Applied Mathematics & Information Sciences
*An International Journal*

http://dx.doi.org/10.18576/amis/150101

# The Minkowski Inequality for Generalized Fractional Integrals

*Juan Galeano Delgado*[1], *Juan E. Nápoles Valdés* [2], *Edgardo Pérez Reyes* [1] *and Miguel Vivas-Cortez*[3,*]

[1]Faculty of Science and Engineering, University of Sinú Eliás Bechara Zainúm, Montería, Colombia
[2]UNNE, FaCENA, Ave. Libertad 5450, Corrientes 3400, Argentina
[3]Faculty of Exact and Natural Sciences, School of Physical Sciences and Mathematics, Pontificia Universidad Católica del Ecuador, Av.12 de octubre 1076 y Roca, Quito, 17012184, Ecuador

Received: 2 Sep. 2020, Revised: 2 Nov. 2020, Accepted: 15 Dec. 2020
Published online: 1 Jan. 2021

**Abstract:** In this work, the well-known Minkowski inequality is studied, using a generalized fractional integral operator, defined and studied by authors in a previous work. Relationships with known results are established throughout the work and in conclusions.

**Keywords:** Minkowski inequality, fractional integral

## 1 Introduction

One of the best-known integral inequalities is the Minkowski inequality, a generalization of the well-known triangular inequality. The best known formulation is as follows.

Minkowski inequality. If $p \geq 1$ is a real number, and $f$ and $g$ are functions of class $L_p[a_1,a_2]$. Then the following inequality, on $[a_1,a_2]$, is satisfied

$$\left(\int |f+g|^p dx\right)^{\frac{1}{p}} \leq \left(\int |f|^p dx\right)^{\frac{1}{p}} + \left(\int |g|^p dx\right)^{\frac{1}{p}}.$$

This inequality, along with Holder's inequality, the inequalities of the arithmetic mean and the geometric mean, have played dominant roles in the theory of inequalities. Today, it is in very common use, in various areas, these and other inequalities, hence, it is not surprising that numerous studies have been carried out linked to these inequalities and, in recent years, the subject has generated considerable interest on the part of many mathematicians, which has turned this area into a useful and important tool in the current development of different branches, pure and applied, of mathematics.

The following definitions specify the type of functions that we will consider in this work (see [1]).

**Definition 1.** *A function $\varphi(u)$ is said to be in $L_p[a_1,a_2]$ if*

$$\left(\int_{a_1}^{a_2} |\varphi(u)|^p du\right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty.$$

**Definition 2.** *A function $\varphi(u)$ is said to be in $L_{p,s}[a_1,a_2]$ if*

$$\left(\int_{a_1}^{a_2} |\varphi(u)|^p u^s du\right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty,\ s \geq 0.$$

On the other hand, we know that fractional calculus, the calculus with derivatives and integrals of non-integer order, has been gaining attention in the last 40 years and has become one of the most active areas in mathematics today. This has brought about the emergence of new integral operators that are natural generalizations of the classical Riemann-Liouville fractional integral. In a previous work (see [2]). The authors defined a generalized operator that contains as a particular case several of those reported in the literature.

**Definition 3.** *The k-generalized fractional Riemann-Liouville integral of order $\alpha$ with $\alpha \in \mathbb{R}$, and $s \neq -1$ of an integrable function $\chi(u)$ on $[0,\infty)$, are given as follows (right and left, respectively):*

$$ {}^sJ_{F,a_1+}^{\frac{\alpha}{k}}\varphi(u) = \frac{1}{k\Gamma_k(\alpha)}\int_{a_1}^{u} \frac{F(\tau,s)\varphi(\tau)d\tau}{[\mathbb{F}(u,\tau)]^{1-\frac{\alpha}{k}}} \quad \text{and} \tag{1}$$

* Corresponding author e-mail: MJVIVAS@puce.edu.ec

$$ {}^{s}J^{\frac{\alpha}{k}}_{F,a_2-}\varphi(u)=\frac{1}{k\Gamma_k(\alpha)}\int_u^{a_2}\frac{F(\tau,s)\varphi(\tau)d\tau}{[F(\tau,u)]^{1-\frac{\alpha}{k}}}, \quad (2) $$

*with* $F(\tau,0)=1$, $\mathbb{F}(u,\tau)=\int_\tau^u F(\theta,s)d\theta$ *and* $\mathbb{F}(\tau,u)=\int_u^\tau F(\theta,s)d\theta$.

With the functions $\Gamma$ (see [3–5] and [6]) and $\Gamma_k$ defined by (cf. [7]):

$$ \Gamma(z)=\int_0^\infty \tau^{z-1}e^{-\tau}\,\mathrm{d}\tau, \quad \Re(z)>0, \text{ and} \quad (3) $$

$$ \Gamma_k(z)=\int_0^\infty \tau^{z-1}e^{-\tau^k/k}\,\mathrm{d}\tau, k>0. \quad (4) $$

It is clear that if $k\to 1$, we have $\Gamma_k(z)\to\Gamma(z)$, $\Gamma_k(z)=(k)^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right)$ and $\Gamma_k(z+k)=z\Gamma_k(z)$. Also, we define the $k$-beta function as follows:

$$ B_k(u,v)=\frac{1}{k}\int_0^1 \tau^{\frac{u}{k}-1}(1-\tau)^{\frac{v}{k}-1}d\tau, $$

notice that $B_k(u,v)=\frac{1}{k}B(\frac{u}{k},\frac{v}{k})$, and $B_k(u,v)=\frac{\Gamma_k(u)\Gamma_k(v)}{\Gamma_k(u+v)}$.

Minkowski's inequality is one of the inequalities that has been given the most attention in recent years. To cite just a few examples, directly linked to fractional operators, we recommend consulting [1, 8–15] and references cited therein.

The main purpose of this paper, using the generalized fractional integral operator of the Riemann-Liouville type, from Definition 3, is to establish several integral inequalities of Minkowski type which contain as particular cases, several of those reported in the literature.

## 2 Main Results

Our first result provides a Minkowski reverse inequality, within the generalized operators of Definition 3.

**Theorem 1.** *For* $k>0$, $s\neq -1$, $\alpha>0$ *and* $p\geq 1$. *Let* $\varphi,\psi\in L_{1,s}[a_1,\tau]$ *be two positive functions in* $[0,+\infty)$ *such that for all* $\tau>a_1$, ${}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\varphi^p(\tau)<\infty$ *and* ${}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\psi^p(\tau)<\infty$ *If* $0<m\leq\frac{\varphi(u)}{\psi(u)}\leq M$ *for* $m,M\in\mathbb{R}^+$ *and for all* $u\in[a_1,\tau]$, *then*

$$ \left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\varphi^p(\tau)\right)^{\frac{1}{p}}+\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\psi^p(\tau)\right)^{\frac{1}{p}}\leq c_1\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\varphi+\psi)^p(\tau)\right)^{\frac{1}{p}}, \quad (5) $$

*with* $c_1=\frac{M(m+1)+(M+1)}{(m+1)(M+1)}$.

*Proof.* Using the fact that $\frac{\varphi(u)}{\psi(u)}\leq M$, $a_1\leq u\leq\tau$, we get

$$ \varphi(u)\leq M(\varphi(u)+\psi(v))-M\varphi(u) $$

which is equivalent to

$$ (M+1)^p\varphi^p(u)\leq M^p(\varphi(u)+\psi(u))^p. \quad (6) $$

Multiplying both sides of (6) by $\frac{F(u,s)}{k\Gamma_k(\alpha)[\mathbb{F}(\tau,u)]^{1-\frac{\alpha}{k}}}$, then we integrate the resulting inequality with respect to $u$ over $(a_1,\tau)$, we get

$$ \frac{(M+1)^p}{k\Gamma_k(\alpha)}\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\varphi^p(\tau)\leq\frac{M^p}{k\Gamma_k(\alpha)}\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\varphi+\psi)^p(\tau), \quad (7) $$

thus,

$$ \left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\varphi^p(\tau)\right)^{\frac{1}{p}}\leq\frac{M}{M+1}\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\varphi+\psi)^p(\tau)\right)^{\frac{1}{p}}. \quad (8) $$

On the other hand, as $m\psi(u)\leq\varphi(u)$, it follows that

$$ \left(1+\frac{1}{m}\right)^p\psi^p(u)\leq\left(\frac{1}{m}\right)^p(\varphi(u)+\psi(u))^p. \quad (9) $$

Now, multiplying both sides of (9) by $\frac{F(u,s)}{k\Gamma_k(\alpha)[\mathbb{F}(\tau,u)]^{1-\frac{\alpha}{k}}}$, then we integrate the resulting inequality with respect to $u$ over $(a_1,\tau)$, we obtain

$$ \left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\psi^p(\tau)\right)^{\frac{1}{p}}\leq\frac{1}{m+1}\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\varphi+\psi)^p(\tau)\right)^{\frac{1}{p}}. \quad (10) $$

Adding Eqs. (8) and (10), we obtain the result.

*Remark.* If we consider the kernel $F(\tau,s)=\tau^{s-1}$, this theorem becomes Theorem 9 of [1].

**Theorem 2.** *For* $k>0$, $s\neq -1$, $\alpha>0$ *and* $p\geq 1$. *Let* $\varphi,\psi\in L_{1,s}[a_1,\tau]$ *be two positive functions in* $[0,\infty)$ *such that for all* $\tau>a_1$, ${}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\varphi^p(\tau)<\infty$ *and* ${}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\psi^p(\tau)<\infty$ *If* $0<m\leq\frac{\varphi(u)}{\psi(u)}\leq M$ *for* $m,M\in\mathbb{R}^+$ *and for all* $u\in[a_1,\tau]$, *then*

$$ \begin{aligned} &\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\varphi^p(\tau)\right)^{\frac{2}{p}}+\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\psi^p(\tau)\right)^{\frac{2}{p}} \\ &\quad\geq c_2\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\varphi^p(\tau)\right)^{\frac{1}{p}}\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\psi^p(\tau)\right)^{\frac{1}{p}}, \end{aligned} \quad (11) $$

*with* $c_2=\frac{(m+1)(M+1)}{M+1}-2$.

*Proof.* Multiplying the equations (8) and (10) we have

$$ \begin{aligned} &\frac{(m+1)(M+1)}{M}\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\varphi^p(\tau)\right)^{\frac{1}{p}}\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\psi^p(\tau)\right)^{\frac{1}{p}} \\ &\quad\leq\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\varphi+\psi)^p(\tau)\right)^{\frac{2}{p}}. \end{aligned} \quad (12) $$

Now, using the Minkowski inequality, we obtain

$$ \begin{aligned} &\frac{(m+1)(M+1)}{M}\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\varphi^p(\tau)\right)^{\frac{1}{p}}\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\psi^p(\tau)\right)^{\frac{1}{p}} \\ &\quad\leq\left(\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\varphi^p(\tau)\right)^{\frac{1}{p}}+\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\psi^p(\tau)\right)^{\frac{1}{p}}\right)^2. \end{aligned} \quad (13) $$

Therefore, we obtain

$$\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\varphi^p(\tau)\right)^{\frac{2}{p}}+\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\psi^p(\tau)\right)^{\frac{2}{p}} \geq \left(\frac{(m+1)(M+1)}{M}-2\right)\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\varphi^p(\tau)\right)^{\frac{1}{p}}\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\psi^p(\tau)\right)^{\frac{1}{p}}. \tag{14}$$

*Remark.* Theorem 10 of [1] is a particular case of the previous result if we put $F(\tau,s)=\tau^{s-1}$.

Other inequalities of the Minkowski reverse type are given in the results that we present below.

**Theorem 3.** *For $k>0$, $s\neq -1$, $\alpha>0$ and $p,q\geq 1$ and $\frac{1}{p}+\frac{1}{q}=1$. Let $\varphi,\psi\in L_{1,s}[a_1,\tau]$ be two positive functions in $[0,\infty)$ such that for all $\tau>a_1$, ${}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\varphi^p(\tau)<\infty$ and ${}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\psi^p(\tau)<\infty$ If $0<m\leq\frac{\varphi(u)}{\psi(u)}\leq M$ for $m,M\in\mathbb{R}^+$ and for all $u\in[a_1,\tau]$, then*

$$\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\varphi(\tau)\right)^{\frac{1}{p}}\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\psi(\tau)\right)^{\frac{1}{q}} \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}}\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\varphi^{\frac{1}{p}}(\tau)\psi^{\frac{1}{q}}(\tau)\right). \tag{15}$$

*Proof.* Since $\frac{\varphi(u)}{\psi(u)}\leq M$, $a_1\leq u\leq\tau$, we have

$$\psi^{\frac{1}{q}}(u)\geq M^{-\frac{1}{q}}\varphi^{\frac{1}{q}}(u). \tag{16}$$

Now, multiplying both sides of (16) by $\frac{F(u,s)}{k\Gamma_k(\alpha)[\mathbb{F}(\tau,u)]^{1-\frac{\alpha}{k}}}$, then we integrate the resulting inequality with respect to $u$ over $(a_1,\tau)$, we obtain

$$M^{-\frac{1}{pq}}\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\varphi(\tau)\right)^{\frac{1}{p}}\leq\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}[\varphi^{\frac{1}{p}}(\tau)\psi^{\frac{1}{q}}(\tau)]\right)^{\frac{1}{p}}. \tag{17}$$

On the other hand, as $m\psi(u)\leq\varphi(u)$, we get

$$m^{\frac{1}{p}}\psi^{\frac{1}{p}}(u)\leq\varphi^{\frac{1}{p}}(u). \tag{18}$$

Then, multiplying both sides of (18) by $\psi^{\frac{1}{q}}$ and using $\frac{1}{p}+\frac{1}{q}=1$, it follows that

$$m^{\frac{1}{p}}\psi(u)\leq\varphi^{\frac{1}{p}}(u)\psi^{\frac{1}{q}}(u). \tag{19}$$

Now, multiplying both sides of (19) by $\frac{F(u,s)}{k\Gamma_k(\alpha)[\mathbb{F}(\tau,u)]^{1-\frac{\alpha}{k}}}$, then we integrate the resulting inequality with respect to $u$ over $(a_1,\tau)$, we have

$$m^{\frac{1}{pq}}\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\psi(\tau)\right)^{\frac{1}{q}}\leq\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}[\varphi^{\frac{1}{p}}(\tau)\psi^{\frac{1}{q}}(\tau)]\right)^{\frac{1}{q}}. \tag{20}$$

Finally, multiplying equations (17) and (20), we obtain the result.

**Theorem 4.** *For $k>0$, $s\neq -1$, $\alpha>0$ and $p,q\geq 1$ and $\frac{1}{p}+\frac{1}{q}=1$. Let $\varphi,\psi\in L_{1,s}[a_1,\tau]$ be two positive functions in $[0,\infty)$ such that for all $\tau>a_1$, ${}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\varphi^p(\tau)<\infty$ and ${}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\psi^p(\tau)<\infty$. If $0<m\leq\frac{\varphi(u)}{\psi(u)}\leq M$ for $m,M\in\mathbb{R}^+$ and for all $u\in[a_1,\tau]$, then*

$$\begin{aligned}&{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}[\varphi(\tau)\psi_2(\tau)]\\&\leq c_3\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}[\varphi^p+\psi^p](\tau)\right)+c_4\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}[\varphi^q+\psi^q](\tau)\right),\end{aligned} \tag{21}$$

*where $c_3=\frac{2^{p-1}M^p}{p(M+1)^p}$ and $c_4=\frac{2^{q-1}}{q(m+1)^q}$.*

*Proof.* Using $\varphi(u)\leq M\psi(u)$, we obtain the following inequality:

$$(M+1)^p\varphi^p(u)\leq M^p(\varphi+\psi)^p(u). \tag{22}$$

Now, multiplying both sides of (22) by $\frac{F(u,s)}{k\Gamma_k(\alpha)[\mathbb{F}(\tau,u)]^{1-\frac{\alpha}{k}}}$, then we integrate the resulting inequality with respect to $u$ over $(a_1,\tau)$, we have

$${}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\varphi^p(u)\leq\frac{M^p}{(M+1)^p}\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\varphi+\psi)^p(\tau). \tag{23}$$

Also, we know that $0<m\leq\frac{\varphi(u)}{\psi(u)}$, $a_1<u<\tau$, thus we get

$$(m+1)^q\psi^q(u)\leq(\varphi+\psi)^q(u). \tag{24}$$

Then, multiplying both sides of (24) by $\frac{F(u,s)}{k\Gamma_k(\alpha)[\mathbb{F}(\tau,u)]^{1-\frac{\alpha}{k}}}$, then we integrate the resulting inequality with respect to $u$ over $(a_1,\tau)$, we get

$${}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\psi^q(\tau)\leq\frac{1}{(m+1)^q}\,{}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\varphi+\psi)^q(\tau). \tag{25}$$

Now, using the Young's inequality, we have

$$\varphi(u)\psi(u)\leq\frac{\varphi^p(u)}{p}+\frac{\psi^q(u)}{q}. \tag{26}$$

Then, multiplying both sides of (26) by $\frac{F(u,s)}{k\Gamma_k(\alpha)[\mathbb{F}(\tau,u)]^{1-\frac{\alpha}{k}}}$, then we integrate the resulting inequality with respect to $u$ over $(a_1,\tau)$, it follows that

$${}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\varphi\psi)(\tau)\leq\frac{1}{p}\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\varphi^p(\tau)\right)+\frac{1}{q}\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\psi^q(\tau)\right). \tag{27}$$

From (23), (25), (27), and the inequality $(u+v)^s \leq 2^{s-1}(u^s+v^s),\ s>1, u,v>0$, we have

$$ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\varphi\psi)(\tau) \leq \frac{1}{p}\left( {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\varphi^p(\tau)\right)+\frac{1}{q}\left( {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\psi^q(\tau)\right) \quad (28)$$
$$\leq \frac{M^p}{p(M+1)^p}\, {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\varphi+\psi)^p(\tau) + \frac{1}{q(m+1)^q}\, {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\varphi+\psi)^q(\tau)$$
$$\leq \frac{2^{p-1}M^p}{p(M+1)^p}\, {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\varphi^p+\psi^p)(\tau) + \frac{2^{q-1}}{q(m+1)^q}\, {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\varphi^q+\psi^q)(\tau).$$

Therefore, the proof is completed.

**Theorem 5.** *For $k>0,\ s\neq -1,\ \alpha>0$ and $p,q\geq 1$ and $\frac{1}{p}+\frac{1}{q}=1$. Let $\varphi,\psi\in L_{1,s}[a_1,\tau]$ be two positive functions in $[0,+\infty)$ such that for all $\tau>a_1$, ${}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\varphi^p(\tau)<\infty$ and ${}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\psi^p(\tau)<\infty$ If $0<c<m\leq\frac{\varphi(u)}{\psi(u)}\leq M$ for $c,m,M\in\mathbb{R}^+$ and for all $u\in[a_1,\tau]$, then*

$$\frac{M+1}{M-c}\left( {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\varphi(\tau)-c\psi(\tau))^p\right)^{\frac{1}{p}} \quad (29)$$
$$\leq \left( {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\varphi^p(\tau)\right)^{\frac{1}{p}}+\left( {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\psi^p(\tau)\right)^{\frac{1}{p}} \leq \frac{m+1}{m-c}\left( {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\varphi(\tau)-c\psi(\tau))^p\right)^{\frac{1}{p}}.$$

*Proof.* Using the hypothesis $0<c<m\leq\frac{\varphi(u)}{\psi(u)}\leq M$, we obtain the following inequalities:

$$\frac{(\varphi(u)-c\psi(u))^p}{(M-c)^p}\leq\psi^p(u)\leq\frac{(\varphi(u)-c\psi(u))^p}{(m-c)^p} \quad (30)$$

and

$$\left(\frac{M}{M-c}\right)^p(\varphi(u)-c\psi(u))^p \quad (31)$$
$$\leq\varphi^p(u)\leq\left(\frac{m}{m-c}\right)^p(\varphi(u)-c\psi(u))^p.$$

Now, multiplying both sides of (30) and (31) by $\frac{F(u,s)}{k\Gamma_k(\alpha)[\mathbb{F}(\tau,u)]^{1-\frac{\alpha}{k}}}$, then we integrate the resulting inequality with respect to $u$ over $(a_1,\tau)$, we obtain

$$\frac{1}{M-c}\left( {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\varphi(\tau)-c\psi(\tau))^p\right)^{\frac{1}{p}} \quad (32)$$
$$\leq\left( {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\psi^p(\tau)\right)^{\frac{1}{p}} \leq\frac{1}{m-c}\left( {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\varphi(\tau)-c\psi(\tau))^p\right)^{\frac{1}{p}}$$

and

$$\frac{M}{M-c}\left( {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\varphi(\tau)-c\psi(\tau))^p\right)^{\frac{1}{p}} \quad (33)$$
$$\leq\left( {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\varphi^p(\tau)\right)^{\frac{1}{p}} \leq\frac{m}{m-c}\left( {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\varphi(\tau)-c\psi(\tau))^p\right)^{\frac{1}{p}}.$$

Then, adding (32) and (33), we obtain the result.

**Theorem 6.** *For $k>0,\ s\neq-1,\ \alpha>0$ and $p\geq 1$. Let $\varphi,\psi\in L_{1,s}[a_1,\tau]$ be two positive functions in $[0,+\infty)$ such that for all $\tau>a_1$, ${}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\varphi^p(\tau)<\infty$ and ${}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\psi^p(\tau)<\infty$. If $0\leq a_1\leq\varphi(u)\leq A_1$ and $0<a_2\leq\psi(u)\leq A_2$ for $a_1,a_2,A_1,A_2\in\mathbb{R}^+$ and for all $u\in[a_1,\tau]$, then*

$$\left( {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\varphi^p(\tau)\right)^{\frac{1}{p}}+\left( {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\psi^p(\tau)\right)^{\frac{1}{p}} \quad (34)$$
$$\leq\frac{A_1(a_1+A_2)+A_2(a_2+A_1)}{(a_1+A_2)(a_2+A_1)}\left( {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\varphi(\tau)+\psi(\tau))^p\right)^{\frac{1}{p}}.$$

*Proof.* Of the hypothesis $0<a_2\leq\psi(u)\leq A_2$ it is followed that

$$\frac{1}{A_2}\leq\frac{1}{\psi(u)}\leq\frac{1}{a_2}. \quad (35)$$

Multiplying member to member (35) and $0\leq a_1\leq\varphi(u)\leq A_1$, we conclude that

$$\frac{a_1}{A_2}\leq\frac{\varphi(u)}{\psi(u)}\leq\frac{A_1}{a_2}. \quad (36)$$

From (36), we deduce

$$\varphi^p(u)\leq\left(\frac{A_2}{a_1+A_2}\right)^p(\psi(u)+\varphi(u))^p \quad (37)$$

and

$$\psi^p(u)\leq\left(\frac{A_1}{a_2+A_1}\right)^p(\psi(u)+\varphi(u))^p. \quad (38)$$

Now, multiplying both sides of (37) and (38) by $\frac{F(u,s)}{k\Gamma_k(\alpha)[\mathbb{F}(\tau,u)]^{1-\frac{\alpha}{k}}}$, then we integrate the resulting inequality with respect to $u$ over $(a_1,\tau)$, we have

$$\left( {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\varphi^p(\tau)\right)^{\frac{1}{p}}\leq\left(\frac{A_2}{a_1+A_2}\right)\left( {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\varphi(\tau)+\psi(\tau))^p\right)^{\frac{1}{p}} \quad (39)$$

and

$$\left( {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\varphi^p(\tau)\right)^{\frac{1}{p}}\leq\left(\frac{A_1}{a_2+A_1}\right)\left( {}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\varphi(\tau)+\psi(\tau))^p\right)^{\frac{1}{p}}. \quad (40)$$

Finally, adding (39) and (40), we obtain the result.

**Theorem 7.** *For $k>0,\ s\neq-1,\ \alpha>0$ and $p\geq1$. Let $\varphi,\psi\in L_{1,s}[a_1,\tau]$ be two positive functions in $[0,+\infty)$ such that for all $\tau>a_1$, ${}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\varphi^p(\tau)<\infty$ and*

${}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\psi^p(\tau)<\infty$ *If* $0<m\leq\frac{\varphi(u)}{\psi(u)}\leq M$ *for* $m,M\in\mathbb{R}^+$ *and for all* $u\in[a_1,\tau]$, *then*

$$\frac{1}{M}\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}[\varphi(\tau)\psi(\tau)]\right) \tag{41}$$
$$\leq\frac{1}{(m+1)(M+1)}\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\varphi(\tau)+\psi(\tau))^2\right)$$
$$\leq\frac{1}{m}\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}[\varphi(\tau)\psi(\tau)]\right).$$

*Proof.*Using the hypothesis $0<m\leq\frac{\varphi(u)}{\psi(u)}\leq M$, we obtain the following inequalities:

$$\psi(\tau)(m+1)\leq\varphi(u)+\psi(u)\leq\psi(\tau)(M+1) \tag{42}$$

and

$$\frac{1}{M}\leq\frac{\psi(u)}{\varphi(u)}\leq\frac{1}{m}. \tag{43}$$

From (43), it is followed that

$$\varphi(\tau)\left(\frac{M+1}{M}\right)\leq\varphi(u)+\psi(u)\leq\varphi(\tau)\left(\frac{m+1}{m}\right). \tag{44}$$

Multiplying member to member (43) and (44), we conclude that

$$\frac{1}{M}[\varphi(\tau)\psi(\tau)] \tag{45}$$
$$\leq\frac{1}{(m+1)(M+1)}(\varphi(\tau)+\psi(\tau))^2\leq\frac{1}{m}[\varphi(\tau)\psi(\tau)].$$

Now, multiplying both sides of (45) by $\frac{F(u,s)}{k\Gamma_k(\alpha)[\mathbb{F}(\tau,u)]^{1-\frac{\alpha}{k}}}$, then we integrate the resulting inequality with respect to $u$ over $(a_1,\tau)$, we have the inequality (41).

**Theorem 8.** *For* $k>0$, $s\neq-1$, $\alpha>0$ *and* $p\geq1$. *Let* $\varphi,\psi\in L_{1,s}[a_1,\tau]$ *be two positive functions in* $[0,\infty)$ *such that for all* $\tau>a_1$, ${}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\varphi^p(\tau)<\infty$ *and* ${}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\psi^p(\tau)<\infty$ *If* $0<m\leq\frac{\varphi(u)}{\psi(u)}\leq M$ *for* $m,M\in\mathbb{R}^+$ *and for all* $u\in[a_1,\tau]$, *then*

$$\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\varphi^p(\tau)\right)^{\frac{1}{p}}+\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\psi^p(\tau)\right)^{\frac{1}{p}} \tag{46}$$
$$\leq2\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\eta^p[\varphi(\tau),\psi(\tau)])^{\frac{1}{p}}\right)$$

*with*

$$\eta[\varphi(\tau),\psi(\tau)]=$$
$$\max\{(\frac{M}{m}+1)\varphi(\tau)-M\psi(\tau),\frac{(m+M)\psi(\tau)-\varphi(\tau)}{m}\}$$

*Proof.*Of the hypothesis $0<m\leq\frac{\varphi(u)}{\psi(u)}\leq M$ and $a_1\leq u\leq\tau$. we have the following inequalities:

$$0<m\leq M+m-\frac{\varphi(u)}{\psi(u)}\leq M \tag{47}$$

and

$$0<m\leq M+m-\frac{\varphi(u)}{\psi(u)}\leq M. \tag{48}$$

From (47), (48) and remembering that $\eta[\varphi(\tau),\psi(\tau)]=\max\{(\frac{M}{m}+1)\varphi(\tau)-M\psi(\tau),\frac{(m+M)\psi(\tau)-\varphi(\tau)}{m}\}$, we deduce

$$\psi(\tau)\leq\frac{(m+M)\psi(\tau)-\varphi(\tau)}{m}\leq\eta[\varphi(\tau),\psi(\tau)]. \tag{49}$$

On the other hand, from the hypothesis, it also follows that $0<\frac{1}{M}\leq\frac{\psi(u)}{\varphi(u)}\leq\frac{1}{m}$ then

$$\frac{1}{M}\leq\frac{1}{M}+\frac{1}{m}-\frac{\psi(u)}{\varphi(u)} \tag{50}$$

and

$$\frac{1}{M}+\frac{1}{m}-\frac{\psi(u)}{\varphi(u)}\leq\frac{1}{m}. \tag{51}$$

From (50) and (51), we have

$$\frac{1}{M}\leq\frac{(\frac{1}{M}+\frac{1}{m})\varphi(u)-\psi(u)}{\varphi(u)}\leq\frac{1}{m}, \tag{52}$$

which implies that

$$\varphi(u)\leq M\left[\left(\frac{1}{M}+\frac{1}{m}\right)\varphi(u)-\psi(u)\right]$$
$$\leq\frac{M(M+m)\varphi(u)-M^2m\psi(u)}{mM} \tag{53}$$
$$\leq\left(\frac{M}{m}+1\right)\varphi(u)-M\psi(u)$$
$$\leq\eta[\varphi(\tau),\psi(\tau)].$$

By raising both of inequalities (49) and (53) to $p$, we obtain

$$\psi^p(\tau)\leq\eta^p[\varphi(\tau),\psi(\tau)] \tag{54}$$

and

$$\varphi^p(u)\leq\eta^p[\varphi(\tau),\psi(\tau)]. \tag{55}$$

Multiplying both sides of (54) and (55) by $\frac{F(u,s)}{k\Gamma_k(\alpha)[\mathbb{F}(\tau,u)]^{1-\frac{\alpha}{k}}}$, then we integrate the resulting inequality with respect to $u$ over $(a_1,\tau)$, we have

$$\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\psi^p(\tau)\right)^{\frac{1}{p}}\leq\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\eta^p[\varphi(\tau),\psi(\tau)])^{\frac{1}{p}}\right). \tag{56}$$

and

$$\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}\varphi^p(\tau)\right)^{\frac{1}{p}}\leq\left({}^{s}J^{\frac{\alpha}{k}}_{F,a_1}(\eta^p[\varphi(\tau),\psi(\tau)])^{\frac{1}{p}}\right). \tag{57}$$

Finally, adding inequalities (56) and (57), we conclude the result (46).

*Remark.*The previous observations are still valid for Theorems 3, 4, 5, 6, 7 and 8, containing as particular cases Theorems 11-16, respectively, of [1].

## 3 Conclusions

As we said at the beginning of the work, the generalized operators of Definition 3 contain, for a proper choice of kernel $F$, several well-known integral operators. For example, let us consider the kernel $F(\tau,s)=\tau^s$, then we will have successively:

$$\mathbb{F}(u,\tau)=\frac{u^{s+1}-\tau^{s+1}}{s+1} \tag{58}$$

$$(\mathbb{F}(u,\tau))^{1-\frac{\alpha}{k}}=\left[\frac{u^{s+1}-\tau^{s+1}}{s+1}\right]^{1-\frac{\alpha}{k}}. \tag{59}$$

Regarding the (k,s)-Riemann-Liouville fractional integral in Definition 2.1 of [16], analogously, if $s\equiv 0$ and $k\equiv 1$, we obtain the classic Riemann-Liouville operator. In addition to the previous one, it is clear that our operator is also a successive generalization of the generalized integral k-fractional (see [**?**, 17]) of the Integral Katugampola (see [18]) of the (k,s)-Riemann-Liouville fractional integral in Definition 2.1 of [16] of the Riemann-Liouville k-integral of a function with respect to another function (a variation of the $\psi$-integral, see [**?**, 19]) and the classic Riemann Integral.

It is clear, then, that the results obtained in [**?**, **?**, 20] achieved within the framework of the Riemann integral [10], where we worked with the generalized k-fractional integrals [15], in the framework of integral Katugampola [8], with the $\psi$-integral among others, can be obtained as particular cases of our results.

Obviously, it remains an open problem, the obtaining of other integral inequalities in this generalized framework such as Gruss, Chebyshev, Opial, etc.

## Acknowledgement

Professor Miguel Vivas-Cortez thanks the research directorate of the Pontifical Catholic University of Ecuador for their support through the project entitled: ”Algunas desigualdades integrales para funciones con algntipo de convexidad generalizada y aplicaciones”.
The authors are grateful to the anonymous referee for the careful checking of the details and for helpful comments that improved this paper.

## Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

## References

[1] S. Mubeen, G. M. Habibullah and M. N. Naeem, The Minkowski inequality involving generalized k-fractional conformable integral, J. Inequal. Appl. **2019** **(81)**, 1-12 (2019).

[2] Juan D. Galeano, Juan E. Nápoles and Edgardo Perez, On a general formulation of the fractional operator Riemann-Liouville and related inequalities, submitted.

[3] F. Qi and B. N. Guo, Integral representations and complete monotonicity of remainders of the Binet and Stirling formulas for the gamma function. Rev. R. Acad. Cienc. Exactas Fis. Nat., Ser. A Mat. **111** **(2)**, 425-434 (2017).

[4] E. D. Rainville, Special Functions. Macmillan Co., New York (1960).

[5] Z. H. Yang and J. F. Tian, Monotonicity and inequalities for the gamma function. J. Inequal. Appl. **2017**, 317-329 (2017).

[6] Z. H. Yang and J. F. Tian, Monotonicity and sharp inequalities related to gamma function. J. Math. Inequal. **12(1)**, 1-22 (2018).

[7] R. Díaz and E. Pariguan, On hypergeometric functions and Pochhammer k-symbol. Divulg. Mat. **15** **(2)**, 179-192 (2007).

[8] T. A. Aljaaidi and D. B. Pachpatte, The Minkowski's inequalities via $\Psi$-Riemann-Liouville fractional integral operators, Rendiconti del Circolo Matematico di Palermo Series, **2**, 2020.

[9] L. Bougoffa, On Minkowski and Hardy integral inequalities, J. Inequal. Pure Appl. Math. **7**, 1-3 (2006)

[10] V. L. Chinchane, New approach to Minkowski fractional inequalities using generalized k-fractional integral operator, Journal of the Indian Mathematical Society, **85(1-2)**, 1-10 (2017).

[11] V. L. Chinchane and D. B. Pachpatte, New fractional inequalities via Hadamard fractional integral, Int. J. Funct. Anal. **5**, 165-176 (2013).

[12] Z. Dahmani, On Minkowski and Hermite Hadamard integral inequalities via fractional integration, Ann. Funct. Anal. **1**, 51-58 (2010).

[13] O. Hutnik, Some integral inequalities of Hölder and Minkowski type, Colloq. Math. **108**, 247-261 (2007).

[14] H. Khan, T. Abdeljawad, C. Tunc, A. Alkhazzan and A. Khan, Minkowski's inequality for the AB-fractional integral operator, Journal of Inequalities and Applications **2019:96**, 1-12 (2019).

[15] J. Sousa and E. C. De Oliveira, The Minkowski's inequality by means of a generalized fractional integral, arXiv:1705.07191, 2017.

[16] M. Z. Sarikaya, Z. Dahmani, M. E. Kiris and F. Ahmad, (k,s)-Riemann-Liouville fractional integral and applications, Hacettepe Journal of Mathematics and Statistics **45** **(1)**, 77-89 (2016).

[17] S. Kilinc, H. Yildirim, Generalized fractional integral inequalities involving hypergeometic operators, International Journal of Pure and Applied Mathematics **101** **(1)**, 71-82 (2015).

[18] U. N. Katugampola, New fractional integral unifying six existing fractional integrals, arXiv.org/abs/1612.08596, (2016).

[19] U. N. Katugampola, New Approach Generalized Fractional Integral, Applied Math and Comp. **218**, 860-865 (2011).

[20] B. Benaissa, More on reverses of Minkowski's inequalities and Hardy's integral inequalities, Asian-European Journal of Mathematics **13(1)**, 1-7 (2020).

**Juan Gabriel Galeano Delgado** has an undergraduate degree in mathematics from the University of Cordoba-Colombia (2005), a master's degree in mathematical sciences National university of Colombia (2009) and doctorate in applied mathematics from the State University of Campinas Brazil (2016). He has research experience in mathematics, working in the following subjects: partial differential equations of fluid mechanics qualitatively and quantitatively,and fractional calculus.

**Juan E. Nápoles Valdés** has a bachelor degree of education, specializing in mathematics in 1983, studied two specialties and finished a doctorate in mathematical sciences in 1994, in Universidad de Oriente (Santiago de Cuba). In 1997 he was elected President of the Cuban Society of Mathematics and Computing until 1998 when it established residence in the Argentine Republic. He has directed postgraduate careers in Cuba and Argentina and held management positions at various universities, Cuban and Argentine. He has participated in various scientific conferences of Cuba, Argentina, Brazil and Colombia and published different works in scientific magazines specialized in the topics of qualitative theory of equations ordinary differentials, math education, problem solving, and history and philosophy of mathematics. For his teaching and research work, he has received several awards and distinctions, both in Cuba and Argentina.

**Edgardo Perez Reyes** earned his degree in mathematics, 2005, from the University of Cordoba (Colombia); Msc. degree in applied mathematics, 2010, from the University of Puerto Rico at Mayaguez; Ph.D. in applied mathematics, 2015, from the University of Sao Paulo (Brazil). Research interest: dynamical system, differential equations, fractional calculus and mathematical modeling. He is currently a full professor at the University of Sinu (Colombia).

**Miguel J. Vivas C.** earned his Ph.D. degree from Universidad Central de Venezuela, Caracas, Distrito Capital (2014) in the field of pure mathematics (nonlinear analysis), and earned his master degree in pure mathematics in the area of differential equations (ecological models). He has vast experience of teaching and research at university levels. It covers many areas of mathematical such as inequalities, bounded ariation functions and ordinary differential equations. He has written and published several research articles in reputed international journals and textbooks of mathematics. He was titular professor in Decanato de Ciencias y Tecnología of Universidad Centroccidental Lisandro Alvarado (UCLA), Barquisimeto, Lara state, Venezuela, and a guest professor in Facultad de Ciencias Naturales y Matemáticas from Escuela Superior Politécnica del Litoral (ESPOL), Guayaquil, Ecuador, and he is actually a principal professor and researcher in Pontificia Universidad Católica del Ecuador. Sede Quito, Ecuador.