The Minkowski Inequality for Generalized Fractional Integrals

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Abstract: In this work, the well-known Minkowski inequality is studied, using a generalized fractional integral operator, defined and studied by authors in a previous work. Relationships with known results are established throughout the work and in conclusions.

Keywords: Minkowski inequality, fractional integral

1 Introduction

One of the best-known integral inequalities is the Minkowski inequality, a generalization of the well-known triangular inequality. The best known formulation is as follows.

Minkowski inequality. If \( p \geq 1 \) is a real number, and \( f \) and \( g \) are functions of class \( L_p[a_1, a_2] \). Then the following inequality, on \( [a_1, a_2] \), is satisfied

\[
\left( \int_{a_1}^{a_2} |f + g|^p \, dx \right)^{\frac{1}{p}} \leq \left( \int_{a_1}^{a_2} |f|^p \, dx \right)^{\frac{1}{p}} + \left( \int_{a_1}^{a_2} |g|^p \, dx \right)^{\frac{1}{p}}.
\]

This inequality, along with Hölder’s inequality, the inequalities of the arithmetic mean and the geometric mean, have played dominant roles in the theory of inequalities. Today, it is in very common use, in various areas, these and other inequalities, hence, it is not surprising that numerous studies have been carried out linked to these inequalities and, in recent years, the subject has generated considerable interest on the part of many mathematicians, which has turned this area into a useful and important tool in the current development of different branches, pure and applied, of mathematics.

The following definitions specify the type of functions that we will consider in this work (see [1]).

\textbf{Definition 1.} A function \( \varphi(u) \) is said to be in \( L_p[a_1, a_2] \) if

\[
\left( \int_{a_1}^{a_2} |\varphi(u)|^p \, du \right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty.
\]

\textbf{Definition 2.} A function \( \varphi(u) \) is said to be in \( L_p,[a_1, a_2] \) if

\[
\left( \int_{a_1}^{a_2} |\varphi(u)|^p \, du \right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty, \quad s \geq 0.
\]

On the other hand, we know that fractional calculus, the calculus with derivatives and integrals of non-integer order, has been gaining attention in the last 40 years and has become one of the most active areas in mathematics today. This has brought about the emergence of new integral operators that are natural generalizations of the classical Riemann-Liouville fractional integral. In a previous work (see [2]). The authors defined a generalized operator that contains as a particular case several of those reported in the literature.

\textbf{Definition 3.} The \( k \)-generalized fractional Riemann-Liouville integral of order \( \alpha \) with \( \alpha \in \mathbb{R}, \) and \( s \neq -1 \) of an integrable function \( \varphi(u) \) on \([0, \infty)\), are given as follows (right and left, respectively):

\[
J^{\frac{\alpha}{s}}_{F, a_1+} \varphi(u) = \frac{1}{k \Gamma(\alpha)} \int_{a_1}^{a} F(\tau, s) \varphi(\tau) d\tau \quad \text{and} \quad (1)
\]

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\[ s_{F,F}^\alpha \varphi(u) = \frac{1}{k\Gamma_k(\alpha)} \int_0^{s_1} F(\tau,u) \varphi(\tau)d\tau, \]  
\[ \text{with } F(\tau,0) = 1, \varphi(u,\tau) = \int_0^\tau F(\theta,s)d\theta \text{ and } \varphi(u,\tau) = \int_0^\tau F(\theta,s)d\theta. \]

With the functions \( \Gamma \) (see [3–5] and [6]) and \( \Gamma_k \) defined by (cf. [7]):

\[ \Gamma(z) = \int_0^\infty \tau^{z-1}e^{-\tau}d\tau, \quad \Re(z) > 0, \quad \text{and} \]
\[ \Gamma_k(z) = \int_0^\infty \tau^{z-1}e^{-\tau^{1/k}}d\tau, \quad k > 0. \]

It is clear that if \( k \rightarrow 1 \), we have \( \Gamma_k(z) \rightarrow \Gamma(z) \). Also, we define the \( k \)-beta function as follows:

\[ B_k(u,v) = \frac{1}{k} \int_0^1 \tau^{u-1}(1-\tau)^{v-1}d\tau, \]

notice that \( B_k(u,v) = \frac{1}{k}B(u,\frac{v}{k}) \), and \( B_k(u,v) = \frac{\Gamma_k(u)\Gamma_k(v)}{\Gamma_k(u+v)} \).

Minkowski’s inequality is one of the inequalities that has been given the most attention in recent years. To cite just a few examples, directly linked to fractional operators, we recommend consulting [1, 8–15] and references cited therein.

The main purpose of this paper, using the generalized fractional integral operator of the Riemann-Liouville type, from Definition 3, is to establish several integral inequalities of Minkowski type which contain as particular cases, several of those reported in the literature.

2 Main Results

Our first result provides a Minkowski reverse inequality, within the generalized operators of Definition 3.

**Theorem 1.** For \( k > 0, s \neq -1, \alpha > 0 \) and \( p \geq 1 \). Let \( \varphi, \psi \in L_{1,s}[a_1,\tau] \) be two positive functions in \([0,\infty)\) such that for all \( \tau > a_1 \), \( s_{F,F}^\alpha \varphi(\tau) < \infty \) and \( s_{F,F}^\alpha \psi(\tau) < \infty \) if \( 0 < M \leq \frac{\varphi(a)}{\psi(a)} \leq M \) for all \( u \in [a_1,\tau] \), then

\[ (s_{F,F}^\alpha \varphi(\tau))^\frac{1}{s} + (s_{F,F}^\alpha \psi(\tau))^\frac{1}{s} \leq c_1 (s_{F,F}^\alpha (\varphi + \psi)(\tau))^\frac{1}{s}, \quad \text{(5)} \]

with \( c_1 = \frac{M(M+1)+(M+1)}{M+1}. \)

**Proof.** Using the fact that \( \frac{\varphi(u)}{\psi(u)} \leq M, a_1 \leq u \leq \tau, \) we get

\[ \varphi(u) \leq M(\varphi(u) + \psi(u)) - M\varphi(u), \]

which is equivalent to

\[ (M+1)^p\varphi^p(u) \leq M^p(\varphi(u) + \psi(u))^p. \]

Multiplying both sides of (5) by \( \frac{F(u,s)}{s_{F,F}^\alpha (\varphi(\tau) + \psi(\tau))^\frac{1}{s}} \), then we integrate the resulting inequality with respect to \( u \) over \( (a_1,\tau) \), we get

\[ \frac{(M+1)^p}{k\Gamma_k(\alpha)} s_{F,F}^\alpha \varphi^p(\tau) \leq \frac{M^p}{k\Gamma_k(\alpha)} s_{F,F}^\alpha (\varphi + \psi)^p(\tau), \quad \text{(7)} \]

thus,

\[ (s_{F,F}^\alpha \varphi^p(\tau))^\frac{1}{p} \leq \frac{M}{M+1} (s_{F,F}^\alpha (\varphi + \psi)^p(\tau))^\frac{1}{p}. \]

On the other hand, as \( m\varphi(u) \leq \varphi(u) \), it follows that

\[ \left(1 + \frac{1}{m}\right) \varphi(u) \leq \frac{1}{m} \varphi(u) + \psi(u)^p. \quad \text{(9)} \]

Now, multiplying both sides of (9) by \( \frac{F(u,s)}{s_{F,F}^\alpha (\varphi(\tau) + \psi(\tau))^\frac{1}{s}} \), then we integrate the resulting inequality with respect to \( u \) over \( (a_1,\tau) \), we obtain

\[ (s_{F,F}^\alpha \varphi^p(\tau))^\frac{1}{p} \leq \frac{1}{m+1} (s_{F,F}^\alpha (\varphi + \psi)^p(\tau))^\frac{1}{p}. \]

Adding Eqs. (8) and (10), we obtain the result.

**Remark.** If we consider the kernel \( F(\tau,s) = \tau^{-1} \), this theorem becomes Theorem 9 of [1].

**Theorem 2.** For \( k > 0, s \neq -1, \alpha > 0 \) and \( p \geq 1 \). Let \( \varphi, \psi \in L_{1,s}[a_1,\tau] \) be two positive functions in \([0,\infty)\) such that for all \( \tau > a_1 \), \( s_{F,F}^\alpha \varphi^p(\tau) < \infty \) and \( s_{F,F}^\alpha \psi^p(\tau) < \infty \). If \( 0 < M \leq \frac{\varphi(a)}{\psi(a)} \leq M \) for all \( u \in [a_1,\tau] \), then

\[ \left(s_{F,F}^\alpha \varphi^p(\tau)ight)^\frac{1}{p} + \left(s_{F,F}^\alpha \psi^p(\tau)ight)^\frac{1}{p} \leq c_2 \left(s_{F,F}^\alpha (\varphi + \psi)^p(\tau)ight)^\frac{1}{p}, \quad \text{(11)} \]

with \( c_2 = \frac{(m+1)(M+1)}{M+1} - 2. \)

**Proof.** Multiplying the equations (8) and (10) we have

\[ \frac{(m+1)(M+1)}{M} \left(s_{F,F}^\alpha \varphi^p(\tau)ight)^\frac{1}{p} \left(s_{F,F}^\alpha \psi^p(\tau)ight)^\frac{1}{p} \leq \left(s_{F,F}^\alpha (\varphi + \psi)^p(\tau)ight)^\frac{1}{p}. \quad \text{(12)} \]

Now, using the Minkowski inequality, we obtain

\[ \frac{(m+1)(M+1)}{M} \left(s_{F,F}^\alpha \varphi^p(\tau)ight)^\frac{1}{p} \left(s_{F,F}^\alpha \psi^p(\tau)ight)^\frac{1}{p} \leq \left((s_{F,F}^\alpha \varphi^p(\tau))^\frac{1}{p} + (s_{F,F}^\alpha \psi^p(\tau))^\frac{1}{p}\right)^2. \quad \text{(13)} \]
Therefore, we obtain
\[
\left( J_{F_{\alpha},1}^\alpha \phi^p (\tau) \right)^{\frac{1}{p}} + \left( J_{F_{\alpha},1}^\alpha \psi^p (\tau) \right)^{\frac{1}{p}} \geq \left( \frac{(m+1)(M+1)}{M} - 2 \right) \left( J_{F_{\alpha},1}^\alpha \phi^p (\tau) \right)^{\frac{1}{p}} \left( J_{F_{\alpha},1}^\alpha \psi^p (\tau) \right)^{\frac{1}{p}}.
\]
(14)

Remark. Theorem 10 of [1] is a particular case of the previous result if we put \( F(\tau, s) = \tau^{s-1} \).

Other inequalities of the Minkowski reverse type are given in the results.

**Theorem 3.** For \( k > 0, s \neq -1, \alpha > 0 \) and \( p, q \geq 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). Let \( \phi, \psi \in L_{1,s}[a_1, \tau] \) be two positive functions in \([0, \infty)\) such that for all \( \tau > a_1 \), \( J_{F_{\alpha},1}^\alpha \phi^p (\tau) < \infty \) and \( J_{F_{\alpha},1}^\alpha \psi^p (\tau) < \infty \) if \( 0 < m \leq \frac{\phi(u)}{\psi(u)} \leq M \) for \( m, M \in \mathbb{R}^+ \) and for all \( u \in [a_1, \tau] \), then
\[
\left( J_{F_{\alpha},1}^\alpha \phi^p (\tau) \right)^{\frac{1}{p}} \left( J_{F_{\alpha},1}^\alpha \psi^p (\tau) \right)^{\frac{1}{p}} \leq \left( M \frac{\psi^{k-1}}{\psi^k} \right) \left( J_{F_{\alpha},1}^\alpha \phi^p (\tau) \right)^{\frac{1}{p}} \left( J_{F_{\alpha},1}^\alpha \psi^p (\tau) \right)^{\frac{1}{p}}.
\]
(15)

Proof. Since \( \frac{\phi(u)}{\psi(u)} \leq M, a_1 \leq u \leq \tau \), we have
\[
\psi^k (u) \geq M^{\frac{1}{p}} \psi^j (u).
\]
(16)

Now, multiplying both sides of (16) by \( \frac{F(u, s)}{k \Gamma (\alpha)(\tau, u)^{1+\frac{1}{q}}} \), then we integrate the resulting inequality with respect to \( u \) over \([a_1, \tau]\), we obtain
\[
M^{-\frac{1}{p}} \left( J_{F_{\alpha},1}^\alpha \psi^k (\tau) \right)^{\frac{1}{p}} \leq \left( J_{F_{\alpha},1}^\alpha \left[ \psi^\frac{k}{q} (\tau) \psi^\frac{1}{q} (\tau) \right]^\frac{1}{p} \right)^{\frac{1}{p}}.
\]
(17)

On the other hand, as \( m \psi(u) \leq \phi(u) \), we get
\[
m^{\frac{1}{p}} \psi^\frac{1}{q} (u) \leq \psi^k (u).
\]
(18)

Then, multiplying both sides of (18) by \( \psi^\frac{1}{q} \) and using \( \frac{1}{p} + \frac{1}{q} = 1 \), it follows that
\[
m^{\frac{1}{p}} \psi(u) \leq \psi^k (u) \psi^\frac{1}{q} (u).
\]
(19)

Now, multiplying both sides of (19) by \( \frac{F(u, s)}{k \Gamma (\alpha)(\tau, u)^{1+\frac{1}{q}}} \), then we integrate the resulting inequality with respect to \( u \) over \([a_1, \tau]\), we have
\[
m^{\frac{1}{p}} \left( J_{F_{\alpha},1}^\alpha \psi^k (\tau) \right)^{\frac{1}{p}} \leq \left( J_{F_{\alpha},1}^\alpha \left[ \psi^\frac{k}{q} (\tau) \psi^\frac{1}{q} (\tau) \right]^\frac{1}{p} \right)^{\frac{1}{p}}.
\]
(20)

Finally, equating inequalities (17) and (20), we obtain the result.

**Theorem 4.** For \( k > 0, s \neq -1, \alpha > 0 \) and \( p, q \geq 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). Let \( \phi, \psi \in L_{1,s}[a_1, \tau] \) be two positive functions in \([0, \infty)\) such that for all \( \tau > a_1 \), \( J_{F_{\alpha},1}^\alpha \phi^p (\tau) < \infty \) and \( J_{F_{\alpha},1}^\alpha \psi^p (\tau) < \infty \). If \( 0 < m \leq \frac{\phi(u)}{\psi(u)} \leq M \) for \( m, M \in \mathbb{R}^+ \) and for all \( u \in [a_1, \tau] \), then
\[
J_{F_{\alpha},1}^\alpha \left[ \phi^p (\tau) \psi^p (\tau) \right] \leq c_3 \left( J_{F_{\alpha},1}^\alpha \left[ \phi^p + \psi^p \right] (\tau) \right) + c_4 \left( J_{F_{\alpha},1}^\alpha \phi^p (\tau) \right),
\]
(21)

where \( c_3 = \frac{2p-1}{p(M+1)} \) and \( c_4 = \frac{2p-1}{q(M+1)} \).

Proof. Using \( \phi(u) \leq M \psi(u) \), we obtain the following inequality:
\[
(M + 1)^p \phi^p (u) \leq M^p (\phi + \psi)^p (u).
\]
(22)

Now, multiplying both sides of (22) by \( \frac{F(u, s)}{k \Gamma (\alpha)(\tau, u)^{1+\frac{1}{q}}} \), then we integrate the resulting inequality with respect to \( u \) over \([a_1, \tau]\), we have
\[
J_{F_{\alpha},1}^\alpha \phi^p (\tau) \leq \frac{M^p}{(M + 1)^p} J_{F_{\alpha},1}^\alpha (\phi + \psi)^p (\tau).
\]
(23)

Also, we know that \( 0 < m \leq \frac{\phi(u)}{\psi(u)} , a_1 < u < \tau \), thus we get
\[
(m + 1)^q \psi^q (u) \leq (\phi + \psi)^q (u).
\]
(24)

Then, multiplying both sides of (24) by \( \frac{F(u, s)}{k \Gamma (\alpha)(\tau, u)^{1+\frac{1}{q}}} \), then we integrate the resulting inequality with respect to \( u \) over \([a_1, \tau]\), we get
\[
J_{F_{\alpha},1}^\alpha \psi^q (\tau) \leq \frac{1}{(m + 1)^q} J_{F_{\alpha},1}^\alpha (\phi + \psi)^q (\tau).
\]
(25)

Now, using the Young’s inequality, we have
\[
\phi(u) \psi(u) \leq \frac{\phi^p (u)}{p} + \frac{\psi^q (u)}{q}.
\]
(26)

Then, multiplying both sides of (26) by \( \frac{F(u, s)}{k \Gamma (\alpha)(\tau, u)^{1+\frac{1}{q}}} \), then we integrate the resulting inequality with respect to \( u \) over \([a_1, \tau]\), it follows that
\[
J_{F_{\alpha},1}^\alpha \left[ \phi \psi \right] (\tau) \leq \frac{1}{p} \left( J_{F_{\alpha},1}^\alpha \phi^p (\tau) \right) + \frac{1}{q} \left( J_{F_{\alpha},1}^\alpha \psi^q (\tau) \right).
\]
(27)
Now, multiplying both sides of (33) by \( \frac{M}{M-c} \), we obtain the result.

**Theorem 6.** For \( k > 0 \), \( s \neq -1 \), \( \alpha > 0 \) and \( p \geq 1 \). Let \( \varphi, \psi \in L_{1,1}[a_1, \tau] \) be two positive functions in \( [0, +\infty) \) such that for all \( \tau > a_1 \), \( J_{F,a_{1}}^{s}\varphi^{\alpha}(\tau) \in \infty \) and

\[
J_{F,a_{1}}^{s}\psi^{p}(\tau) < \infty.
\]

Then, adding (32) and (33), we obtain the result.

**Theorem 7.** For \( k > 0 \), \( s \neq -1 \), \( \alpha > 0 \) and \( p \geq 1 \). Let \( \varphi, \psi \in S_{1,1}[a_1, \tau] \) be two positive functions in \( [0, +\infty) \) such that for all \( \tau > a_1 \), \( J_{F,a_{1}}^{s}\varphi^{\alpha}(\tau) \in \infty \) and

\[
J_{F,a_{1}}^{s}\psi^{p}(\tau) < \infty.
\]
\[ J_{F,a_1}^\mu \psi^p(\tau) < \infty \] if \( 0 < m \leq \frac{\varphi(u)}{\psi(u)} \leq M \) for \( m,M \in \mathbb{R}^+ \) and for all \( u \in [a_1, \tau] \), then

\[
\frac{1}{M} \left(J_{F,a_1}^\mu \phi(\tau)\psi(\tau)\right) \leq \frac{1}{(m+1)(M+1)} \left(J_{F,a_1}^\mu (\phi(\tau) + \psi(\tau))^2\right) \leq \frac{1}{m} \left(J_{F,a_1}^\mu \phi(\tau)\psi(\tau)\right).
\]

**Proof.** Using the hypothesis \( 0 < m \leq \frac{\varphi(u)}{\psi(u)} \leq M \), we obtain the following inequalities:

\[
\psi(\tau)(m+1) \leq \phi(u) + \psi(u) \leq \psi(\tau)(M+1)
\]

and

\[
\frac{1}{M} \leq \frac{\psi(u)}{\phi(u)} \leq \frac{1}{m}.
\]

From (43), it is followed that

\[
\phi(\tau) \left(\frac{M+1}{M}\right) \leq \phi(u) + \psi(u) \leq \phi(\tau) \left(\frac{m+1}{m}\right).
\]

Multiplying member to member (43) and (44), we conclude that

\[
\frac{1}{M} \left[J_{F,a_1}^\mu \phi(\tau)\psi(\tau)\right] \leq \frac{1}{(m+1)(M+1)} (\phi(\tau) + \psi(\tau))^2 \leq \frac{1}{m} \left[J_{F,a_1}^\mu \phi(\tau)\psi(\tau)\right].
\]

Now, multiplying both sides of (45) by \( \frac{F(u)}{\kappa_{\mu}(a_1)[\psi(u)]^{(1-\tau)}} \), then we integrate the resulting inequality with respect to \( u \) over \((a_1, \tau)\), we have the inequality (41).

**Theorem 8.** For \( k > 0 \), \( s \neq -1 \), \( \alpha > 0 \) and \( p \geq 1 \). Let \( \phi, \psi \in L_1([a_1, \tau]) \) be two positive functions in \([0, \infty)\) such that for all \( \tau > a_1 \), \( J_{F,a_1}^\mu \phi^p(\tau) < \infty \) and \( J_{F,a_1}^\mu \psi^p(\tau) < \infty \).

If \( 0 < m \leq \frac{\varphi(u)}{\psi(u)} \leq M \) for \( m,M \in \mathbb{R}^+ \) and for all \( u \in [a_1, \tau] \), then

\[
\left(J_{F,a_1}^\mu \phi^p(\tau)\right)^{\frac{1}{p}} + \left(J_{F,a_1}^\mu \psi^p(\tau)\right)^{\frac{1}{p}} \leq 2\left(J_{F,a_1}^\mu \eta^p[\varphi(\tau),\psi(\tau)]\right)^{\frac{1}{p}}
\]

with

\[
\eta[\varphi(\tau),\psi(\tau)] = \max \left\{\frac{M+1}{m} \varphi(\tau) - M\psi(\tau), \frac{(m+M)\psi(\tau) - \varphi(\tau)}{m}\right\}
\]

**Proof.** Of the hypothesis \( 0 < m \leq \frac{\varphi(u)}{\psi(u)} \leq M \) and \( a_1 \leq u \leq \tau \), we have the following inequalities:

\[
0 < m \leq M + m - \frac{\varphi(u)}{\psi(u)} \leq M
\]

and

\[
0 < m \leq M + m - \frac{\varphi(u)}{\psi(u)} \leq M.
\]

From (47), (48) and remembering that \( \eta[\varphi(\tau),\psi(\tau)] = \max \left\{\frac{M+1}{m} \varphi(\tau) - M\psi(\tau), \frac{(m+M)\psi(\tau) - \varphi(\tau)}{m}\right\} \), we deduce

\[
\psi(\tau) \leq \frac{(m+M)\psi(\tau) - \varphi(\tau)}{m} \leq \eta[\varphi(\tau),\psi(\tau)].
\]

On the other hand, from the hypothesis, it also follows that

\[
\frac{1}{M} \leq \frac{\psi(u)}{\phi(u)} \leq \frac{1}{m}
\]

and

\[
\frac{1}{M} \leq \frac{\psi(u)}{\phi(u)} \leq \frac{1}{m}.
\]

From (50) and (51), we have

\[
\frac{1}{M} \leq \left(\frac{1}{m} + \frac{1}{m}\right) \phi(u) - \psi(u) \leq \frac{1}{m},
\]

which implies that

\[
\phi(u) \leq M \left[\frac{1}{M} + \frac{1}{m}\right] \phi(u) - \psi(u) \leq \frac{m(M+m)\psi(u) - M^2m\psi(u)}{m}\]

\[
\leq \frac{m}{m+1} \phi(u) - M\psi(u) \leq \eta[\varphi(\tau),\psi(\tau)].
\]

By raising both of inequalities (49) and (53) to \( p \), we obtain

\[
\psi^p(\tau) \leq \eta^p[\varphi(\tau),\psi(\tau)]
\]

and

\[
\phi^p(u) \leq \eta^p[\varphi(\tau),\psi(\tau)].
\]

Multiplying both sides of (54) and (55) by \( \frac{F(u)}{\kappa_{\mu}(a_1)[\psi(u)]^{(1-\tau)}} \), then we integrate the resulting inequality with respect to \( u \) over \((a_1, \tau)\), we have

\[
\left(J_{F,a_1}^\mu \psi^p(\tau)\right)^{\frac{1}{p}} \leq \left(J_{F,a_1}^\mu \eta^p[\varphi(\tau),\psi(\tau)]\right)^{\frac{1}{p}}.
\]

and

\[
\left(J_{F,a_1}^\mu \phi^p(\tau)\right)^{\frac{1}{p}} \leq \left(J_{F,a_1}^\mu \eta^p[\varphi(\tau),\psi(\tau)]\right)^{\frac{1}{p}}.
\]

Finally, adding inequalities (56) and (57), we conclude the result (46).

**Remark.** The previous observations are still valid for Theorems 3, 4, 5, 6, 7 and 8, containing as particular cases Theorems 11-16, respectively, of [1].
3 Conclusions
As we said at the beginning of the work, the generalized operators of Definition 3 contain, for a proper choice of kernel $F$, several well-known integral operators. For example, let us consider the kernel $F(\tau, s) = \tau^s$, then we will have successively:

$$
F(u, \tau) = \frac{u^{r+1} - \tau^{s+1}}{s+1} \quad (58)
$$

$$(F(u, \tau))^{1-\frac{\alpha}{2}} = \left[ \frac{u^{r+1} - \tau^{s+1}}{s+1} \right]^{1-\frac{\alpha}{2}}. \quad (59)
$$

Regarding the $(k,s)$-Riemann-Liouville fractional integral in Definition 2.1 of [16], analogously, if $s \equiv 0$ and $k \equiv 1$, we obtain the classic Riemann-Liouville operator. In addition to the previous one, it is clear that our operator is also a successive generalization of the generalized integral $k$-fractional (see [?, 17]) of the Integral Katugampola (see [18]) of the $(k,s)$-Riemann-Liouville fractional integral in Definition 2.1 of [16] of the Riemann-Liouville $k$-integral of a function with respect to another function (a variation of the $\psi$-integral, see [?, 19]) and the classic Riemann Integral.

It is clear, then, that the results obtained in [?, ?, 20] achieved within the framework of the generalized integral $k$-fractional [10], where we worked with the generalized $k$-fractional integrals [15], in the framework of integral Katugampola [8], with the $\psi$-integral among others, can be obtained as particular cases of our results.

Obviously, it remains an open problem, the obtaining of other integral inequalities in this generalized framework such as Gruss, Chebyshev, Opial, etc.

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Conflict of Interest
The authors declare that there is no conflict of interest regarding the publication of this article.

References
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