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The Precontinuity Property via $\mathscr{G}_{\mathscr{N}}-$Preopen Sets in Grill Topological Spaces

Amin Saif[1] *and Maged Abdulwahab Al-Muntaser*[2,*],

[1]Department of Mathematics, Faculty of Applied Sciences, Taiz University, Taiz, Yemen
[2]Department of Mathematics,Faculty of Education, University of Saba Region, Marib, Yemen

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Abstract: This paper introduces and investigates the notions of precontinuity property, contra and almost continuity via $\mathscr{G}_{\mathscr{N}}-$preopen sets. They will be introduced in grill topological spaces by using the class of $\mathscr{G}_{\mathscr{N}}-$preopen sets. Their relationships with the other known classes of continuous maps will be studied, too.

Keywords: Preopen sets, precontinuous, grill topological Spaces.

1 Introduction

The idea of grill on a topological space, given by Choquet [1], goes as follows: A non-null collection $\mathscr{G}$ of subsets of a topological spaces X is said to be a *grill* on X if
(i) $A \in \mathscr{G}$ and $A \subseteq B \Longrightarrow B \in \mathscr{G}$
(ii) $A, B \subseteq X$ and $A \cup B \in \mathscr{G} \Longrightarrow A \in \mathscr{G}$ or $B \in \mathscr{G}$.

For a topological space X, the operator $\Phi : P(X) \to P(X)$ from the power set $P(X)$ of X to $P(X)$ was first defined in [2] in terms of grill; the latter concept being defined by Choquet [1] several decades back. Interestingly, it is found from subsequent investigations that the notion of grills as an appliance like nets and filters, turns out to be extremely useful towards the study of certain specific topological problems (see for instance [3], [4] and [5]). For a grill $\mathscr{G}$ on a topological space X, an operator from the power set $P(X)$ of X to $P(X)$ was defined in [6] in the following manner : For any $A \in P(X)$,

$$\Phi(A) = \{x \in X : U \cap A \in \mathscr{G},$$

for each open neighborhood U of $x\}$.

Then the operator $\Psi : P(X) \to P(X)$, given by $\Psi(A) = A \cup \Phi(A)$, for $A \in P(X)$, was also shown in [6] to be a Kuratowski closure operator, defining a unique topology $\tau_{\mathscr{G}}$ on X such that $\tau \subseteq \tau_{\mathscr{G}}$. If (X, τ) is a topological space and $\mathscr{G}$ is a grill on X, then the triple $(X, \tau, \mathscr{G})$ will be called a *grill topological space*.

In 1982, Mashhour [7] introduced the notion of a precontinuous function. In 2009, Al-Omari and Noiri [8] introduced the notions of $\mathscr{N}-$precontinuous function. In 2002, Jafari and Noiri [9] introduced the notions of contra precontinuous function. In 2004, Ekici [10] introduced the notions of almost contra-precontinuous function.

Under the notion of grill topological space and its operators above, several authors defined and investigated the notions in this part. In 2010, Hatir and Jafari [11] introduced the notions of $\mathscr{G}-$precontinuous function.

This paper is organized as follows. In Section 2, we introduce the concept of $\mathscr{G}_{\mathscr{N}}-$precontinuous functions. Furthermore, the relationship with the other known continuous functions will be studied. In Section 3, we introduce the notion of contra-almost precontinuous functions.

For a topological space (X, τ) and $A \subseteq X$, throughout this paper, we mean $Cl(A)$ and $Int(A)$ the closure set and the interior set of A, respectively. A subset A of a topological space X is called a preopen set, [7] if $A \subseteq Int(Cl(A))$. The complement of preopen set is called preclosed set. A subset A of topological space (X, τ) is called a $\mathscr{N}-$preopen set, [8] if for each $x \in A$, there exists a preopen set U_x containing x such that $U_x - A$ is a finite set.

* Corresponding author e-mail: majedabuaceamh@gmail.com

The complement of $\mathscr{N}-$preopen set is called $\mathscr{N}-$preclosed set. A subset A of a grill topological space $(X,\tau,\mathscr{G})$ is called a $\mathscr{G}-$preopen set, [11] if $A \subseteq Int(\Psi(A))$. The complement of $\mathscr{G}-$preopen set is called $\mathscr{G}-$preclosed set.

Definition 11 [7]. A function $f:(X,\tau)\to(Y,\rho)$ of a topological space (X,τ) into a topological space (Y,ρ) is called *precontinuous function* if $f^{-1}(U)$ is a preopen set in X for every open set U in Y.

Definition 12 [8]. A function $f:(X,\tau)\to(Y,\rho)$ is called $\mathscr{N}-$precontinuous function if $f^{-1}(U)$ is a $\mathscr{N}-$preopen set in X for every open set U in Y.

Definition 13 [9]. A function $f:(X,\tau)\to(Y,\rho)$ is called contra-precontinuous function if $f^{-1}(V)$ is a preclosed set in X for every open set V in Y.

Definition 14 [10]. A function $f:(X,\tau)\to(Y,\rho)$ is called almost contra-precontinuous function if $f^{-1}(V)$ is a preclosed set in X for every $r-$open set V in Y.

Definition 15 [11]. A function $f:(X,\tau,\mathscr{G})\to(Y,\rho)$ of a grill topological space $(X,\tau,\mathscr{G})$ into a space (Y,ρ) is called $\mathscr{G}-$precontinuous function if $f^{-1}(U)$ is $\mathscr{G}-$preopen set in $(X,\tau,\mathscr{G})$ for every open set U in Y.

Theorem 16[11]. Every $\mathscr{G}-$precontinuous function is a precontinuous function.

Definition 17 [12]. A function $f:(X,\tau)\to(Y,\rho)$ of a topological space (X,τ) into a space (Y,ρ) is called contra-continuous function, if $f^{-1}(V)$ is a closed set in X for every open set V in Y .

Definition 18 [13]. A function $f:(X,\tau)\to(Y,\rho)$ of a topological space (X,τ) into a space (Y,ρ) is called almost continuous function, if for each $x\in X$ and each neighborhood U of $f(x)$ in Y , there exists a neighborhood V of x in X such that $f(V)\subseteq Int[Cl(U)]$.

Theorem 19 [9]. Every contra-continuous function is a contra-precontinuous.

Theorem 110 [10]. Every contra-precontinuous function is a almost contra-precontinuous.

Theorem 111 [14]. A topological space (X,τ) is regular space if and only if for each $x\in X$ and for each open set V in X containing x, there exists an open set W in X containing x such that $Cl(W)\subseteq V$.

Definition 112 [15]. A function $f:(X,\tau)\to(Y,\rho)$ of a topological space (X,τ) into a space (Y,ρ) is called slightly continuous function if for each $x\in X$ and each clopen set U in Y containing $f(x)$, there exists an open set V in X containing x such that $f(V)\subseteq U$.

Definition 113 [16]. A function $f:(X,\tau)\to(Y,\rho)$ of a topological space (X,τ) into a space (Y,ρ) is called slightly precontinuous function if for each $x\in X$ and each clopen set U in Y containing $f(x)$, there exists preopen set V in X containing x such that $f(V)\subseteq U$.

2 $\mathscr{G}_{\mathscr{N}}-$Precontinuous functions

Definition 21 A function $f:(X,\tau,\mathscr{G})\to(Y,\rho)$ of a grill topological space $(X,\tau,\mathscr{G})$ into a space (Y,ρ) is called $\mathscr{G}_{\mathscr{N}}-$*precontinuous* if $f^{-1}(U)$ is a $\mathscr{G}_{\mathscr{N}}-$preopen set in $(X,\tau,\mathscr{G})$ for every open set U in Y.

Theorem 22 A function $f:(X,\tau,\mathscr{G})\to(Y,\rho)$ of a grill topological space $(X,\tau,\mathscr{G})$ into a space (Y,ρ) is $\mathscr{G}_{\mathscr{N}}-$precontinuous if and only if $f^{-1}(F)$ is a $\mathscr{G}_{\mathscr{N}}-$preclosed set in $(X,\tau,\mathscr{G})$ for every closed set F in Y.

*Proof.*Let $f:(X,\tau,\mathscr{G})\to(Y,\rho)$ be a $\mathscr{G}_{\mathscr{N}}-$precontinuous and F be any closed set in Y. Then $f^{-1}(Y-F)=X-f^{-1}(F)$ is a $\mathscr{G}_{\mathscr{N}}-$preopen set in $(X,\tau,\mathscr{G})$, that is, $f^{-1}(F)$ is $\mathscr{G}_{\mathscr{N}}-$preclosed set in $(X,\tau,\mathscr{G})$.
Conversely, suppose that $f^{-1}(F)$ is a $\mathscr{G}_{\mathscr{N}}-$preclosed set in $(X,\tau,\mathscr{G})$ for every closed set F in Y. Let U be any open set in Y. Then by the hypothesis, $f^{-1}(Y-U)=X-f^{-1}(U)$ is a $\mathscr{G}_{\mathscr{N}}-$preclosed set in $(X,\tau,\mathscr{G})$, that is, $f^{-1}(U)$ is a $\mathscr{G}_{\mathscr{N}}-$preopen set in $(X,\tau,\mathscr{G})$. Hence f is a $\mathscr{G}_{\mathscr{N}}-$precontinuous.

Theorem 23 A function $f:(X,\tau,\mathscr{G})\to(Y,\rho)$ is a $\mathscr{G}_{\mathscr{N}}-$precontinuous function if and only if for each $x\in X$ and each open set U in Y with $f(x)\in U$, there exists a $\mathscr{G}_{\mathscr{N}}-$preopen set V in $(X,\tau,\mathscr{G})$ such that $x\in V$ and $f(V)\subseteq U$.

*Proof.*Suppose function f is a $\mathscr{G}_{\mathscr{N}}-$precontinuous function. Let $x\in X$ and U be any open set in Y containing $f(x)$. Put $V=f^{-1}(U)$. Since f is a $\mathscr{G}_{\mathscr{N}}-$precontinuous, then V is a $\mathscr{G}_{\mathscr{N}}-$preopen set in $(X,\tau,\mathscr{G})$ such that $x\in V$ and $f(V)\subseteq U$.
Conversely, let U be any open set in Y and $x\in f^{-1}(U)$. Then $f(x)\in U$ and hence by the hypothesis, there exists a $\mathscr{G}_{\mathscr{N}}-$preopen set V in $(X,\tau,\mathscr{G})$ such that $x\in V$ and $f(V)\subseteq U$. Hence $x\in V\subseteq f^{-1}(U)$ and $f^{-1}(U)$ is a $\mathscr{G}_{\mathscr{N}}-$preopen set in $(X,\tau,\mathscr{G})$. That is, f is a $\mathscr{G}_{\mathscr{N}}-$precontinuous.

Theorem 24 Every $\mathscr{G}-$precontinuous function is $\mathscr{G}_{\mathscr{N}}-$precontinuous function.

*Proof.*Let $f:(X,\tau,\mathscr{G})\to(Y,\rho)$ be a $\mathscr{G}-$precontinuous function and U be any open set in Y. Then $f^{-1}(U)$ is a $\mathscr{G}-$preopen set in $(X,\tau,\mathscr{G})$, hence $f^{-1}(U)$ is a $\mathscr{G}_{\mathscr{N}}-$preopen set in $(X,\tau,\mathscr{G})$. That is, f is a $\mathscr{G}_{\mathscr{N}}-$precontinuous function.

The converse of last theorem need not be true

Example 25 Let $f:(X,\tau,\mathscr{G})\to(Y,\rho)$ be a function defined by $f(a)=f(b)=1$ and $f(c)=2$ where $X=\{a,b,c\}$, $Y=\{1,2\}$, $\mathscr{G}=P(X)-\{\emptyset\}$.

$$\tau=\{\emptyset,X,\{a\},\{a,b\}\} \text{ and } \rho=\{\emptyset,Y,\{2\}\}.$$

The function f is a $\mathscr{G}_{\mathscr{N}}-$precontinuous, since $f^{-1}(\{2\})=\{c\}$ and $f^{-1}(Y)=X$ are $\mathscr{G}_{\mathscr{N}}-$preopen sets in $(X,\tau,\mathscr{G})$. The function f is not $\mathscr{G}-$precontinuous, since $f^{-1}(\{2\})=\{c\}$ is not $\mathscr{G}-$preopen set in $(X,\tau,\mathscr{G})$.

Theorem 26 Every $\mathscr{G}_{\mathscr{N}}-$precontinuous function is $\mathscr{N}-$precontinuous function.

*Proof.*Let $f:(X,\tau,\mathscr{G})\to(Y,\rho)$ be a $\mathscr{G}_{\mathscr{N}}-$precontinuous function and U be any open set in Y. Then $f^{-1}(U)$ is a $\mathscr{G}_{\mathscr{N}}-$preopen set in $(X,\tau,\mathscr{G})$, hence $f^{-1}(U)$ is a $\mathscr{N}-$preopen set in $(X,\tau,\mathscr{G})$. That is, f is a $\mathscr{N}-$precontinuous function.

The converse of above theorem need not be true

Example 27 Let $f:(\mathbb{R},\tau,\mathscr{G})\to(Y,\rho)$ be a function defined by

$$f(x)=\begin{cases}2, & x=2\\ 1, & x\neq 2\end{cases}$$

where $Y=\{1,2\}$, $\mathscr{G}=\{\mathbb{R}\}$

$$T=\{\emptyset,\mathbb{R}\}\text{ and }\rho=\{\emptyset,Y,\{2\}\}.$$

The function f is a $\mathscr{N}-$precontinuous, since $f^{-1}(\{2\})=\{2\}$ and $f^{-1}(Y)=\mathbb{R}$ are $\mathscr{N}-$preopen set in $(\mathbb{R},\tau,\mathscr{G})$. The function f is not a $\mathscr{G}_{\mathscr{N}}-$precontinuous, since $f^{-1}(\{2\})=\{2\}$ is not $\mathscr{G}_{\mathscr{N}}-$preopen set in $(\mathbb{R},\tau,\mathscr{G})$.

Let $(X,\tau,\mathscr{G})$ be a grill topological space and $A\subseteq X$. The $\mathscr{G}_{\mathscr{N}}-$closure set of A is defined as the intersection of all $\mathscr{G}_{\mathscr{N}}-$preclosed subsets of $(X,\tau,\mathscr{G})$ containing A and is denoted by ${}_{\mathscr{G}\mathscr{N}}Cl(A)$. The $\mathscr{G}_{\mathscr{N}}-$interior set of A is defined as the union of all $\mathscr{G}_{\mathscr{N}}-$preopen subsets of $(X,\tau,\mathscr{G})$ contained in A and is denoted by ${}_{\mathscr{G}\mathscr{N}}Int(A)$.

Theorem 28 Let $f:(X,\tau,\mathscr{G})\to(Y,\rho)$ be a function of a space $(X,\tau,\mathscr{G})$ into a space (Y,ρ). Then f is a $\mathscr{G}_{\mathscr{N}}-$precontinuous if and only if $f[{}_{\mathscr{G}\mathscr{N}}Cl(A)]\subseteq Cl(f(A))$ for all $A\subseteq X$.

*Proof.*Let f be a $\mathscr{G}_{\mathscr{N}}-$precontinuous and A be any subset of X. Then $Cl(f(A))$ is a closed set in Y. Since f is a $\mathscr{G}_{\mathscr{N}}-$precontinuous, then by Theorem (22), $f^{-1}[Cl(f(A))]$ is a $\mathscr{G}_{\mathscr{N}}-$preclosed set in $(X,\tau,\mathscr{G})$. That is,

$${}_{\mathscr{G}\mathscr{N}}Cl[f^{-1}[Cl(f(A))]]=f^{-1}[Cl(f(A))].$$

Since $f(A)\subseteq Cl(f(A))$, then $A\subseteq f^{-1}[Cl(f(A))]$. This implies,

$${}_{\mathscr{G}\mathscr{N}}Cl(A)\subseteq{}_{\mathscr{G}\mathscr{N}}Cl[f^{-1}[Cl(f(A))]]=f^{-1}[Cl(f(A))].$$

Hence $f[{}_{\mathscr{G}\mathscr{N}}Cl(A)]\subseteq Cl(f(A))$.
Conversely, let H be any closed set in Y, then $Cl(H)=H$. Since $f^{-1}(H)\subseteq X$. Then by the hypothesis,

$$f[{}_{\mathscr{G}\mathscr{N}}Cl[f^{-1}(H)]]\subseteq Cl[f(f^{-1}(H))]\subseteq Cl(H)=H.$$

This implies, ${}_{\mathscr{G}\mathscr{N}}Cl[f^{-1}(H)]\subseteq f^{-1}(H)$. Hence ${}_{\mathscr{G}\mathscr{N}}Cl[f^{-1}(H)]=f^{-1}(H)$, that is, $f^{-1}(H)$ is a $\mathscr{G}_{\mathscr{N}}-$preclosed set in $(X,\tau,\mathscr{G})$. Therefore f is a $\mathscr{G}_{\mathscr{N}}-$precontinuous.

Theorem 29 Let $f:(X,\tau,\mathscr{G})\to(Y,\rho)$ be a function of a space $(X,\tau,\mathscr{G})$ into a space (Y,ρ). Then f is $\mathscr{G}_{\mathscr{N}}-$precontinuous if and only if ${}_{\mathscr{G}\mathscr{N}}Cl(f^{-1}(B))\subseteq f^{-1}(Cl(B))$ for all $B\subseteq Y$.

*Proof.*Let f be a $\mathscr{G}_{\mathscr{N}}-$precontinuous and B be any subset of Y. Then $Cl(B)$ is a closed set in Y. Since f is a $\mathscr{G}_{\mathscr{N}}-$precontinuous, then by Theorem(22), $f^{-1}[Cl(B)]$ is a $\mathscr{G}_{\mathscr{N}}-$preclosed set in $(X,\tau,\mathscr{G})$. That is,

$${}_{\mathscr{G}\mathscr{N}}Cl[f^{-1}[Cl(B)]]=f^{-1}[Cl(B)].$$

Since $B\subseteq Cl(B)$ then $f^{-1}(B)\subseteq f^{-1}[Cl(B)]$. This implies,

$${}_{\mathscr{G}\mathscr{N}}Cl(f^{-1}(B))\subseteq{}_{\mathscr{G}\mathscr{N}}Cl[f^{-1}[Cl(B)]]=f^{-1}[Cl(B)].$$

Hence ${}_{\mathscr{G}\mathscr{N}}Cl(f^{-1}(B))\subseteq f^{-1}[Cl(B)]$.
Conversely, let H be any closed set in Y, then $Cl(H)=H$. Since $H\subseteq Y$. Then by the hypothesis,

$${}_{\mathscr{G}\mathscr{N}}Cl(f^{-1}(H))\subseteq f^{-1}(Cl(H))=f^{-1}(H).$$

This implies, ${}_{\mathscr{G}\mathscr{N}}Cl[f^{-1}(H)]\subseteq f^{-1}(H)$. Hence ${}_{\mathscr{G}\mathscr{N}}Cl[f^{-1}(H)]=f^{-1}(H)$, that is, $f^{-1}(H)$ is a $\mathscr{G}_{\mathscr{N}}-$preclosed set in $(X,\tau,\mathscr{G})$. Hence f is a $\mathscr{G}_{\mathscr{N}}-$precontinuous.

Theorem 210 Let $f:(X,\tau,\mathscr{G})\to(Y,\rho)$ be a function of a space $(X,\tau,\mathscr{G})$ into a space (Y,ρ). Then f is $\mathscr{G}_{\mathscr{N}}-$precontinuous if and if $f^{-1}(Int(B))\subseteq{}_{\mathscr{G}\mathscr{N}}Int[f^{-1}(B)]$ for all $B\subseteq Y$.

*Proof.*Let f be a $\mathscr{G}_{\mathscr{N}}-$precontinuous and B be any subset of Y. Then $Int(B)$ is an open set in Y. Since f is a $\mathscr{G}_{\mathscr{N}}-$precontinuous, then $f^{-1}[Int(B)]$ is a $\mathscr{G}_{\mathscr{N}}-$preopen set in $(X,\tau,\mathscr{G})$. That is,

$${}_{\mathscr{G}\mathscr{N}}Int[f^{-1}[Int(B)]]=f^{-1}[Int(B)].$$

Since $Int(B)\subseteq B$ then $f^{-1}[Int(B)]\subseteq f^{-1}(B)$. This implies,

$$f^{-1}[Int(B)]={}_{\mathscr{G}\mathscr{N}}Int[f^{-1}[Int(B)]]\subseteq{}_{\mathscr{G}\mathscr{N}}Int(f^{-1}(B)).$$

Hence $f^{-1}(Int(B))\subseteq{}_{\mathscr{G}\mathscr{N}}Int[f^{-1}(B)]$.
Conversely, let U be any open set in Y, then $Int(U)=U$. Since $U\subseteq Y$. Then by the hypothesis,

$$f^{-1}(U)=f^{-1}(Int(U))\subseteq{}_{\mathscr{G}\mathscr{N}}Int[f^{-1}(U)].$$

This implies, $f^{-1}(U)\subseteq{}_{\mathscr{G}\mathscr{N}}Int[f^{-1}(U)]$. Hence $f^{-1}(U)={}_{\mathscr{G}\mathscr{N}}Int[f^{-1}(U)]$, that is, $f^{-1}(U)$ is a $\mathscr{G}_{\mathscr{N}}-$preopen set in $(X,\tau,\mathscr{G})$. Hence f is $\mathscr{G}_{\mathscr{N}}-$precontinuous.

Definition 211 A function $f:(X,\tau,\tau')\to(Y,\rho)$ of a bitopological space (X,τ,τ') into a space (Y,ρ) is called *$B_{2\rho}-$precontinuous function* if $f^{-1}(U)$ is $(\tau\tau')_{\mathscr{N}}-$preopen set in (X,τ,τ') for every open set U in Y.

Theorem 212 A function $f:(X,\tau,\mathscr{G})\rightarrow(Y,\rho)$ of a grill topological space $(X,\tau,\mathscr{G})$ into a space (Y,ρ) is $\mathscr{G}_{\mathscr{N}}-$precontinuous if and only if it is a $B_{2\rho}-$precontinuous function of bitopological space $(X,\tau,\tau_{\mathscr{G}})$ into (Y,ρ).

*Proof.*It is clear from the definitions.

Theorem 213 If $f:(X,\tau,\mathscr{G})\rightarrow(Y,\rho)$ is a $\mathscr{G}_{\mathscr{N}}-$precontinuous function and A is an open subspace of a grill topological space $(X,\tau,\mathscr{G})$, then the restriction function $f|_A:(A,\tau|_A,\tau_{\mathscr{G}}|_A)\rightarrow(Y,\rho)$ of f on A is a $B_{2\rho}-$precontinuous function of bitopological space $(A,\tau|_A,\tau_{\mathscr{G}}|_A)$ into (Y,ρ).

*Proof.*Let U be an open set in Y. since f is a $\mathscr{G}_{\mathscr{N}}-$precontinuous, then $f^{-1}(U)$ is a $\mathscr{G}_{\mathscr{N}}-$preopen set in $(X,\tau,\mathscr{G})$. Since A is an open in $(X,\tau,\mathscr{G})$ then, $f^{-1}(U)\cap A=(f|_A)^{-1}(U)$ is a $\mathscr{G}_{\mathscr{N}}-$preopen set in $(X,\tau,\mathscr{G})$. Then $(f|_A)^{-1}(U)\cap A=(f|_A)^{-1}(U)$ is a $\mathscr{G}_{\mathscr{N}}-$preopen set in A. Hence

$$(f|_A)^{-1}(U)\cap A=(f|_A)^{-1}(U)$$

is a $(\tau|_A\tau_{\mathscr{G}}|_A)_{\mathscr{N}}-$preopen set in $(A,\tau|_A,\tau_{\mathscr{G}}|_A)$. That is, $f|_A$ is a $B_{2\rho}-$precontinuous.

Theorem 214 Let $(X,\tau,\mathscr{G})$ be a grill topological space and X be the union of two open subsets A and B. Then the function $f:(X,\tau,\mathscr{G})\rightarrow(Y,\rho)$ is a $\mathscr{G}_{\mathscr{N}}-$precontinuous if the restriction functions $f|_A:(A,\tau|_A,\tau_{\mathscr{G}}|_A)\rightarrow(Y,\rho)$ and $f|_B:(B,\tau|_B,\tau_{\mathscr{G}}|_B)\rightarrow(Y,\rho)$ are $B_{2\rho}-$precontinuous.

*Proof.*Let U be any open set in Y . Since $f|_A$ is $B_{2\rho}-$precontinuous, then $(f|_A)^{-1}(U)$ is a $(\tau|_A\tau_{\mathscr{G}}|_A)_{\mathscr{N}}-$preopen set in $(A,\tau|_A,\tau_{\mathscr{G}}|_A)$. Since A is an open set in X, then $(f|_A)^{-1}(U)\subseteq A$ is a $\mathscr{G}_{\mathscr{N}}-$preopen set in $(X,\tau,\mathscr{G})$. Similarly, $(f|_B)^{-1}(U)$ is a $\mathscr{G}_{\mathscr{N}}-$preopen set in $(X,\tau,\mathscr{G})$. Hence

$$(f|_A)^{-1}(U)\cup(f|_B)^{-1}(U)=f^{-1}(U)$$

is a $\mathscr{G}_{\mathscr{N}}-$preopen set in $(X,\tau,\mathscr{G})$. That is, f is a $\mathscr{G}_{\mathscr{N}}-$precontinuous.

Definition 215 A function $f:(X,\tau,\mathscr{G})\rightarrow(Y,\rho,\mathscr{G}')$ of a grill topological space $(X,\tau,\mathscr{G})$ into a grill topological space $(Y,\rho,\mathscr{G}')$ is called *$\mathscr{G}_{\mathscr{N}}-$preirresolute function* if $f^{-1}(U)$ is a $\mathscr{G}_{\mathscr{N}}-$preopen set in $(X,\tau,\mathscr{G})$ for every $\mathscr{G}_{\mathscr{N}}-$preopen set U in $(Y,\rho,\mathscr{G}')$.

Theorem 216 A function $f:(X,\tau,\mathscr{G})\rightarrow(Y,\rho,\mathscr{G}')$ of a grill topological space $(X,\tau,\mathscr{G})$ into a grill topological space $(Y,\rho,\mathscr{G}')$ is $\mathscr{G}_{\mathscr{N}}-$preirresolute if and only if $f^{-1}(R)$ is a $\mathscr{G}_{\mathscr{N}}-$preclosed set in $(X,\tau,\mathscr{G})$ for every $\mathscr{G}_{\mathscr{N}}-$preclosed set R in $(Y,\rho,\mathscr{G}')$.

*Proof.*Let $f:(X,\tau,\mathscr{G})\rightarrow(Y,\rho,\mathscr{G}')$ be a $\mathscr{G}_{\mathscr{N}}-$preirresolute and F be any $\mathscr{G}_{\mathscr{N}}-$preclosed set in $(Y,\rho,\mathscr{G}')$. Then $f^{-1}(Y-F)=X-f^{-1}(F)$ is a $\mathscr{G}_{\mathscr{N}}-$preopen set in $(X,\tau,\mathscr{G})$, that is, $f^{-1}(F)$ is $\mathscr{G}_{\mathscr{N}}-$preclosed set in $(X,\tau,\mathscr{G})$.
Conversely, suppose that $f^{-1}(F)$ is a $\mathscr{G}_{\mathscr{N}}-$preclosed set in $(X,\tau,\mathscr{G})$ for every $\mathscr{G}_{\mathscr{N}}-$preclosed set F in $(Y,\rho,\mathscr{G}')$. Let U be any $\mathscr{G}_{\mathscr{N}}-$preopen set in $(Y,\rho,\mathscr{G}')$. Then by the hypothesis, $f^{-1}(Y-U)=X-f^{-1}(U)$ is a $\mathscr{G}_{\mathscr{N}}-$preclosed set in $(X,\tau,\mathscr{G})$, that is, $f^{-1}(U)$ is a $\mathscr{G}_{\mathscr{N}}-$preopen set in $(X,\tau,\mathscr{G})$. Hence f is a $\mathscr{G}_{\mathscr{N}}-$preirresolute.

Theorem 217 A function $hof:(X,\tau,\mathscr{G})\rightarrow(Z,\gamma)$ is a $\mathscr{G}_{\mathscr{N}}-$precontinuous if $f:(X,\tau,\mathscr{G})\rightarrow(Y,\rho,\mathscr{G}')$ is $\mathscr{G}_{\mathscr{N}}-$preirresolute and $h:(X,\rho,\mathscr{G}')\rightarrow(Z,\gamma)$ is a $\mathscr{G}_{\mathscr{N}}-$precontinuous.

*Proof.*Let U be any open set in (Z,γ). Since h is a $\mathscr{G}_{\mathscr{N}}-$precontinuous, then $h^{-1}(U)$ is a $\mathscr{G}_{\mathscr{N}}-$preopen set in $(Y,\rho,\mathscr{G}')$. Since f is $\mathscr{G}_{\mathscr{N}}-$preirresolute, then $f^{-1}[h^{-1}(U)]=(hof)^{-1}(U)$ is $\mathscr{G}_{\mathscr{N}}-$preopen set in $(X,\tau,\mathscr{G})$. That is, hof is a $\mathscr{G}_{\mathscr{N}}-$precontinuous.

Theorem 218 A function $hof:(X,\tau,\mathscr{G})\rightarrow(Z,\gamma)$ is a $\mathscr{G}_{\mathscr{N}}-$precontinuous if $f:(X,\tau,\mathscr{G})\rightarrow(Y,\rho)$ is $\mathscr{G}_{\mathscr{N}}-$precontinuous and $h:(Y,\rho)\rightarrow(Z,\gamma)$ is a continuous.

*Proof.*Let U be any open set in (Z,γ). Since h is a continuous, then $h^{-1}(U)$ is open set in (Y,ρ). Since f is $\mathscr{G}_{\mathscr{N}}-$precontinuous, then $f^{-1}[h^{-1}(U)]=(hof)^{-1}(U)$ is $\mathscr{G}_{\mathscr{N}}-$preopen set in $(X,\tau,\mathscr{G})$. That is, hof is a $\mathscr{G}_{\mathscr{N}}-$precontinuous.

3 Contra and almost $\mathscr{G}_{\mathscr{N}}-$Precontinuous functions

Definition 31 A function $f:(X,\tau,\mathscr{G})\rightarrow(Y,\rho)$ of a grill topological space $(X,\tau,\mathscr{G})$ into a space (Y,ρ) is called *contra $\mathscr{G}_{\mathscr{N}}-$precontinuous function* if $f^{-1}(V)$ is a $\mathscr{G}_{\mathscr{N}}-$preclosed set in $(X,\tau,\mathscr{G})$ for every open set V in Y.

Theorem 32 A function $f:(X,\tau,\mathscr{G})\rightarrow(Y,\rho)$ is contra $\mathscr{G}_{\mathscr{N}}-$precontinuous if and only if $f^{-1}(F)$ is a $\mathscr{G}_{\mathscr{N}}-$preopen set in $(X,\tau,\mathscr{G})$ for every closed set F in Y.

*Proof.*Suppose that f is contra $\mathscr{G}_{\mathscr{N}}-$precontinuous. Let G be any closed set in Y. Then $Y-G$ is an open set in Y. Since f is contra $\mathscr{G}_{\mathscr{N}}-$precontinuous, then $X-f^{-1}(G)=f^{-1}(Y-G)$ is a $\mathscr{G}_{\mathscr{N}}-$preclosed set in $(X,\tau,\mathscr{G})$. That is, $f^{-1}(G)$ is a $\mathscr{G}_{\mathscr{N}}-$preopen set in $(X,\tau,\mathscr{G})$.
Conversely, let G be any open set in Y, then $Y-G$ is an closed set in Y . Then by the hypothesis, $f^{-1}(Y-G)=X-f^{-1}(G)$ is a $\mathscr{G}_{\mathscr{N}}-$preopen set in $(X,\tau,\mathscr{G})$. That is, $f^{-1}(G)$ is a $\mathscr{G}_{\mathscr{N}}-$preclosed set in $(X,\tau,\mathscr{G})$. Hence f is contra $\mathscr{G}_{\mathscr{N}}-$precontinuous.

Theorem 33 A function $f:(X,\tau,\mathscr{G})\rightarrow(Y,\rho)$ is contra $\mathscr{G}_{\mathscr{N}}-$precontinuous if and only if for each $x\in X$ and each closed set G in Y containing $f(x)$, there is a $\mathscr{G}_{\mathscr{N}}-$preopen set U in $(X,\tau,\mathscr{G})$ containing x such that $f(U)\subseteq G$.

*Proof.*Suppose that f is contra $\mathscr{G}_{\mathscr{N}}-$precontinuous. Let $x \in X$ and G be a closed set in Y containing $f(x)$. Then by the last theorem, $U = f^{-1}(G)$ is a $\mathscr{G}_{\mathscr{N}}-$preopen set in $(X,\tau,\mathscr{G})$. Since $f(x) \in G$, then $x \in f^{-1}(G) = U$ and $f(U) = f(f^{-1}(G)) \subseteq G$.
Conversely, let G be a closed set in Y. For each $x \in f^{-1}(G)$, $f(x) \in G$. Then by the hypothesis, there is a $\mathscr{G}_{\mathscr{N}}-$preopen set U_x in $(X,\tau,\mathscr{G})$ containing x such that $f(U_x) \subseteq G$. Therefore, we obtain

$$f^{-1}(G) = \cup\{U_x : x \in f^{-1}(G)\}.$$

Then $f^{-1}(G)$ is a $\mathscr{G}_{\mathscr{N}}-$preopen set in $(X,\tau,\mathscr{G})$. Hence by the last theorem, f is a contra $\mathscr{G}_{\mathscr{N}}-$precontinuous.

Theorem 34 The set of all points x in X at which a function $f : (X,\tau,\mathscr{G}) \to (Y,\rho)$ is not a contra $\mathscr{G}_{\mathscr{N}}-$precontinuous, it is identical with the union of the $\mathscr{G}_{\mathscr{N}}-$frontier of the inverse images of closed sets of Y containing $f(x)$.

*Proof.*Suppose that f is not contra $\mathscr{G}_{\mathscr{N}}-$precontinuous at $x \in X$. Then by Theorem (33), there is a closed set G in Y containing $f(x)$ such that $f(U) \nsubseteq G$ for all $\mathscr{G}_{\mathscr{N}}-$preopen set U in $(X,\tau,\mathscr{G})$ containing x. That is, for all $\mathscr{G}_{\mathscr{N}}-$preopen set U in $(X,\tau,\mathscr{G})$ containing x, $f(U) \cap (Y-G) \neq \emptyset$ and this implies

$$U \cap f^{-1}(Y-G) \neq \emptyset.$$

Therefore, we have

$$x \in {}_{\mathscr{G}_{\mathscr{N}}}Cl[f^{-1}(Y-G)] = {}_{\mathscr{G}_{\mathscr{N}}}Cl[X - f^{-1}(G)].$$

However, since $f(x) \in G$, then

$$x \in f^{-1}(G) \subseteq {}_{\mathscr{G}_{\mathscr{N}}}Cl[f^{-1}(G)].$$

Then

$$x \in {}_{\mathscr{G}_{\mathscr{N}}}Cl[X - f^{-1}(G)] \cap {}_{\mathscr{G}_{\mathscr{N}}}Cl[f^{-1}(G)] = {}_{\mathscr{G}_{\mathscr{N}}}\Gamma(f^{-1}(G)).$$

Conversely, suppose that $x \in X$ and $x \in {}_{\mathscr{G}_{\mathscr{N}}}\Gamma(f^{-1}(G))$ for some closed sets G in Y containing $f(x)$. If f is a contra $\mathscr{G}_{\mathscr{N}}-$precontinuous at x, then there is $\mathscr{G}_{\mathscr{N}}-$preopen set U in $(X,\tau,\mathscr{G})$. containing x such that $f(U) \subseteq G$. Therefore, we have $x \in U \subseteq f^{-1}(G)$. That is,

$$x \in {}_{\mathscr{G}_{\mathscr{N}}}Int[f^{-1}(G)] \subseteq {}_{\mathscr{G}_{\mathscr{N}}}\Gamma(f^{-1}(G)).$$

This is a contradiction. Hence by Theorem (33), f is not contra $\mathscr{G}_{\mathscr{N}}-$precontinuous at x.

Theorem 35 If a function $f : (X,\tau,\mathscr{G}) \to (Y,\rho)$ is contra $\mathscr{G}_{\mathscr{N}}-$precontinuous and Y is a regular space, then f is a $\mathscr{G}_{\mathscr{N}}-$precontinuous.

*Proof.*Let $x \in X$ be an arbitrary point in X and V be any open subset of Y containing $f(x)$. Since Y is a regular space, then there is an open set W in Y containing $f(x)$ such that $Cl(W) \subseteq V$. Since $Cl(W)$ is a closed in Y and f is a contra $\mathscr{G}_{\mathscr{N}}-$precontinuous, then there is a $\mathscr{G}_{\mathscr{N}}-$preopen set U in $(X,\tau,\mathscr{G})$ containing x such that $f(U) \subseteq V$. Hence by Theorem (22), f is a $\mathscr{G}_{\mathscr{N}}-$precontinuous.

A subset A of a grill topological space $(X,\tau,\mathscr{G})$ is called $r-$open set if $A = Int(Cl(A))$. The complement of $r-$open set is called $r-$closed set. A subset of a grill topological space is called *a $\mathscr{G}_{\mathscr{N}}-$preclopen set* if it is both $\mathscr{G}_{\mathscr{N}}-$preopen and $\mathscr{G}_{\mathscr{N}}-$preclosed set.

Definition 36 A function $f : (X,\tau,\mathscr{G}) \to (Y,\rho)$ of a grill topological space $(X,\tau,\mathscr{G})$ into a space (Y,ρ) is called:

1. *almost $\mathscr{G}_{\mathscr{N}}-$precontinuous* if for each $x \in X$ and each open set V in Y containing $f(x)$, there is a $\mathscr{G}_{\mathscr{N}}-$preopen set U in $(X,\tau,\mathscr{G})$ containing x such that $f(U) \subseteq Int[Cl(V)]$.

2. *almost contra $\mathscr{G}_{\mathscr{N}}-$precontinuous function* if $f^{-1}(V)$ is a $\mathscr{G}_{\mathscr{N}}-$preclosed set in $(X,\tau,\mathscr{G})$ for every $r-$open set V in Y.

3. *weakly $\mathscr{G}_{\mathscr{N}}-$precontinuous function*, if for each $x \in X$ and each open set V in Y containing $f(x)$, there is a $\mathscr{G}_{\mathscr{N}}-$preopen set U in $(X,\tau,\mathscr{G})$ containing x such that $f(U) \subseteq Cl(V)$.

4. *slightly $\mathscr{G}_{\mathscr{N}}-$precontinuous function* if for each $x \in X$ and each clopen set V in Y containing $f(x)$, there exists $\mathscr{G}_{\mathscr{N}}-$preopen set U in $(X,\tau,\mathscr{G})$ containing x such that $f(U) \subseteq V$.

It is clear that every a $\mathscr{G}_{\mathscr{N}}-$precontinuous function is almost $\mathscr{G}_{\mathscr{N}}-$precontinuous and every almost $\mathscr{G}_{\mathscr{N}}-$precontinuous is weakly $\mathscr{G}_{\mathscr{N}}-$precontinuous.

Theorem 37 A function $f : (X,\tau,\mathscr{G}) \to (Y,\rho)$ is almost contra $\mathscr{G}_{\mathscr{N}}-$precontinuous if and only if $f^{-1}(F)$ is a $\mathscr{G}_{\mathscr{N}}-$preopen set in $(X,\tau,\mathscr{G})$ for every $r-$closed set F in Y.

*Proof.*Similar for the proof of Theorem(32).

Theorem 38 A function $f : (X,\tau,\mathscr{G}) \to (Y,\rho)$ is almost contra $\mathscr{G}_{\mathscr{N}}-$precontinuous if and only if for each $x \in X$ and each $r-$closed set F in Y containing $f(x)$, there is a $\mathscr{G}_{\mathscr{N}}-$preopen set U in $(X,\tau,\mathscr{G})$ containing x such that $f(U) \subseteq F$.

*Proof.*Suppose that f is almost contra $\mathscr{G}_{\mathscr{N}}-$precontinuous. Let $x \in X$ and F be a $r-$closed set in Y containing $f(x)$. Then by the last theorem, $U = f^{-1}(F)$ is a $\mathscr{G}_{\mathscr{N}}-$preopen set in $(X,\tau,\mathscr{G})$. Since $f(x) \in F$, then $x \in f^{-1}(F) = U$ and $f(U) = f(f^{-1}(F)) \subseteq F$.
Conversely, let F be a $r-$closed set in Y. For each $x \in f^{-1}(F)$, $f(x) \in F$. Then by the hypothesis, there is a $\mathscr{G}_{\mathscr{N}}-$preopen set U_x in $(X,\tau,\mathscr{G})$ containing x such that $f(U_x) \subseteq F$. Therefore, we obtain

$$f^{-1}(F) = \cup\{U_x : x \in f^{-1}(F)\}.$$

Then $f^{-1}(F)$ is a $\mathscr{G}_{\mathscr{N}}-$preopen set in $(X,\tau,\mathscr{G})$. Hence by the last theorem, f is an almost contra $\mathscr{G}_{\mathscr{N}}-$precontinuous.

It is clear that every contra $\mathscr{G}_{\mathscr{N}}-$precontinuous function is almost contra $\mathscr{G}_{\mathscr{N}}-$precontinuous, since every $r-$open set is open.

Theorem 39 Every almost contra $\mathscr{G}_{\mathscr{N}}-$precontinuous function is a weakly $\mathscr{G}_{\mathscr{N}}-$precontinuous.

*Proof.*Let $f:(X,\tau,\mathscr{G})\rightarrow(Y,\rho)$ be almost contra $\mathscr{G}_{\mathscr{N}}-$precontinuous. Let $x\in X$ be any point in $(X,\tau,\mathscr{G})$ and V be any open set in Y containing $f(x)$. Then

$$Cl(V)=Cl[Int(V)]\subseteq Cl[Int(Cl(V))]$$

and

$$Cl[Int(Cl(V))]\subseteq Cl[Cl(V)]=Cl(V),$$

this implies, $Cl(V)=Cl[Int(Cl(V))]$. That is, $Cl(V)$ is $r-$closed set in Y containing $f(x)$. Since f is almost contra $\mathscr{G}_{\mathscr{N}}-$precontinuous, then by Theorem (38), there is a $\mathscr{G}_{\mathscr{N}}-$preopen set U in $(X,\tau,\mathscr{G})$ containing x such that $f(U)\subseteq Cl(V)$. That is, f is a weakly $\mathscr{G}_{\mathscr{N}}-$precontinuous.

The converse of the last theorem need not be true.

Example 310 Let $f:(\mathbb{N},\tau,\mathscr{G})\rightarrow(Y,\rho)$ be a function defined by

$$f(n)=\begin{cases}3, & n=\{2,3\}\\ 1, & n\neq\{2,3\}\end{cases}$$

, where , $Y=\{1,2,3\}$,

$$\tau=\{\emptyset,\mathbb{N},\{3\}\},\mathscr{G}=\{\mathbb{N}\}\text{ and }\rho=\{\emptyset,Y,\{3\},\{1,2\}\}.$$

The function f is a weakly $\mathscr{G}_{\mathscr{N}}-$precontinuous, but it is not almost contra $\mathscr{G}_{\mathscr{N}}-$precontinuous, since $\{3\}$ is $r-$closed in Y, but $f^{-1}(\{3\})=\{2,3\}$ is not $\mathscr{G}_{\mathscr{N}}-$preopen set in $(X,\tau,\mathscr{G})$.

Theorem 311 A function $f:(X,\tau,\mathscr{G})\rightarrow(Y,\rho)$ is almost contra $\mathscr{G}_{\mathscr{N}}-$precontinuous if and only if for each $x\in X$ and each $r-$open set V in Y non-containing $f(x)$, there is a $\mathscr{G}_{\mathscr{N}}-$preclosed set U in $(X,\tau,\mathscr{G})$ non-containing x such that $f^{-1}(V)\subseteq U$.

*Proof.*Suppose that f is almost contra $\mathscr{G}_{\mathscr{N}}-$precontinuous. Let $x\in X$ and V be a $r-$open set in Y non-containing $f(x)$. Then $Y-V$ is a $r-$closed set in Y containing $f(x)$. Then by Theorem (38), there is a $\mathscr{G}_{\mathscr{N}}-$preopen set G in $(X,\tau,\mathscr{G})$ containing x such that $f(G)\subseteq Y-V$. That is, $U=X-G$ is a $\mathscr{G}_{\mathscr{N}}-$preclosed set in $(X,\tau,\mathscr{G})$ non-containing x and so

$$G\subseteq f^{-1}[Y-V]=X-f^{-1}(V).$$

Hence $f^{-1}(V)\subseteq X-G=U$.
Conversely, let $x\in X$ any point in X and F be any $r-$closed set in Y containing $f(x)$. $Y-F$ is $r-$open set in Y non-containing $f(x)$. Then by the hypothesis, there is a $\mathscr{G}_{\mathscr{N}}-$preclosed set G in $(X,\tau,\mathscr{G})$ non-containing x such that $f^{-1}(Y-F)\subseteq G$. Hence $U=X-G$ is a $\mathscr{G}_{\mathscr{N}}-$preopen set in $(X,\tau,\mathscr{G})$ containing x and

$$f(U)=f(X-G)\subseteq f[X-(f^{-1}(Y-F))]=f[f^{-1}(F)]\subseteq F.$$

Then by Theorem (38), f is an almost contra $\mathscr{G}_{\mathscr{N}}-$precontinuous.

A function $f:(X,\tau,\mathscr{G})\rightarrow(Y,\rho)$ of a grill topological space $(X,\tau,\mathscr{G})$ into a space (Y,ρ) is called *$\mathscr{G}_{\mathscr{N}}-$preopen function* if $f(V)$ is an open set in Y for every $\mathscr{G}_{\mathscr{N}}-$preopen set V in $(X,\tau,\mathscr{G})$.

Theorem 312 If a function $f:(X,\tau,\mathscr{G})\rightarrow(Y,\rho)$ is a $\mathscr{G}_{\mathscr{N}}-$preopen function and contra $\mathscr{G}_{\mathscr{N}}-$precontinuous, then f is an almost $\mathscr{G}_{\mathscr{N}}-$precontinuous.

*Proof.*Let $x\in X$ be any point in X and V be any open set in Y containing $f(x)$. Since f is a contra $\mathscr{G}_{\mathscr{N}}-$precontinuous and $Cl(V)$ is a closed set in Y containing $f(x)$, then by Theorem (33), there is a $\mathscr{G}_{\mathscr{N}}-$preopen set U in $(X,\tau,\mathscr{G})$ containing x such that $f(U)\subseteq Cl(V)$. Since f is a $\mathscr{G}_{\mathscr{N}}-$preopen and U is a $\mathscr{G}_{\mathscr{N}}-$preopen set in $(X,\tau,\mathscr{G})$, then $f(U)$ is open set in Y and

$$f(U)=Int[f(U)]\subseteq Int[Cl(f(U))]\subseteq Int[Cl(V)].$$

This shows that f is an almost $\mathscr{G}_{\mathscr{N}}-$precontinuous.

Theorem 313 Let (Y,ρ) be an extremally disconnected space. A function $f:(X,\tau,\mathscr{G})\rightarrow(Y,\rho)$ is almost contra $\mathscr{G}_{\mathscr{N}}-$precontinuous if and only if f is an almost $\mathscr{G}_{\mathscr{N}}-$precontinuous.

*Proof.*Let f be almost contra $\mathscr{G}_{\mathscr{N}}-$precontinuous. Let $x\in X$ be any point in X and V be any open set in Y containing $f(x)$. Then

$$Cl(V)=Cl[Int(V)]\subseteq Cl[Int(Cl(V))]$$

and

$$Cl[Int(Cl(V))]\subseteq Cl[(Cl(V)]=Cl(V),$$

this implies, $Cl(V)=Cl[Int(Cl(V))]$. That is, $Cl(V)$ is $r-$closed set in Y containing $f(x)$. Since f is a almost contra $\mathscr{G}_{\mathscr{N}}-$precontinuous, then by Theorem (38), there is a $\mathscr{G}_{\mathscr{N}}-$preopen set U in $(X,\tau,\mathscr{G})$ containing x such that $f(U)\subseteq Cl(V)$. Since Y is an extremally disconnected space, then

$$f(U)\subseteq Cl(V)=Int[Cl(V)].$$

This shows that f is an almost $\mathscr{G}_{\mathscr{N}}-$precontinuous.
Conversely, let f be almost $\mathscr{G}_{\mathscr{N}}-$precontinuous. Let $x\in X$ be any point in X and F be any $r-$closed set in Y containing $f(x)$. Since Y is an extremally disconnected space, then

$$Int(F)=Int[Cl(Int(F))]=Cl(Int(F))=F,$$

that is, F is open set in Y containing $f(x)$. Since f is a almost $\mathscr{G}_{\mathscr{N}}-$precontinuous, then there is a $\mathscr{G}_{\mathscr{N}}-$preopen set U in $(X,\tau,\mathscr{G})$ containing x such that $f(U)\subseteq Int[Cl(F)]$. Then

$$f(U)\subseteq Int[Cl(F)]=Cl(F)=F.$$

Then by Theorem (38), f is an almost contra $\mathscr{G}_{\mathscr{N}}-$precontinuous.

Theorem 314 A function $f : (X,\tau,\mathcal{G}) \to (Y,\rho)$ is a slightly $\mathcal{G}_{\mathcal{N}}-$precontinuous if and only if $f^{-1}(V)$ is a $\mathcal{G}_{\mathcal{N}}-$preopen set in $(X,\tau,\mathcal{G})$ for every clopen set V in Y.

*Proof.*Suppose that f is slightly $\mathcal{G}_{\mathcal{N}}-$precontinuous and let V be a clopen set in Y. For each $x \in f^{-1}(V)$, $f(x) \in V$. Since f is slightly $\mathcal{G}_{\mathcal{N}}-$precontinuous, then there exists $\mathcal{G}_{\mathcal{N}}-$preopen set U_x in $(X,\tau,\mathcal{G})$ containing x such that $f(U_x) \subseteq V$. This implies, $x \in U_x \subseteq f^{-1}(V)$. Hence

$$f^{-1}(V) = \cup\{U_x : x \in f^{-1}(V)\}.$$

That is, $f^{-1}(V)$ is a $\mathcal{G}_{\mathcal{N}}-$preopen set in $(X,\tau,\mathcal{G})$. Conversely, it is trivial.

Corollary 315 A function $f : (X,\tau,\mathcal{G}) \to (Y,\rho)$ is a slightly $\mathcal{G}_{\mathcal{N}}-$precontinuous if and only if $f^{-1}(V)$ is a $\mathcal{G}_{\mathcal{N}}-$preclosed set in $(X,\tau,\mathcal{G})$ for every clopen set V in Y.

Corollary 316 A function $f : (X,\tau,\mathcal{G}) \to (Y,\rho)$ is a slightly $\mathcal{G}_{\mathcal{N}}-$precontinuous if and only if $f^{-1}(V)$ is a $\mathcal{G}_{\mathcal{N}}-$preclopen set in $(X,\tau,\mathcal{G})$ for every clopen set V in Y.

It is clear that if f is a contra $\mathcal{G}_{\mathcal{N}}-$precontinuous function or $\mathcal{G}_{\mathcal{N}}-$precontinuous function, then f is a slightly $\mathcal{G}_{\mathcal{N}}-$precontinuous.

Theorem 317 Every weakly $\mathcal{G}_{\mathcal{N}}-$precontinuous is slightly $\mathcal{G}_{\mathcal{N}}-$precontinuous.

*Proof.*Let $f : (X,\tau,\mathcal{G}) \to (Y,\rho)$ be a weakly $\mathcal{G}_{\mathcal{N}}-$precontinuous function. Let $x \in X$ be any point in X and V be any clopen set in Y containing $f(x)$. Then V is an open set in Y containing $f(x)$. Then there is a $\mathcal{G}_{\mathcal{N}}-$preopen set U in $(X,\tau,\mathcal{G})$ containing x such that $f(U) \subseteq Cl(V) = V$. Hence f is slightly $\mathcal{G}_{\mathcal{N}}-$precontinuous.

4 Conclusion

The applications of $\mathcal{G}_{\mathcal{N}}-$Precontinuous functions and Contra and almost $\mathcal{G}_{\mathcal{N}}-$Precontinuous functions are well-known and important in the area of mathematics, computer science and other areas. The our notions in this article be development for the last notions in topological space into grill topological space by giving concept is a strong. And will play significant role in solving some mathematical problems.

Competing Interests

The authors have declared that no competing interests exist.

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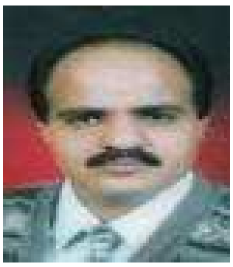
References

[1] G. Choquet; Sur les notions de filtre et grille, Comptes Rendus Acad. Sci. Paris, 224, 171-173, 1947.
[2] N. Levine; A decomposition of continuity in topological spaces, Amer Math. Monthly, 68, 36-41, 1961.
[3] K. C. Chattopadhyay, O. Njastad and W. J. Thron; Merotopic spaces and extensions of closure spaces, Canad. J. Math., 4, 613-629, 1983.
[4] K. C. Chattopadhyay and W. J. Thron; Extensions of closure spaces, Canad. J. Math., 6, 1277-1286, 1977.
[5] W. J. Thron; Proximity structures and grills, Math. Ann., 206, 35-62, 1973.
[6] B. Roy and M. N. Mukherjee; On a typical topology induced by a grill, Soochow J. Math., 33, 771-786, (2007).
[7] A. Mashhour, M. Abd EL-Monsef and S. ElDeep, On Pre-continuous and Weak Pre-continuous Mappings, Proc. Math. and Phys. Soc. Egypt, 53, 47-53, (1982).
[8] A. Al-Omari and T. Noiri, Characterizations of strongly compact spaces, Int. J. Math. and Math. Sciences, ID 573038, 1-9, (2009).
[9] S. Jafari and T. Noiri, On contra-precontinuous functions, Bull. Malaysian Math. Soc., 25, 115-128, (2002).
[10] E. Ekici, Almost Contra-precontinuous functions, Bull. Malaysian Math. Soc., 1 53-65, (2004).
[11] E. Hatir and S. Jafari; On some new classes of sets and a new decomposition of continuty via grills, J. Advs. Math. Studies., 3 33-40, (2010).
[12] J. Dontchev, Contra-continuous functions and strongly $S-$closed spaces, Int. J. Math. Math. Sci., 19, 303-310, 1996.
[13] M. Singal and A. Singal, Almost continuous mappings, Yokohama Math. J., 16, 63-73, 1968.
[14] J.Dugundji, Topology. New Jersey: Allyn and Bacon, 1966.
[15] R. Jain, The role of regularly open sets in general topology, Ph.D. thesis: Meerut University.
[16] C. Baker, On slightly precontinuous functions, Act. Math. Hunger, 94, 45- 52, (2002).
[17] A. Saif, M.Al-Hawmi and M. Abulwahab, On preopenness property in grill topological spaces, EPH.Int.J.Math, 6, 13-26, (2020).



Amin Saif is an Associate Professor of Mathematical Sciences at Department of Mathematics, Faculty of Applied Sciences, Taiz University, Taiz, Yemen. He received the PhD degree in Algebraic Topology at (UKM) in Malaysia. His main research interests are: General Topology, Algebraic Topology, Homotopy theory for topological semigroups. He has published research articles in reputed international journals of mathematical sciences.

Maged A. Al-Muntaser is a student at Department of Mathematics, Faculty of Education, Arts and Sciences, University of Saba Region, Marib. He has published some research articles in reputed international journals of mathematics.