr{G}(\lambda)} + \frac{T^{\lambda}}{\mathscr{G}(\lambda)\Gamma(\lambda)}\right\}\lambda_{\mathfrak{D}}\tau\\
&= \nabla_1 + \nabla_2\tau \le \tau.
\end{aligned}$$

This shows that $\mathfrak{P}_1\mathrm{V}_1 + \mathfrak{P}_2\mathrm{V}_2 \in \mathfrak{B}_\tau$. Now, we prove that $\mathfrak{P}_1$ is fulfills the criteria of contraction. $\forall\ \mathrm{V}_1, \mathrm{V}_2 \in \mathfrak{B}_\tau$ and utilizing Lipschitz condition, one reach to:

$$\begin{aligned}
\|\mathfrak{P}_1\mathrm{V}_1 - \mathfrak{P}_1\mathrm{V}_2\| &= \sup_{t\in[0,\mathbb{T}]}\frac{(1-\lambda)}{\mathscr{G}(\lambda)}|\mathfrak{D}(t,\mathrm{V}_1(t)) - \mathfrak{D}(t,\mathrm{V}_2(t))|\\
&\le \frac{(1-\lambda)}{\mathscr{G}(\lambda)}L_{\mathfrak{D}}\sup_{t\in[0,\mathbb{T}]}|\mathrm{V}_1(t) - \mathrm{V}_2(t)|\\
&\le \frac{(1-\lambda)}{\mathscr{G}(\lambda)}L_{\mathfrak{D}}\|\mathrm{V}_1 - \mathrm{V}_2\|,
\end{aligned}$$

where $\frac{(1-\lambda)}{\mathscr{G}(\lambda)}L_{\mathfrak{D}} < 1$. Hence we proved that $\mathfrak{P}_1$ satisfy the contraction condition. Now we have to show that $\mathfrak{P}_2$ is relatively compact. For $\mathrm{V} \in \mathfrak{B}_\tau$, consider

$$\begin{aligned}
\|\mathfrak{P}_2\mathrm{V}\| &\le \sup_{t\in[0,\mathbb{T}]}\frac{\lambda}{\mathscr{G}(\lambda)}\frac{1}{\Gamma(\lambda)}\int_0^t (t-\varpi)^{\lambda-1}|\mathfrak{D}(\varpi,\mathrm{V}(\varpi))|\,d\varpi\\
&\le \frac{\lambda}{\mathscr{G}(\lambda)}\frac{1}{\Gamma(\lambda)}\int_0^t (t-\varpi)^{\lambda-1}\sup_{t\in[0,\mathbb{T}]}[\lambda_{\mathfrak{D}}|\mathrm{V}| + \rho_{\mathfrak{D}}]\,d\varpi\\
&\le \frac{\lambda}{\mathscr{G}(\lambda)}\frac{1}{\Gamma(\lambda)}\int_0^t (t-\varpi)^{\lambda-1}[\lambda_{\mathfrak{D}}\|\mathrm{V}\| + \rho_{\mathfrak{D}}]\,d\varpi\\
&\le \frac{T^{\lambda}}{\mathscr{G}(\lambda)\Gamma(\lambda)}[\lambda_{\mathfrak{D}}\tau + \rho_{\mathfrak{D}}].
\end{aligned}$$

Hence, we have showed the uniform boundedness of $\mathfrak{P}_2$ on $\mathfrak{B}_\tau$. Finally, we need to derive that $\mathfrak{P}_2$ is equicontinuous. Let $\mathrm{V} \in \mathfrak{B}_\tau$ and $t_1, t_2 \in [0, \mathbb{T}]$ such that $t_2 > t_1$, then

$$\begin{aligned}
\|\mathfrak{P}_2\mathrm{V}(t_1) - \mathfrak{P}_2\mathrm{V}(t_2)\| &\leq \frac{\lambda}{\mathscr{G}(\lambda)} \frac{1}{\Gamma(\lambda)} \int_{t_1}^{t_2} (t_2 - \varpi)^{\lambda-1} |\mathfrak{D}(\varpi, \mathrm{V}(\varpi))| \, d\varpi \\
&\quad + \frac{\lambda}{\mathscr{G}(\lambda)} \frac{1}{\Gamma(\lambda)} \int_0^{t_2} \left[(t_1 - \varpi)^{\lambda-1} - (t_2 - \varpi)^{\lambda-1}\right] |\mathfrak{D}(\varpi, \mathrm{V}(\varpi))| \, d\varpi \\
&\leq \frac{2[\lambda_{\mathfrak{D}} \tau + \rho_{\mathfrak{D}}]}{\mathscr{G}(\lambda)\Gamma(\lambda)} \left[(t_2 - t_1)^\lambda\right].
\end{aligned}$$

Thus $\|\mathfrak{P}_2\mathrm{V}(t_1) - \mathfrak{P}_2\mathrm{V}(t_2)\| \to 0$ as $t_2 \to t_1$. As a result, according to the Arzela-Ascoli outcome, the function $\mathfrak{P}_2$ is relatively compact, indicating that it is equicontinuous. As a result, the equation (7) has at least one solution. As a result, there is at least one solution to the subsequent model,(12).

Theorem 2.*If the condition (9) and*

$$\left\{\frac{(1-\lambda)}{\mathscr{G}(\lambda)} + \frac{T^\lambda}{\mathscr{G}(\lambda)\Gamma(\lambda)}\right\} L_{\mathfrak{D}} < 1.$$

Then the HBV model (7) has one and only one solution.

*Proof.*Let $\mathrm{V}, \mathrm{V}^* \in \mathscr{B}$ and consider

$$\begin{aligned}
\|\mathfrak{P}\mathrm{V}(t) - \mathfrak{P}\mathrm{V}^*(t)\| &\leq \max_{t\in[0,\mathbb{T}]} \frac{(1-\lambda)}{\mathscr{G}(\lambda)} |\mathfrak{D}(t, \mathrm{V}(t)) - \mathfrak{D}(t, \mathrm{V}^*(t))| \\
&\quad + \max_{t\in[0,\mathbb{T}]} \frac{\lambda}{\mathscr{G}(\lambda)\Gamma(\lambda)} \int_0^t (t-\varpi)^{\lambda-1} |\mathfrak{D}(t, \mathrm{V}(t)) - \mathfrak{D}(t, \mathrm{V}^*(t))| \, d\varpi \\
&\leq \left(\frac{(1-\lambda)}{\mathscr{G}(\lambda)} + \frac{T^\lambda}{\mathscr{G}(\lambda)\Gamma(\lambda)}\right) L_{\mathfrak{D}} \|\mathrm{V} - \mathrm{V}^*\|.
\end{aligned}$$

Since, $\left\{\frac{(1-\lambda)}{\mathscr{G}(\lambda)} + \frac{T^\lambda}{\mathscr{G}(\lambda)\Gamma(\lambda)}\right\} L_{\mathfrak{D}} < 1$. The contraction requirement is met by the $\mathfrak{P}$. Thanks to the Banach fixed point principle, the integral equation Eq. (7) has a single solution. CThis leads to an unique solution for the system (2).

2.2 *Hyres-Ulam (HU) stability*

The HU-stability of (5) will be discussed in this part..

Definition 4.*The proposed model (2) is HU-stable if for any $\varepsilon > 0$ and $\tilde{\mathfrak{D}} \in \mathscr{B}$, the inequality is achieved as:*

$$\left| {}^{\mathbb{ABC}}\mathscr{D}_0^\lambda \tilde{\mathfrak{D}}(t) - \mathfrak{D}\left(t, \tilde{\mathfrak{D}}(t)\right)\right| \leq \varepsilon. \tag{10}$$

Then $\exists \mathrm{V} \in \mathscr{B}$ which is a solution of the model (2) with $\mathrm{V}(0) = \tilde{\mathfrak{D}}(0) = \tilde{\mathfrak{D}}_0$, such that $\left\|\tilde{\mathfrak{D}} - \mathrm{V}\right\| \leq \lambda\varepsilon$, where $\lambda > 0$ and

$$\begin{cases}
\tilde{\mathfrak{D}}(t) & = \left(\widetilde{\mathscr{H}}, \widetilde{\mathscr{I}}, \widetilde{\mathscr{D}}, \widetilde{\mathscr{V}}, \widetilde{\mathscr{W}}\right)^T \\
\tilde{\mathfrak{D}}_0 & = \left(\widetilde{\omega_1}, \widetilde{\omega_2}, \widetilde{\omega_3}, \widetilde{\omega_4}, \widetilde{\omega_5}\right)^T \\
\mathfrak{D}\left(t, \tilde{\mathfrak{D}}(t)\right) & = \left(\widetilde{\mathscr{U}_1}, \widetilde{\mathscr{U}_2}, \widetilde{\mathscr{U}_3}, \widetilde{\mathscr{U}_4}, \widetilde{\mathscr{U}_5}\right)^T \\
\varepsilon & = \max(\varepsilon_i)^T, \quad i = 1, 2, \cdots, 5 \\
\lambda & = \max(\lambda_i)^T, \quad i = 1, 2, \cdots, 5.
\end{cases} \tag{11}$$

*Remark.*Let's consider a little perturbation $h \in C[0, \mathbb{T}]$ so that the following criteria are satisfied:

1. $|h(t)| \leq \tilde{\varepsilon}$ for $t \in [0, \mathbb{T}]$ and $\tilde{\varepsilon} > 0$,

2. For $t \in [0, \mathbb{T}]$, one achieve
$$ {}^{\mathbb{ABC}}\mathscr{D}_0^{\lambda} \tilde{\mathfrak{D}}(t) = \mathfrak{D}\left(t, \tilde{\mathfrak{D}}(t)\right) + h(t), $$
where $h(t) = (h_1(t), h_2(t), h_3(t), h_4(t), h_5(t))^T$.

Lemma 1. *The solution $\tilde{\mathfrak{D}}_h(t)$ of the equation:*
$$ \begin{cases} {}^{\mathbb{ABC}}\mathscr{D}_0^{\lambda} \tilde{\mathfrak{D}}(t) & = \mathfrak{D}\left(t, \tilde{\mathfrak{D}}(t)\right) + h(t) \\ \mathrm{V}(0) & = \tilde{\mathfrak{D}}_0 \end{cases} \tag{12} $$
satisfies:
$$ \left|\tilde{\mathfrak{D}}_h(t) - \tilde{\mathfrak{D}}(t)\right| \leq \left[\frac{(1-\lambda)\Gamma(\lambda) + T^{\lambda}}{\mathscr{G}(\lambda)\Gamma(\lambda)}\right] \tilde{\varepsilon}. \tag{13} $$

Proof. Equivalently, one can express the solution of (12) as follows:
$$ \tilde{\mathfrak{D}}_h(t) = \begin{cases} \tilde{\mathfrak{D}}_0 + \frac{(1-\lambda)}{\mathscr{G}(\lambda)} \mathfrak{D}\left(t, \tilde{\mathfrak{D}}(t)\right) + \frac{\lambda}{\mathscr{G}(\lambda)\Gamma(\lambda)} \int_0^t (t-\varpi)^{\lambda-1} \mathfrak{D}\left(\varpi, \tilde{\mathfrak{D}}(\varpi)\right) d\varpi \\ + \frac{(1-\lambda)}{\mathscr{G}(\lambda)} h(t) + \frac{\lambda}{\mathscr{G}(\lambda)\Gamma(\lambda)} \int_0^t (t-\varpi)^{\lambda-1} h(\varpi) d\varpi. \end{cases} \tag{14} $$
Also
$$ \tilde{\mathfrak{D}}(t) = \tilde{\mathfrak{D}}_0 + \frac{(1-\lambda)}{\mathscr{G}(\lambda)} \mathfrak{D}\left(t, \tilde{\mathfrak{D}}(t)\right) + \frac{\lambda}{\mathscr{G}(\lambda)} \frac{1}{\Gamma(\lambda)} \int_0^t (t-\varpi)^{\lambda-1} \mathfrak{D}\left(\varpi, \tilde{\mathfrak{D}}(\varpi)\right) d\varpi. \tag{15} $$
With the help of remark (i), one achieve
$$ \begin{aligned} \left|\tilde{\mathfrak{D}}_h(t) - \tilde{\mathfrak{D}}(t)\right| &\leq \frac{(1-\lambda)}{\mathscr{G}(\lambda)} |h(t)| + \frac{\lambda}{\mathscr{G}(\lambda)} \frac{1}{\Gamma(\lambda)} \int_0^t (t-\varpi)^{\lambda-1} |h(\varpi)| d\varpi \\ &\leq \left[\frac{(1-\lambda)\Gamma(\lambda) + T^{\lambda}}{\mathscr{G}(\lambda)\Gamma(\lambda)}\right] \tilde{\varepsilon}. \end{aligned} $$
Thus, the expression $\tilde{\mathfrak{D}}h(t)$ fulfils the requirement (13).

Theorem 3. *Assuming the Theorem 2 presumptions are true. The HBV model (2) is then HU-stable.*

Proof. Let $\tilde{\mathfrak{D}} \in \mathscr{B}$ satisfies the model (10) and $\mathfrak{H} \in \mathscr{B}$ be a unique solution of model (2) with $\mathrm{V}(0) = \tilde{\mathfrak{D}}_0$.
$$ \mathrm{V}(t) = \tilde{\mathfrak{D}}_0 + \frac{(1-\lambda)}{\mathscr{G}(\lambda)} \mathfrak{D}(t, \mathrm{V}(t)) + \frac{\lambda}{\mathscr{G}(\lambda)} \frac{1}{\Gamma(\lambda)} \int_0^t (t-\varpi)^{\lambda-1} \mathfrak{D}(\varpi, \mathrm{V}(\varpi)) d\varpi. $$
Using above Lemma, we have
$$ \begin{aligned} \left|\tilde{\mathfrak{D}}(t) - \mathrm{V}(t)\right| &\leq \left|\tilde{\mathfrak{D}}_h(t) - \tilde{\mathfrak{D}}(t)\right| + \left|\tilde{\mathfrak{D}}_h(t) - \mathrm{V}(t)\right| \\ &\leq 2\left[\frac{(1-\lambda)\Gamma(\lambda) + T^{\lambda}}{\mathscr{G}(\lambda)\Gamma(\lambda)}\right] \tilde{\varepsilon} + \frac{1-\lambda}{\mathscr{G}(\lambda)} \left|\mathfrak{D}\left(t, \tilde{\mathfrak{D}}(t)\right) - \mathfrak{D}(t, \mathrm{V}(t))\right| \\ &\quad + \frac{\lambda}{\mathscr{G}(\lambda)} \frac{1}{\Gamma(\lambda)} \int_0^t (t-\varpi)^{\lambda-1} \left|\mathfrak{D}\left(t, \tilde{\mathfrak{D}}(t)\right) - \mathfrak{D}(t, \mathrm{V}(t))\right| d\varpi \\ &\leq 2\left[\frac{(1-\lambda)\Gamma(\lambda) + T^{\lambda}}{\mathscr{G}(\lambda)\Gamma(\lambda)}\right] \tilde{\varepsilon} + \left(\frac{(1-\lambda)\Gamma(\lambda) + T^{\lambda}}{\mathscr{G}(\lambda)\Gamma(\lambda)}\right) L_{\mathfrak{D}} \left\|\tilde{\mathfrak{D}} - \mathrm{V}\right\|, \end{aligned} $$
which means that
$$ \left\|\tilde{\mathfrak{D}} - \mathrm{V}\right\| \leq \frac{2\varsigma\tilde{\varepsilon}}{1-\Psi}, \tag{16} $$
where
$$ \begin{cases} \varsigma = & \frac{(1-\lambda)\Gamma(\lambda) + T^{\lambda}}{\mathscr{G}(\lambda)\Gamma(\lambda)}, \\ \Psi = & \left(\frac{(1-\lambda)\Gamma(\lambda) + T^{\lambda}}{\mathscr{G}(\lambda)\Gamma(\lambda)}\right) L_{\mathfrak{D}}. \end{cases} \tag{17} $$
Eq. (16) becomes $\left\|\tilde{\mathfrak{D}} - \mathrm{V}\right\| \leq \lambda\tilde{\varepsilon}$ when $\lambda = \frac{2\varsigma}{1-\Psi}$. Thus, the theorem.

3 Conclusion

In this letter, we explored qualitatively the HBV infection with an antibody immune response under the Mittag-Leffler derivative. We explored the existence and uniqueness results by utilizing the Banach contraction theorem and Krasnoselskii fixed point theorem. Also, we showed that the system's solution is HU-stable under the Mittag-Leffler operator.

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