Journal of Analysis & Number Theory
An International Journal

http://dx.doi.org/10.18576/jant/050107

Dirichlet-Power Series

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Received: 2 Sep. 2016, Revised: 18 Nov. 2016, Accepted: 23 Nov. 2016
Published online: 1 Jan. 2017

Abstract: In this paper we introduce and investigate a new type of series which we call a ”Dirichlet-power” series. This series includes both Dirichlet series and power series as a special case. It arose during the investigation of series representations of the second order hypergeometric zeta function. The Dirichlet-power series has many properties analogous to Dirichlet series. For a Dirichlet-power series we establish Domain of convergence and show the existence of zero free regions to the right half plane.

Keywords: Multiplicative cone b-metric space, fixed point, contractive mapping.

1 Introduction

The Riemann Zeta function $\zeta(s)$ defined by

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

with an integral representation

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx$$

is a special example of a type of series called Dirichlet series. A Dirichlet series is a series of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

with a_n complex number and s complex variable. For a Dirichlet series of this type there is an extended real number σ_a in the extended real line with the property that the series converges absolutely for $Re(s) > \sigma_a$, but not for $Re(s) < \sigma_a$. Moreover, for any positive number ε, the convergence is uniform for $Re(s) > \sigma_a + \varepsilon$, so that the series represents a holomorphic function for all $Re(s) > \sigma_a$. The quantity σ_a is called the abscissa of absolute convergence of the Dirichlet series; it is an analogue of the radius of convergence of a power series [8]. In fact, if we fix a prime p, and only allow a_n to be nonzero when n is a power of p, then we get an ordinary power series in p^{-s}. So in some sense, Dirichlet series are a strict generalization of an ordinary power series.

In this paper we introduce and investigate a new type of series which we call a ”Dirichlet-power series”. This series includes both Dirichlet series and power series as a special case. As in the case of Dirichlet series it has a kind of abscissa of absolute convergence is attached to it with a right half plane convergence. On the other hand as a power series it has a kind of radius of convergence with D-shaped region of convergence. This kind of series actually arose during the investigation of series representations of the second order hypergeometric zeta function. The hypergeometric zeta function of order N where N is a positive integer, defined as

$$\zeta_N(s) = \frac{1}{\Gamma(s+N-1)} \int_0^{\infty} \frac{x^{s+N-2}}{e^x - T_{N-1}(x)} dx$$

which admits a Dirichlet-type series given by,

$$\zeta_N(s) = \sum_{n=1}^{\infty} \frac{\mu_N(n,s)}{n^{s+N-1}}$$

for $\sigma > 1$, where

$$\mu_N(n,s) = \sum_{k=0}^{(N-1)(n-1)} \frac{a_k(N,n)\Gamma(s+N+k-1)}{n^k \Gamma(s+N-1)}$$

and $a_k(N,n)$ is generated by

$$(T_{N-1}(x))^{n-1} = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{N-1}}{(N-1)!}\right)^{n-1} = \sum_{k=0}^{(N-1)(n-1)} a_k(N,n) x^k$$

Originally introduced and investigated by Abdul Hassen and Heiu D. Nuygen as a generalization of the

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Riemann Zeta function, via the integral representation in several papers like [2] , [4], [5]. Recently we studied series representation of the second order hypergeometric zeta function in [6] which can be given as:

$$\zeta_2(s) = \sum_{n=1}^{\infty} D_m(s)s^{m-1}$$

where the coefficient $D_m(s)$ is a Dirichlet series for each $m = 1,2,3,\cdots$. This series representation of the second order hypergeometric zeta function is a source for the title of this paper.

The Dirichlet-power series has many properties analogous to Dirichlet series. For a Dirichlet-power series we have a D-shaped region of absolute convergence which can be extended to right half plane. In this extension there is a unique extended real number σ_a in the extended real line with the property that the series converges absolutely for $Re(s) > \sigma_a$, but not for $Re(s) < \sigma_a$ just analogous to Dirichlet series. In any compact subset of its absolute convergence the series represents an analytic function. Moreover; this series has a zero free regions in the right half-plane.

2 Preliminaries and Auxiliary Results

In this section we review some main known results on Dirichlet series which we need for our consumption, formally define a Dirichlet-power series and discuss on the domain of absolute convergence.

2.1 Review on Dirichlet Series

We summarize some of the properties of Dirichlet series as follows:

Theorem 21 *Suppose a Dirichlet series*

$$D(s) = \sum_{m=1}^{\infty} a_m m^{-s}$$

is absolutely convergent at a point $s_0 = \sigma_0 + it_0$. Then it is also absolutely convergent at all points s with $\sigma = Re(s) > \sigma_0$.

Theorem 22 (Uniqueness theorem for Dirichlet series)

Let

$$D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

and

$$E(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

where $D(s)$ and $E(s)$ are both absolutely convergent Dirichlet series in the half-plane $\sigma = Re(s) > \sigma_0$.

Suppose there exists an infinite sequence $\{s_k\}_{k=1}^{\infty}$ of complex numbers, with $\sigma_k = Re(s_k) > \sigma_0$ for all k and $\sigma_k \to \infty$ such that $D(s_k) = E(s_k)$ for all k . Then $D(s) = E(s)$ for all s with $\sigma = Re(s) > \sigma_0$.

Corollary 23 *Let*

$$D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

be a Dirichlet series with abscissa of absolute convergence σ_a . Suppose that for some s with $\sigma = Re(s) > \sigma_a$ we have $D(s) \neq 0$. Then there exists a half-plane in which the Diriclet series is absolutely convergent and never zero.

From the these properties of Dirichlet series it is known that the Domain of absolute convergence of a Dirichlet series is one of the following .

1. empty set, that is for no s does the series absolutely converges.
2. $(-\infty,\infty)$, that is the series converges for every complex number s.
3. (σ,∞) that is the series converges for every complex number s with $Re(s) > \sigma$
4. $[\sigma,\infty)$ that is the series converges for every complex number s with $Re(s) \geq \sigma$

Notice that in all cases, there is a unique number $\sigma_a \in [-\infty,\infty]$ such that for all s with $Re(s) > \sigma_a$, the Dirichlet series is absolutely convergent. And for all s with $Re(s) < \sigma_a$, the Dirichlet series is not absolutely convergent.

2.2 Definition and Examples of Dirichlet-Power Series

Definition 24 *Let $D_m(s)$ be a Dirichlet series for each $m = 1,2,\ldots$. We define a Dirichlet-Power series to be a series of the form*

$$\sum_{m=1}^{\infty} D_m(s)s^{m-1} \qquad (2.1)$$

Remark 25

1. *A Dirichlet-power series has a form of a power series but with a Dirichlet series as a coefficient.*
2. *If a Dirichlet-power series converges absolutely for $Re(s) > \sigma_0$, then $D_m(s)$ converges absolutely for each $m = 1,2,3,...$, for $Re(s) > \sigma_0$.*
3. *If $D_m(s)$ is identically zero for $m \geq 2$, then the Dirichlet-power series becomes a Dirichlet series.*
4. *If $D_m(s) = D_m$, a constant sequence, then the Dirichlet-power series becomes a power series.*

Proposition 26 *Suppose the function $F(s)$ is represented by a Dirichlet-power series, for $Re(s) > \sigma_a$, and suppose*

$$\sum_{m=2}^{\infty} D_m(s)s^{m-1} \to 0$$

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as $\sigma = Re(s) \to \infty$, independent of t. Then

$$F(s) \to a_{11}$$

as $\sigma = Re(s) \to \infty$

Proof. By the hypothesis of the proposition we have,

$$\sum_{m=2}^{\infty} D_m(s)s^{m-1} \to 0$$

as $\sigma = Re(s) \to \infty$, independent of t. Then we are left only with the first term of the Dirichlet-power series D_1. But D_1 is a Dirichlet series and hence by the property of Dirichlet series it converges to its first term which is a_{11} independent of t as $\sigma = Re(s) \to \infty$.

Some examples of a Dirichlet-power series are given here:

1.the second order Hypergeometric zeta function given by,

$$\zeta_2(s) = \sum_{n=1}^{\infty} D_m(s)s^{m-1}$$

where the coefficient $D_m(s)$ is a Dirichlet series for each $m = 1, 2, 3, \cdots$, where the $D_m(s)$ is given as in [6].

2.a series given by,

$$F(s) = \zeta(s)e^s$$

3.a series given by,

$$G(s) = \zeta(s)\cosh(s)$$

2.3 Domain of Absolute Convergence

In this section we establish domain of absolute convergence for a Dirichlet-power series and characterize its region of absolute convergence.

Theorem 27 *Suppose a Dirichlet-power series*

$$F(s) = \sum_{m=1}^{\infty} D_m(s)s^{m-1}$$

is absolutely convergent at a point $s_0 = \sigma_0 + it_0$. Then $F(s)$ converges absolutely at all points $s = \sigma + it$ with $\sigma > \sigma_0$ and $\sigma^2 + t^2 < \sigma_0^2 + t_0^2$.

Proof. From the hypothesis of the theorem, we have: $|s| \leq |s_0|$ and since $\sigma > \sigma_0$ we have also $|D_m(s)| < |D_m(s_0)|$. Thus we have,

$$|D_m(s)s^{m-1}| < |D_m(s_0)s_0^{m-1}|$$

Hence

$$\sum_{m=1}^{\infty} |D_m(s)s^{m-1}| < \sum_{m=1}^{\infty} |D_m(s_0)s_0^{m-1}| < \infty$$

Therefore, the Dirichlet-power series converges absolutely at all pointss $= \sigma + it$,with $\sigma = Re(s) > \sigma_0$ and $\sigma^2 + t^2 < \sigma_0^2 + t_0^2$.

Theorem 28 *Suppose the series*

$$\sum_{m=1}^{\infty} |D_m(s)s^{m-1}|$$

does not converge for all t or diverge for all t on a line with $Re(s) = \sigma_0$. Then there exists a real number t_0, such that the series

$$\sum_{m=1}^{\infty} |D_m(s)s^{m-1}|$$

converges for all t such that $0 \leq t \leq t_0$ and for any s with $\sigma > \sigma_0$ with the property $\sigma^2 + t^2 < \sigma_a^2 + t_a^2$, but does not converge otherwise.

Proof. Let $E = \{t/t \;\; is \;\; nonnegative \;\; and \sum_{m=1}^{\infty} |D_m(s)s^{m-1}| converges\}$ where $s = \sigma_0 + it$. Let t_0 be the supremum of t in E such that

$$\sum_{m=1}^{\infty} |D_m(s)s^{m-1}| converges\}$$

Such supremum exists since the series does not converge for all t.

Observe that from the theorem we have a segment $E = \{\sigma_0 + it / -t_0 \leq t \leq t_0\}$. Moreover, the series converges absolutely for all t with the property $\sigma = Re(s) > \sigma_0$ and $\sigma^2 + t^2 < \sigma_0^2 + t_0^2$. If the series converges for all t we define $t_0 = \infty$ and if it does not converge for any t we define $t_0 = -\infty$.

Theorem 29 *Suppose the series*

$$\sum_{m=1}^{\infty} |D_m(s)s^{m-1}|$$

does not converge for all σ or diverge for all σ. Then there exists a real number σ_a, called point of absolute convergence, such that the series

$$\sum_{m=1}^{\infty} |D_m(s)s^{m-1}|$$

converges if $\sigma > \sigma_a$ with the property $\sigma^2 + t^2 < \sigma_a^2 + t_a^2$, but does not converge otherwise.

Proof. Let $E = \{\sigma / \sum_{m=1}^{\infty} |D_m(s)s^{m-1}| diverges\}$ where $s = \sigma + it$. Then E is not empty as the series does not converge for all σ. E is bounded above because the series does not diverge for all σ. Therefore, E has a least upper bound call it σ_a. If $\sigma < \sigma_a$, then σ is in E, otherwise σ would be an upper bound for E smaller than the least upper bound. If $\sigma > \sigma_a$ with the property $\sigma^2 + t^2 < \sigma_a^2 + t_a^2$, then σ is not in E, since σ_a is an upper bound for E.

Observe that if we know the absolute convergence of the Dirichlet-power series at a point $s_0 = \sigma_0 + it_0$, then its domain of absolute convergence D is given as follows:

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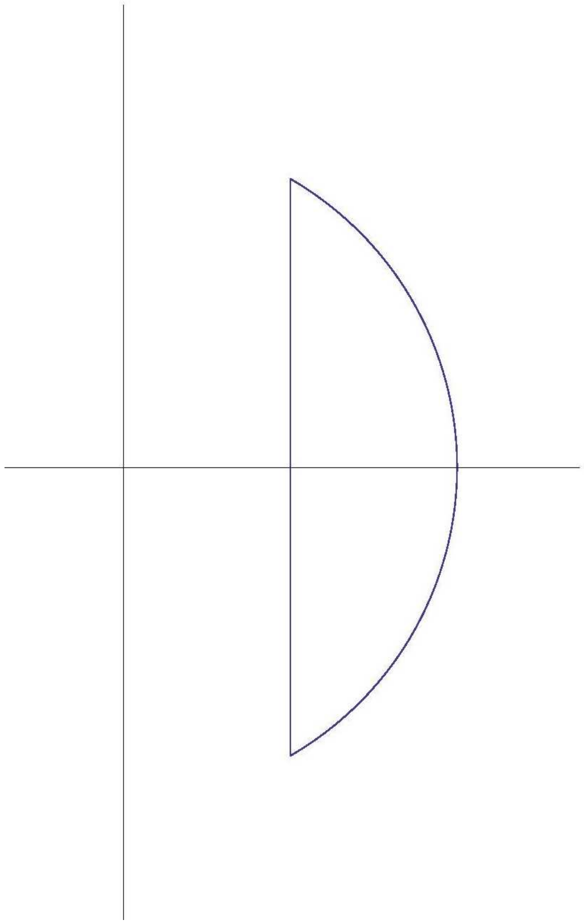

Fig. 1: Caption for D-Shape

$$D = \{s = \sigma + it/\sigma > \sigma_0, \sigma^2 + t^2 < \sigma_0^2 + t_0^2\}$$

It is a D-shaped region given as in the following figure:

So from this we can conclude that the domain of absolute convergence of a Dirichlet-power series is one of the following:

1. empty set, that is for no s does the series absolutely converge.
2. a point, that is the series converges for s_0 only, actually this is the case where it converges only on a point on a real line.
3. a D-shaped region, that is the series converges for every complex number $s = \sigma + it$ with $\sigma > \sigma_0$ and $\sigma^2 + t^2 < \sigma_0^2 + t_0^2$.

Now one naturally asks, whether this series has a right half plane as a region of absolute convergence analogous to Dirichlet series. This is the content of the next subsection.

3 Half-Plane of Absolute Convergence

In case of a Dirichlet series, we know that if a Dirichlet series converges absolutely at a point $s_0 = \sigma_0 + it_0$, then it converges absolutely on the whole line with $Re(s) = \sigma_0$ and $\sigma = Re(s) > \sigma_0$. But this is not generally true for a Dirichlet-power series due to the presence of the power s^{m-1} in the expression defining it. But if the Dirichlet-power series converges on a line with $Re(s) = \sigma_0$, then we have a right half plane of convergence analogous to the Dirichlet series. If this is the case then the segment of convergence becomes line of convergence and hence σ_a which is previously defined becomes abscissa of absolute convergence analogous to the Dirichlet series case.

Theorem 31 *Suppose a Dirichlet-power series*

$$F(s) = \sum_{m=1}^{\infty} D_m(s)s^{m-1}$$

is absolutely convergent at every point on a line with $Re(s) = \sigma_0$*. Then it is also absolutely convergent at all points* s *with* $\sigma = Re(s) > \sigma_0$.

Proof. For s with $\sigma = Re(s) > \sigma_0 = Re(s_0)$, we have

$$|D_m(s)| < |D_m(s_0)|$$

But if a complex number $s = \sigma + it$ with $\sigma = Re(s) > \sigma_0 = Re(s_0)$ is given, then we can find a real number t_0, so that $|s| \leq |s_0|$ where $s_0 = \sigma_0 + it_0$. Hence

$$|D_m(s)s^{m-1}| < |D_m(s_0)s_0^{m-1}|$$

Hence we have

$$\sum_{m=1}^{\infty} |D_m(s)s^{m-1}| < \sum_{m=1}^{\infty} |D_m(s_0)s_0^{m-1}| < \infty$$

Therefore, the Dirichlet-power series converges absolutely for $Re(s) > \sigma_0 = Re(s_0)$.

From the theorem it follows that the Domain of absolute convergence of a Dirichlet-power series is one of the following similar to that of a Dirichlet series once the Dirichlet-power series converges absolutely on the whole line $Re(s) = \sigma_0$.

1. empty set, that is for no s does the series absolutely converges.
2. $(-\infty, \infty)$, that is the series converges for every complex number s.
3. (σ, ∞) that is the series converges for every complex number s with $Re(s) > \sigma$
4. $[\sigma, \infty)$ that is the series converges for every complex number s with $Re(s) \geq \sigma$

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Notice that in all cases, there is a unique number $\sigma_a \in [-\infty,\infty]$ such that for all s with $Re(s) > \sigma_a$, the Dirichlet-power series is absolutely convergent. And for all s with $Re(s) < \sigma_a$, the Dirichlet-power series is not absolutely convergent. We can also have a right half plane of absolute convergence if the Dirichlet-power series as a power series has radius of convergence infinity. So the next question would be what are the conditions for which the series has such line of convergence. The D-shaped region of absolute convergence is due to the presence of the power of s^{m-1} as a power series in a Dirichlet-power series. If as a power series we have the radius of convergence to be infinity, then we have right half plane of absolute convergence. Here below we first recall a theorem on Double series and apply for the Dirichlet power series.

Theorem 32 *If the double series,*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|a_{nm}|$$

converges, then each of the following holds:

1.
$$\sum_{n=1}^{\infty} a_{nm}$$
converges absolutely for each m.

2.
$$\sum_{m=1}^{\infty} a_{nm}$$
converges absolutely for each n.

3.
$$\sum_{m=1}^{\infty}(\sum_{n=1}^{\infty} a_{nm})$$
converges absolutely.

4.
$$\sum_{m=1}^{\infty}(\sum_{n=1}^{\infty} a_{nm})$$
converges absolutely.

5.
$$\sum_{n,m=1}^{\infty} a_{nm}$$
converges absolutely; moreover, the last three sums in $3,4,5$ are all the same.

So if the a_{nm}'s are replaced by $\frac{a_{nm}s^{m-1}}{n^s}$, we have the Dirichlet-power series defined earlier, moreover the above theorem can be restated as,

Theorem 33 *If the double series,*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|\frac{a_{nm}s^{m-1}}{n^s}|$$

converges, then each of the following holds:

1.
$$\sum_{n=1}^{\infty}\frac{a_{nm}s^{m-1}}{n^s}$$
converges absolutely for each m.

2.
$$\sum_{m=1}^{\infty}\frac{a_{nm}s^{m-1}}{n^s}$$
converges absolutely for each n.

3.
$$\sum_{m=1}^{\infty}(\sum_{n=1}^{\infty}\frac{a_{nm}s^{m-1}}{n^s})$$
converges absolutely.

4.
$$\sum_{m=1}^{\infty}(\sum_{n=1}^{\infty}\frac{a_{nm}s^{m-1}}{n^s})$$
converges absolutely.

5.
$$\sum_{n,m=1}^{\infty}\frac{a_{nm}s^{m-1}}{n^s}$$
converges absolutely; moreover, the last three sums in $3,4,5$ are all the same.

Therefore, for a Dirichlet-power series

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_{nm}s^{m-1}}{n^s}$$

to have a right half plane of absolute convergence,

1.as a Dirichlet series

$$\sum_{n=1}^{\infty}\frac{a_{nm}s^{m-1}}{n^s}$$

converges absolutely for each m with some abscissa of absolute convergence σ_a.

2.as a power series

$$\sum_{m=1}^{\infty}\frac{a_{nm}s^{m-1}}{n^s}$$

converges absolutely for each n with radius convergence infinity.

Now in particular, if the coefficient $a_{nm} = a(n,m)$ as a function of two variables can be given as $a_{nm} = b_n c_m$ for each m and n. Then

$$F(s) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_{nm}s^{m-1}}{n^s} = \sum_{n=1}^{\infty}\frac{b_n}{n^s}\sum_{m=1}^{\infty} c_m s^{m-1}$$

Thus if $\sum_{n=1}^{\infty}\frac{b_n}{n^s}$ converges absolutely to $D(s)$ with abscissa of absolute convergence σ_a and $\sum_{m=1}^{\infty} c_m s^{m-1}$ represents an entire function $f(s)$ (that means as a power series it has

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radius of convergence infinity). Then the Dirichlet-power series

$$F(s)=\sum_{n=1}^{\infty}\frac{b_n}{n^s}\sum_{m=1}^{\infty}c_m s^{m-1}$$

converges absolutely to $D(s)f(s)$ for $s=\sigma+it$ with $\sigma>\sigma_a$.

Theorem 34 *Let $\sigma_0>0$ and suppose a Dirichlet-power series*

$$F(s)=\sum_{m=1}^{\infty}D_m(s)s^{m-1}$$

is absolutely convergent for some $\sigma>\sigma_0$. The Dirichlet-power series

$$F(s)=\sum_{m=1}^{\infty}D_m(s)s^{m-1}$$

converges absolutely to the right of σ if the power series

$$\sum_{m=1}^{\infty}a_{nm}s^{m-1}$$

represents an entire function.

Proof. Suppose

$$\sum_{m=1}^{\infty}a_{nm}s^{m-1}$$

represents an entire function. Then its radius of convergence is infinity. But the domain of $F(s)$ is the intersection of the domain of the power series and the Dirichlet series attached to it, we have right half plane as a domain of absolute convergence for the Dirichlet-power series.

Observe that On any compact subset in the domain of absolute convergence, the function represented by a Dirichlet-power series is analytic. Hence if we have a right half-plane in which it converges absolutely, then the function represented by a Dirichlet-power series is analytic in the same right half-plane.

4 Zero free Region for a Dirichlet-Power Series

In this section we analyze whether such a series has a zero free region in the right half plane where it converges absolutely in the whole right half plane. Actually in Dirichlet series case the existence of zero free region depends on the unique representation of the Dirichlet series. In this case we show by using the limit of the modulus of $F(s)$ as σ tends to infinity. This is the content of the following theorem:

Theorem 41 *Let*

$$F(s)=\sum_{m=1}^{\infty}D_m(s)s^{m-1}$$

be a Dirichlet-power series with

$$D_m(s)=\sum_{n=1}^{\infty}a_{nm}n^{-s}$$

a Dirichlet series for each $m=1,2,3,\cdots$. If a_{11} is not zero and , $\lim_{\sigma\to\infty}|F(s)|=a_{11}$ uniformly for all t, then $F(s)$ has a zero free region in the right half plane.

Proof. Assume $F(s)$ has no zero free region in the right-half plane. Then there is a sequence of complex numbers $\{s_k\}_{k=1}^{\infty}$ with the property

$$\sigma_1<\sigma_2<\sigma_3<\cdots$$

such that, $F(s_k)=0$ for all k. But this contradicts the fact that as $\sigma_k\to\infty$, we have $|F(s_k)|$ tends to a_{11} which is not equal to zero by hypothesis, since as k tends to infinity σ_k is also tends to infinity.

Theorem 42*(Uniqueness theorem)*
Given two Dirichlet-power series,

$$F(s)=\sum_{m=1}^{\infty}D_m(s)s^{m-1}$$

and

$$G(s)=\sum_{m=1}^{\infty}E_m(s)s^{m-1}$$

both absolutely convergent for $\sigma=Re(s)>\sigma_a$. If $F(s_k)=G(s_k)$ for all k in an infinite sequence $\{s_k\}_{k=1}^{\infty}$ of complex numbers, with $\sigma_k=Re(s_k)>\sigma_a$ such that $\sigma_k\to\infty$, as $k\to\infty$ with the property that $F(s_k)=G(s_k)$ implies that $D_m(s_k)=E_m(s_k)$ for each $m=1,2,3,\cdots$. Then $F(s)=G(s)$ for all s with $\sigma=Re(s)>\sigma_a$.

Proof. Suppose $\{s_k\}_{k=1}^{\infty}$ be an infinite sequence with the property that $\sigma_k\to\infty$, as $k\to\infty$. Assume

$$F(s_k)=G(s_k$$

for each $k=1,2,3,\cdots$, then

$$D_m(s_k)=E_m(s_k)$$

$m=1,2,3,\cdots$ and $k=1,2,3,\cdots$. Hence by identity theorem for Dirichlet series we have

$$D_m(s)=E_m(s)$$

for all s and each $m=1,2,3,\cdots$ Thus we have,

$$D_m(s)s^{m-1}=E_m(s)s^{m-1}$$

Therefore,

$$\sum_{m=1}^{\infty}D_m(s)s^{m-1}=\sum_{m=1}^{\infty}E_m(s)s^{m-1}$$

that is $F(s)=G(s)$ for all s in the domain.

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This theorem implies the existence of a right half plane in which a Dirichlet-power series does not vanish, unless of-course it is identically zero.

Theorem 43 *Suppose*

$$F(s) = \sum_{m=1}^{\infty} D_m(s)s^{m-1}$$

with the property that $F(s_k) = 0$ implies that $D_m(s_k) = 0$ for each $m = 1,2,3,\cdots$ and assume that $F(s) \neq 0$ for some s with $\sigma = Re(s) > \sigma_a$. Then there is a half plane in which $F(s)$ is never zero.

Proof. Assume no such half plane exists. Then for each $k = 1,2,3,\cdots$, there is a point s_k with $\sigma_k = \Re(s_k) > k$ such that $F(s_k) = 0$. Since $\sigma_k \to \infty$, as $k \to \infty$, the uniqueness theorem shows that $F(s) = 0$ for all s with $\sigma = Re(s) > \sigma_a$. This contradicts the hypothesis that $F(s) \neq 0$ for some s with $\sigma = Re(s) > \sigma_a$.

4.1 Open problems

1. can we remove the condition $F(s_k)) = G(s_k)$ implies that $D_m(s_k) = E_m(s_k)$ for each $m = 1,2,3,\cdots$ in the Uniqueness Theorem?
2. can we improve the limit condition on the theorem which guarantees the existence of zero free regions?

The above problem is concerned to the prove or disprove of the following:

Theorem 44 (Uniqueness theorem for Dirichlet-power series)

Given two Dirichlet-power series,

$$F(s) = \sum_{m=1}^{\infty} D_m(s)s^{m-1}$$

and

$$G(s) = \sum_{m=1}^{\infty} E_m(s)s^{m-1}$$

both absolutely convergent for $\sigma = Re(s) > \sigma_a$. If $F(s_k) = G(s_k)$ for all k in an infinite sequence $\{s_k\}_{k=1}^{\infty}$ of complex numbers, with $\sigma_k = Re(s_k) > \sigma_a$ such that $\sigma_k \to \infty$, as $k \to \infty$. Then $F(s) = G(s)$ for all s with $\sigma = Re(s) > \sigma_a$.

Corollary 45

Let

$$F(s) = \sum_{m=1}^{\infty} D_m(s)s^{m-1}$$

be a Dirichlet-power series with abscissa of absolute convergence σ_a. Suppose that for some s with $\sigma = Re(s) > \sigma_a$ we have $F(s) \neq 0$. Then there exists a half-plane in which the Diriclet-power series is absolutely convergent and never zero.

5 Conclusion

As we have seen a new type of series called Dirichlet-Power series is introduced. It has many properties analogous to an ordinary Dirichlet series and power series. The series has a D-shaped region of absolute convergence which can be extended to to right half plane. There is a unique extended real number σ_a with the property that the series converges absolutely and uniformly to the right of it and diverges to the left of it. In any compact subset of its absolute convergence the series represents an analytic function. This series has zero free region in the right half-plane. We have also forwarded some open problems for discussions.

Acknowledgments

I would like to thank Professor Abdulkadir Hassen for introducing me such an interesting topics. I also thank the referees for their insightful comments on the drafts of this paper.

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