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Closed Forms for Odd Moments of Squared Bessel Functions

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Abstract: We investigate the odd moments of squared Bessel functions, $S_{2n+1}(x)=\sum_{k=0}^{\infty}k^{2n+1}J_k^2(2x)$, and derive a systematic finite representation valid for arbitrary nonnegative integer (n). Using Bessel-function recurrence relations and a telescoping construction, we show that each odd moment can be expressed in terms of the three basic products $J_0^2(2x)$, $J_1^2(2x)$, $J_0(2x)J_1(2x)$, together with a terminal contribution involving $J_0(2x)J_{2n}(2x)+J_1(2x)J_{2n+1}(2x)$. The corresponding coefficient functions are polynomials in x^2 and are generated by a finite algebraic procedure based on a polynomial antidifference operator. We derive the associated difference equation, establish the polynomial character and uniqueness of its solutions, and obtain explicit coefficient formulas for the first several odd moments. The resulting construction reproduces the known expressions for the first and third moments and provides, in a unified way, closed finite representations for higher odd moments. This approach also yields a convenient algebraic framework for generating moments of arbitrarily high order without evaluating the original infinite series directly.

Keywords: Polynomials; generalized Bessel functions; special functions; nonlinear coherent states.

1 Introduction

Bessel functions of the first kind, $J_k(z)$, rank among the most widely used special functions of mathematical physics. They arise naturally whenever a problem has circular or cylindrical symmetry, and they play a central role in diffraction theory, in signal processing and in the quantum mechanics of periodic systems [1,2,3,4]. Of particular relevance here, the propagation of classical electromagnetic fields in weakly coupled optical waveguide arrays is described by discrete Schrödinger–type equations whose solutions are precisely the Bessel functions of integer order [5,6]. This fact has given rise to a fruitful analogy between classical light propagation and quantum electrodynamics: with suitably engineered optical lattices, the interaction between quantized light and matter can, under appropriate conditions, be emulated by purely classical means [7,8,9,10].

The Jaynes–Cummings model [11,12] is a prominent example. It and its generalisations (the Rabi model, the Dicke model, and anisotropic or driven variants, among others) can all be simulated experimentally with arrays of evanescently coupled waveguides [13,14,15,16,17,18,19,20,21,22]. The same is true of the atom–field interaction in which only the counter-rotating terms are retained, i.e., the anti-Jaynes–Cummings model [23,24], which can be modelled in this way too. Furthermore, time-dependent couplings, realised by bent or curved waveguide arrays, widen the scope of the analogy still more and make it possible to mimic non-autonomous quantum evolutions [25].

In the simplest geometry, an infinite array with constant nearest-neighbour coupling, the amplitudes of the propagating modes are given exactly by Bessel functions [5,6]. For a semi-infinite array the problem becomes considerably harder and usually calls for phase-operator techniques [26,27]. If, instead, light is allowed to hop to several neighbours, one is naturally led to generalised Bessel functions, which were studied extensively by Dattoli and co-workers [28,29,30] and, more recently, in the context of multi-site waveguide couplers [31].

A second motivation comes from quantum optics, where nonlinear coherent states whose expansion coefficients are Bessel functions have recently attracted attention [32,33,34]. Characterising the statistics of such states, for instance through the Mandel Q parameter [35], requires closed-form expressions for the moments of the number operator, that is,

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for sums of the type

$$\sum_{k=0}^{\infty} k^m J_k^2(2x). \tag{1}$$

For $m=0$ and $m=1$ the sums follow from standard Bessel identities and recurrence relations. The even sum (*i.e.*, the case k^{2m}) of square Bessel functions produces so-called Bessel moment polynomials [36]. The cases $m=2$ and $m=3$ can also be evaluated explicitly, but already for $m=3$ the result is a cumbersome combination of Bessel functions of several orders, and no clear pattern can be recognised for higher odd moments.

In this work we concentrate on the odd moments,

$$S_{2n+1}(x) = \sum_{k=0}^{\infty} k^{2n+1} J_k^2(2x), \qquad n=0,1,2,\ldots, \tag{2}$$

and show that each of them can be written in closed form in terms of only the products J_0^2, J_1^2 and J_0J_1, with polynomial coefficients, plus a short terminal term. These coefficients are generated by a finite sequence of elementary polynomial operations that start from a discrete telescoping identity; the construction is derived in Section 2, and Section 3 condenses it into a step-by-step recipe.

2 Telescoping construction and closed coefficient formulas

We now derive the coefficient polynomials directly from a discrete telescoping identity. This construction avoids the ambiguities that arise when the higher-order Bessel products are reduced too early.

Throughout this section we write

$$u = x^2, \qquad p = 2n+1, \qquad y_k = J_k(2x). \tag{3}$$

2.1 Telescoping identity

Consider the representation

$$k^p y_k^2 = W_{n,k+1} - W_{n,k}, \tag{4}$$

where

$$W_{n,k} = a_n(k) y_k^2 + b_n(k) y_k y_{k+1} + c_n(k) y_{k+1}^2. \tag{5}$$

Using

$$y_{k+2} = \frac{k+1}{x} y_{k+1} - y_k, \tag{6}$$

and equating the coefficients of y_k^2, $y_k y_{k+1}$, and y_{k+1}^2, one obtains

$$c_n(k+1) - a_n(k) = k^p, \tag{7}$$

$$b_n(k+1) + b_n(k) + \frac{2(k+1)}{x} c_n(k+1) = 0, \tag{8}$$

$$a_n(k+1) + \frac{k+1}{x} b_n(k+1) + \frac{(k+1)^2}{x^2} c_n(k+1) - c_n(k) = 0. \tag{9}$$

The polynomial solution is selected by requiring that $W_{n,k} \to 0$ as $k \to \infty$. Consequently,

$$S_p(x) = -\lim_{k\to 0} W_{n,k} = -W_{n,0}. \tag{10}$$

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2.2 The polynomial $Q_n(k,u)$

It is useful to introduce

$$\boxed{c_n(k) = -\frac{2u}{p} Q_n(k,u).} \tag{11}$$

The polynomial Q_n has degree n in u and degree $2n$ in k:

$$Q_n(k,u) = \sum_{r=0}^{n} q_{n,r}(k) u^r. \tag{12}$$

Eliminating $a_n(k)$ and $b_n(k)$ from Eqs. (7)–(9) gives the following exact difference equation for Q_n:

$$\begin{aligned}(k+1)Q_n(k+1,u) &- kQ_n(k,u) + u\,\mathscr{D}_k Q_n(k,u) \\ &= \frac{p}{2}\left[(k+1)^{p-1} + k^{p-1}\right],\end{aligned} \tag{13}$$

where

$$\mathscr{D}_k Q(k,u) = \frac{Q(k,u) - Q(k+2,u)}{k+1} + \frac{Q(k-1,u) - Q(k+1,u)}{k}. \tag{14}$$

Equation (14), together with polynomiality, uniquely determines Q_n.

2.3 Uniqueness of the polynomial solution

We show that the difference equation (14) has a unique solution in the polynomial ring $\mathbb{C}[k,u]$. Suppose that $H(k,u)$ satisfies the corresponding homogeneous equation

$$(k+1)H(k+1,u) - kH(k,u) + u\,\mathscr{D}_k H(k,u) = 0. \tag{15}$$

Assume that H is not identically zero. Let

$$H(k,u) = a_m(u)k^m + \cdots,$$

where $m = \deg_k H$ and $a_m(u) \neq 0$. Since

$$(k+1)H(k+1,u) - kH(k,u) = a_m(u)\big((k+1)^{m+1} - k^{m+1}\big) + \cdots,$$

the coefficient of k^m in this expression is

$$(m+1)a_m(u).$$

On the other hand,

$$u\,\mathscr{D}_k H(k,u)$$

has degree at most $m-1$ in k and therefore does not affect the coefficient of k^m. Hence the coefficient of k^m in the left-hand side of (15) is $(m+1)a_m(u)$. Since the left-hand side vanishes identically, we obtain

$$(m+1)a_m(u) = 0.$$

Because $m+1 \neq 0$, it follows that $a_m(u) = 0$, contradicting the choice of $a_m(u)$. Therefore the homogeneous equation has only the trivial polynomial solution

$$H(k,u) \equiv 0.$$

Now let Q and $\widetilde{Q}$ be two polynomial solutions of (14). Their difference

$$H = Q - \widetilde{Q}$$

satisfies (15), and hence $H \equiv 0$. Therefore

$$Q = \widetilde{Q}.$$

We conclude that (14) has a unique polynomial solution.

Writing

$$Q_n(k,u) = \sum_{r=0}^{n} q_{n,r}(k) u^r, \tag{16}$$

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gives the coefficient equations

$$(k+1)q_{n,0}(k+1)-kq_{n,0}(k)=\frac{p}{2}\left[(k+1)^{p-1}+k^{p-1}\right], \tag{17}$$

$$(k+1)q_{n,r}(k+1)-kq_{n,r}(k)=-\mathscr{D}_k q_{n,r-1}(k), \qquad r\geq 1. \tag{18}$$

The first coefficient has the explicit Bernoulli-polynomial form (for $n>0$)

$$q_{n,0}(k)=\frac{B_p(k+1)+B_p(k)-2B_p}{2k},\ p=2n+1, \tag{19}$$

where $B_p=B_p(0)$ is a Bernoulli number. For the exceptional case $n=0$, the Bernoulli-polynomial expression is not used; instead the solution is simply

$$q_{0,0}(k)=1, \qquad Q_0(k,u)=1. \tag{20}$$

More generally, define the inverse of the operator

$$\mathscr{L}f(k)=(k+1)f(k+1)-kf(k) \tag{21}$$

by

$$\mathscr{L}^{-1}g(k)=\frac{1}{k}\sum_{\ell=0}^{k-1}g(\ell). \tag{22}$$

Then

$$\boxed{q_{n,r}=\left(-\mathscr{L}^{-1}\mathscr{D}\right)^r q_{n,0}, \qquad r=0,\ldots,n.} \tag{23}$$

Thus the complete polynomial Q_n is determined by a finite sequence of polynomial antidifferences. For $n=0$, Eq. (23) is interpreted with $r=0$ and the initial value $q_{0,0}=1$ from Eq. (20). Since every operation in Eq. (23) maps polynomials into polynomials, this is also an algebraic construction of all the coefficients.

2.4 First examples of Q_n

The first four polynomials obtained from Eq. (14) are

$$Q_0(k,u)=1, \tag{24}$$

$$Q_1(k,u)=k^2+8u+\frac{1}{2}, \tag{25}$$

$$Q_2(k,u)=k^4+\frac{5}{3}k^2+\frac{16}{3}k^2u+\frac{128}{3}u^2+32u-\frac{1}{6}, \tag{26}$$

$$Q_3(k,u)=\boxed{\begin{aligned}&k^6+\frac{7}{2}k^4-\frac{7}{6}k^2+\frac{1}{6}\\&+\left(\frac{24}{5}k^4+\frac{160}{3}k^2+\frac{1388}{15}\right)u\\&+\left(\frac{128}{5}k^2+\frac{7744}{15}\right)u^2+\frac{1024}{5}u^3\end{aligned}}. \tag{27}$$

In particular, the coefficient of the highest power of u is

$$q_{n,n}=\frac{16^n(n!)^2}{(2n)!}. \tag{28}$$

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2.5 Coefficients in the reduced three-function representation

The telescoping construction gives a representation involving only J_0^2, J_1^2, and J_0J_1. Define

$$S_{2n+1}(x) = x^2\widetilde{A}_n(u)J_0^2(2x) + x^2\widetilde{B}_n(u)J_1^2(2x) + x^3\widetilde{C}_n(u)J_0(2x)J_1(2x). \tag{29}$$

From Eq. (10), together with Eqs. (7)–(9), one finds

$$\boxed{\widetilde{A}_n(u) = \frac{2}{p}Q_n(1,u),} \tag{30}$$

and

$$\boxed{\widetilde{B}_n(u) = \frac{2}{p}Q_n(0,u).} \tag{31}$$

For the coefficient of J_0J_1 one obtains

$$\boxed{\widetilde{C}_n(u) = \frac{1}{u}\left[1 - \frac{2}{p}Q_n(1,u) + \frac{2u}{p}\left(Q_n(2,u) - Q_n(0,u)\right)\right].} \tag{32}$$

For $n \geq 1$, the numerator in Eq. (32) is divisible by u, so that $\widetilde{C}_n$ is a polynomial of degree $n-1$. For $n = 0$, this divisibility fails and $\widetilde{C}_0(u) = -1/u$; this is the only non-polynomial case in the reduced representation.

For $n \geq 0$, write

$$\widetilde{A}_n(u) = \sum_{j=0}^{n} \widetilde{a}_{n,j}u^j, \qquad \widetilde{B}_n(u) = \sum_{j=0}^{n} \widetilde{b}_{n,j}u^j. \tag{33}$$

For $n \geq 1$, write

$$\widetilde{C}_n(u) = \sum_{j=0}^{n-1} \widetilde{c}_{n,j}u^j. \tag{34}$$

The coefficients are

$$\begin{aligned}
\widetilde{a}_{n,j} &= \frac{2}{2n+1}\,q_{n,j}(1), \\
\widetilde{b}_{n,j} &= \frac{2}{2n+1}\,q_{n,j}(0), \\
\widetilde{c}_{n,j} &= \frac{2}{2n+1}\left[q_{n,j}(2) - q_{n,j}(0) - q_{n,j+1}(1)\right], \\
j &= 0,\ldots,n-1.
\end{aligned} \tag{35}$$

The first two formulas hold for $n \geq 0$, while the third holds for $n \geq 1$. In the exceptional case $n = 0$,

$$\widetilde{A}_0(u) = \widetilde{B}_0(u) = 2, \qquad \widetilde{C}_0(u) = -\frac{1}{u}, \tag{36}$$

so $\widetilde{C}_0$ is not a polynomial and the expansion $\widetilde{C}_n = \sum_{j=0}^{n-1}\widetilde{c}_{n,j}u^j$ is empty for $n = 0$.

These formulas are the direct coefficient formulas generated by the telescoping identity.

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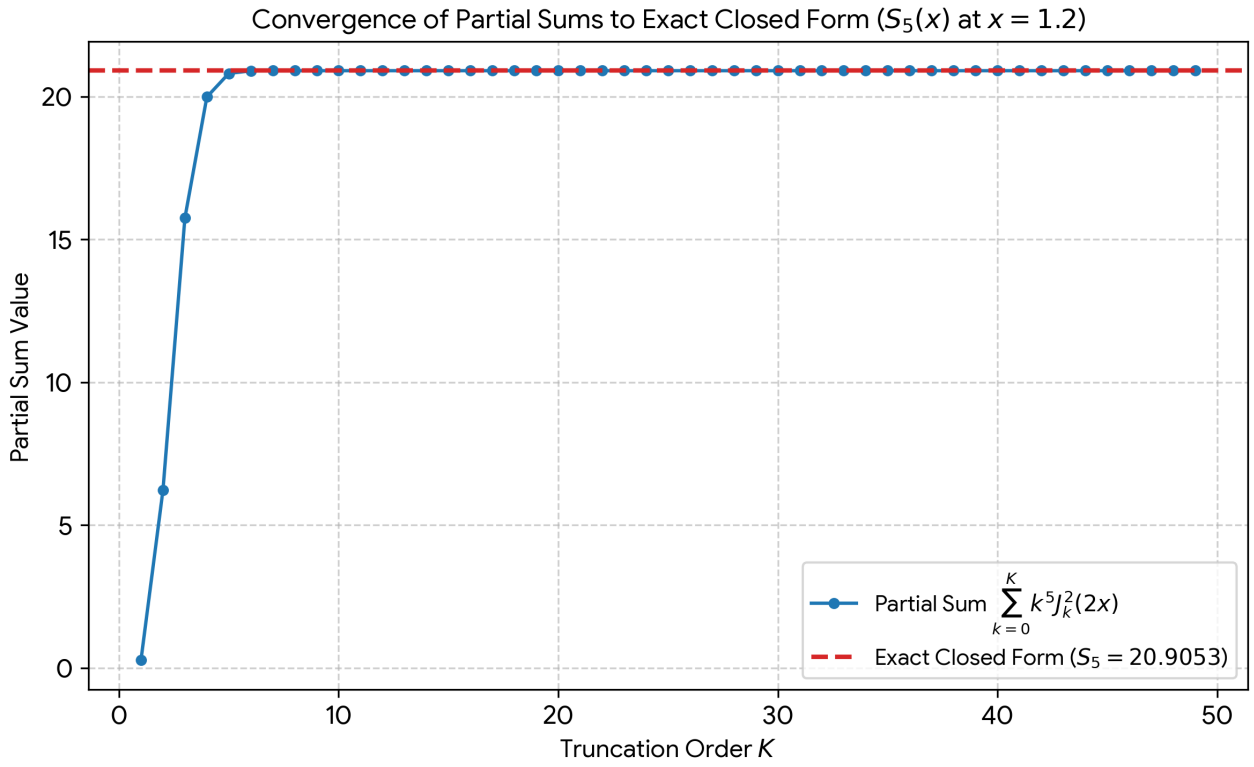


Fig. 1: Convergence of the truncated series $\sum_{k=0}^{K} k^5 J_k^2(2x)$ (dots) to the closed-form value of $S_5(x)$, Eq. (62) (dashed line), at $x = 1.2$, where $S_5 = 20.9053$, as a function of the truncation order K.

Figure 1 illustrates why a closed form is valuable. Since $J_k^2(2x)$ decays super-exponentially once k exceeds $2x$, the series converges quickly, but the number of terms needed still depends on x and on the moment. At $x = 1.2$ the relative error of the partial sum is about 4×10^{-2} for $K = 4$, 3×10^{-4} for $K = 6$ and 4×10^{-7} for $K = 8$. The closed form (62) gives the converged value at once, without truncation.

2.6 Restoring the terminal Bessel product

We now return to the preferred form

$$\begin{aligned} S_{2n+1}(x) = {} & x^2 A_n(u) J_0^2(2x) + x^2 B_n(u) J_1^2(2x) \\ & + x^3 C_n(u) J_0(2x) J_1(2x) \\ & + \frac{2x^{2n+2}}{2n+1} \left[J_0(2x) J_{2n}(2x) + J_1(2x) J_{2n+1}(2x) \right]. \end{aligned} \tag{37}$$

To convert between Eqs. (29) and (37), use the Bessel-order recurrence

$$J_{r+1}(2x) = \frac{r}{x} J_r(2x) - J_{r-1}(2x). \tag{38}$$

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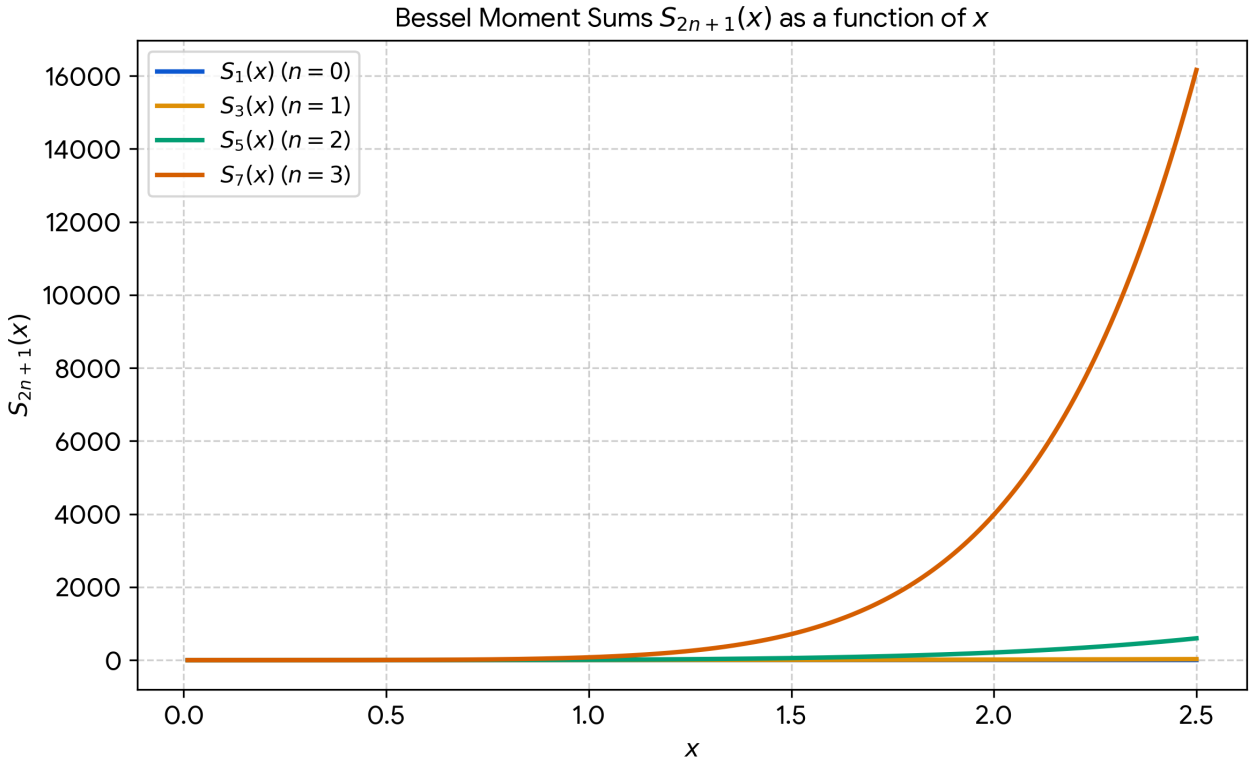


Fig. 2: Odd Bessel moments $S_{2n+1}(x)$ for $n = 0, 1, 2, 3$ (that is, S_1, S_3, S_5 and S_7) as functions of x for $0 \le x \le 2.5$.

Figure 2 shows the moments whose closed forms are given in Eqs. (55), (58), (62) and (66). All of them vanish at $x = 0$ and increase monotonically with x, and the growth becomes much steeper as n increases: at $x = 2.5$ one finds $S_1 \simeq 1.59$, $S_3 \simeq 26.9$, $S_5 \simeq 597$ and $S_7 \simeq 1.6 \times 10^4$. This is expected, because $J_k^2(2x)$ is appreciable only for $k \lesssim 2x$, so higher powers of k weigh the upper edge of this window more heavily. On the scale of the figure S_1 and S_3 are barely distinguishable from the axis. Define $P_r(x)$ and $R_r(x)$ by

$$J_r(2x) = P_r(x)J_0(2x) + R_r(x)J_1(2x), \tag{39}$$

with

$$P_0 = 1, \qquad P_1 = 0, \qquad P_{r+1} = \frac{r}{x}P_r - P_{r-1}, \tag{40}$$

$$R_0 = 0, \qquad R_1 = 1, \qquad R_{r+1} = \frac{r}{x}R_r - R_{r-1}. \tag{41}$$

Then

$$J_0J_{2n} + J_1J_{2n+1} = P_{2n}J_0^2 + R_{2n+1}J_1^2 + (R_{2n} + P_{2n+1})\,J_0J_1. \tag{42}$$

Introduce the three polynomials

$$\Pi_n(u) = u^n P_{2n}(x), \tag{43}$$

$$\mathscr{R}_n(u) = u^n R_{2n+1}(x), \tag{44}$$

$$\Sigma_n(u) = x^{2n-1}\left[R_{2n}(x) + P_{2n+1}(x)\right]. \tag{45}$$

All three are polynomials in u. For $n = 0$, $\Sigma_0 = 0$, while for $n \ge 1$, Σ_n has degree at most $n - 1$.

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The coefficients in Eq. (37) are therefore

$$\boxed{A_n(u) = \widetilde{A}_n(u) - \frac{2}{2n+1}\Pi_n(u),} \tag{46}$$

$$\boxed{B_n(u) = \widetilde{B}_n(u) - \frac{2}{2n+1}\mathscr{R}_n(u),} \tag{47}$$

$$\boxed{C_n(u) = \widetilde{C}_n(u) - \frac{2}{2n+1}\Sigma_n(u).} \tag{48}$$

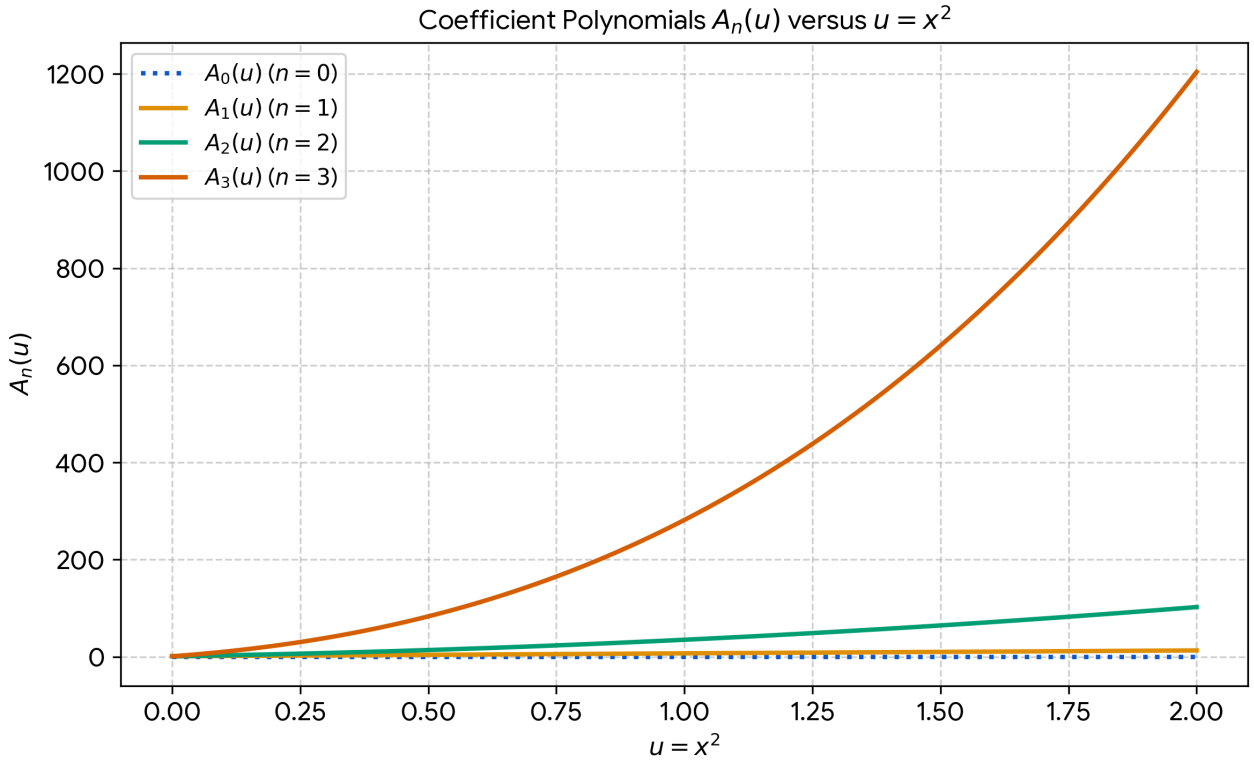


Fig. 3: Coefficient polynomials $A_n(u)$ of Eq. (46) for $n = 0, 1, 2, 3$ as functions of $u = x^2$ for $0 \leq u \leq 2$. The case $n = 0$ is identically zero (dotted line).

Figure 3 displays the polynomial coefficients of $J_0^2(2x)$ in the preferred form (37). The curve for $n = 0$ is the zero polynomial, $A_0 \equiv 0$, Eq. (54). For $n \geq 1$ the degree of A_n is n and, as the explicit cases in Section 2.7 show, $A_n(0) = 1$, so all curves start at the same point. The curves then separate rapidly, for instance $A_n(2) = 13$, 102.3 and 1204.2 for $n = 1, 2, 3$, reflecting the growth of the leading coefficients 6, $50/3$ and $294/5$. Consequently, for $n \geq 0$, write

$$\Pi_n(u) = \sum_{j=0}^{n} \pi_{n,j} u^j, \qquad \mathscr{R}_n(u) = \sum_{j=0}^{n} \rho_{n,j} u^j. \tag{49}$$

For $n \geq 1$, write

$$\Sigma_n(u) = \sum_{j=0}^{n-1} \sigma_{n,j} u^j, \tag{50}$$

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and for $n=0$ set $\Sigma_0=0$. Then the desired coefficient arrays are

$$a_{n,j}=\frac{2}{2n+1}\left[q_{n,j}(1)-\pi_{n,j}\right], \qquad j=0,\ldots,n, \tag{51}$$

$$b_{n,j}=\frac{2}{2n+1}\left[q_{n,j}(0)-\rho_{n,j}\right], j=0,\ldots,n, \tag{52}$$

$$c_{n,j}=\frac{2}{2n+1}\left[q_{n,j}(2)-q_{n,j}(0)-q_{n,j+1}(1)-\sigma_{n,j}\right], \quad j=0,\ldots,n-1. \tag{53}$$

Equations (51) and (52) hold for $n\geq 0$, while Eq. (53) holds for $n\geq 1$. For the exceptional case $n=0$, the coefficient array for C_0 is empty and

$$A_0(u)=B_0(u)=0, \qquad C_0(u)=-\frac{1}{u}. \tag{54}$$

Thus, for $n=0$, the preferred form (37) reads

$$S_1(x)=2x^2\left[J_0^2(2x)+J_1^2(2x)\right]-xJ_0(2x)J_1(2x). \tag{55}$$

Equations (51)–(53) are the corrected coefficient formulas for the normalization used in Eq. (37), with the explicit $n=0$ exception in Eq. (54).

2.7 Explicit checks

For $n=0$, Eqs. (46)–(48) give

$$A_0(u)=0, \qquad B_0(u)=0, \qquad C_0(u)=-\frac{1}{u}, \tag{56}$$

which yields Eq. (55). This illustrates the exceptional non-polynomial nature of C_0.

For $n=1$, Eqs. (46)–(48) give

$$A_1(u)=1+6u, \qquad B_1(u)=-1+6u, \qquad C_1(u)=-2. \tag{57}$$

Consequently,

$$\begin{aligned} S_3(x)&=x^2(1+6x^2)J_0^2(2x)+x^2(-1+6x^2)J_1^2(2x)\\ &-2x^3J_0(2x)J_1(2x)+\frac{2x^4}{3}\left[J_0(2x)J_2(2x)+J_1(2x)J_3(2x)\right]. \end{aligned} \tag{58}$$

For $n=2$ one obtains

$$A_2(u)=1+\frac{52}{3}u+\frac{50}{3}u^2, \tag{59}$$

$$B_2(u)=-\frac{29}{3}+20u+\frac{50}{3}u^2, \tag{60}$$

$$C_2(u)=\frac{4}{3}-\frac{28}{3}u. \tag{61}$$

Hence

$$\begin{aligned} S_5(x)&=x^2\left(1+\frac{52}{3}x^2+\frac{50}{3}x^4\right)J_0^2(2x)\\ &+x^2\left(-\frac{29}{3}+20x^2+\frac{50}{3}x^4\right)J_1^2(2x)\\ &+x^3\left(\frac{4}{3}-\frac{28}{3}x^2\right)J_0(2x)J_1(2x)\\ &+\frac{2x^6}{5}\left[J_0(2x)J_4(2x)+J_1(2x)J_5(2x)\right]. \end{aligned} \tag{62}$$

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For $n=3$ the result is

$$A_3(u)=1+\frac{232}{3}u+\frac{2168}{15}u^2+\frac{294}{5}u^3, \tag{63}$$

$$B_3(u)=-\frac{617}{3}+\frac{2968}{15}u+\frac{1904}{15}u^2+\frac{294}{5}u^3, \tag{64}$$

$$C_3(u)=\frac{484}{3}-\frac{1696}{15}u-\frac{142}{5}u^2. \tag{65}$$

Thus

$$\begin{aligned}S_7(x)=&\,x^2A_3(x^2)J_0^2(2x)+x^2B_3(x^2)J_1^2(2x)\\&+x^3C_3(x^2)J_0(2x)J_1(2x)\\&+\frac{2x^8}{7}\left[J_0(2x)J_6(2x)+J_1(2x)J_7(2x)\right].\end{aligned} \tag{66}$$

These expressions agree with the defining series

$$S_{2n+1}(x)=\sum_{k=0}^{\infty}k^{2n+1}J_k^2(2x). \tag{67}$$

3 Conclusions

We have shown that the odd Bessel moments $S_{2n+1}(x)=\sum_{k\geq 0}k^{2n+1}J_k^2(2x)$ are expressible through $J_0^2(2x)$, $J_1^2(2x)$ and $J_0(2x)J_1(2x)$ multiplied by polynomials in $u=x^2$, and that these polynomials A_n, B_n, C_n follow from a single auxiliary polynomial $Q_n(k,u)$. Everything needed to write down S_{2n+1} for a given n is collected in the following recipe (set $p=2n+1$ and $u=x^2$).

1. *Seed.* For $n\geq 1$ compute $q_{n,0}(k)$ from the Bernoulli polynomials, Eq. (19). (For $n=0$ take $Q_0=1$ and go directly to Step 5.)

2. *Auxiliary polynomial.* Obtain $q_{n,r}=(-\mathscr{L}^{-1}\mathscr{D})^r q_{n,0}$ for $r=1,\ldots,n$, Eq. (23), using $\mathscr{L}^{-1}g(k)=\frac{1}{k}\sum_{\ell=0}^{k-1}g(\ell)$, and assemble $Q_n(k,u)=\sum_{r=0}^{n}q_{n,r}(k)u^r$. This polynomial is unique, Section 2.3.

3. *Three evaluations.* Only $Q_n(0,u)$, $Q_n(1,u)$ and $Q_n(2,u)$ are needed. They give the reduced coefficients

$$\begin{aligned}\widetilde{A}_n&=\tfrac{2}{p}Q_n(1,u),\qquad \widetilde{B}_n=\tfrac{2}{p}Q_n(0,u),\\ \widetilde{C}_n&=\tfrac{1}{u}\left[1-\tfrac{2}{p}Q_n(1,u)+\tfrac{2u}{p}\big(Q_n(2,u)-Q_n(0,u)\big)\right],\end{aligned}$$

Eqs. (30)–(32).

4. *Result in three Bessel products.* Insert them in Eq. (29),

$$S_{2n+1}=x^2\widetilde{A}_nJ_0^2(2x)+x^2\widetilde{B}_nJ_1^2(2x)+x^3\widetilde{C}_nJ_0(2x)J_1(2x).$$

This is already a complete closed form.

5. *Optional terminal form.* To expose the terminal term of Eq. (37), generate P_r and R_r from the recurrences in Eqs. (40)–(41), form Π_n, $\mathscr{R}_n$, Σ_n from Eqs. (43)–(45), and subtract:

$$A_n=\widetilde{A}_n-\tfrac{2}{p}\Pi_n,\qquad B_n=\widetilde{B}_n-\tfrac{2}{p}\mathscr{R}_n,\qquad C_n=\widetilde{C}_n-\tfrac{2}{p}\Sigma_n,$$

Eqs. (46)–(48), so that

$$\begin{aligned}S_{2n+1}=&\,x^2A_nJ_0^2(2x)+x^2B_nJ_1^2(2x)+x^3C_nJ_0(2x)J_1(2x)\\&+\tfrac{2x^{2n+2}}{2n+1}\left[J_0(2x)J_{2n}(2x)+J_1(2x)J_{2n+1}(2x)\right].\end{aligned}$$

Only $n=0$ is special: there $\widetilde{A}_0=\widetilde{B}_0=2$, $\widetilde{C}_0=-1/u$, equivalently $A_0=B_0=0$ and $C_0=-1/u$, and the result is Eq. (55).

As an illustration, take $n=1$ ($p=3$). Step 1 gives $q_{1,0}=k^2+\frac{1}{2}$, and Step 2 gives $\mathscr{D}q_{1,0}=-8$, hence $q_{1,1}=8$ and $Q_1=k^2+8u+\frac{1}{2}$. In Step 3, $Q_1(0,u)=8u+\frac{1}{2}$, $Q_1(1,u)=8u+\frac{3}{2}$ and $Q_1(2,u)=8u+\frac{9}{2}$, so that

$$\widetilde{A}_1=1+\tfrac{16}{3}u,\qquad \widetilde{B}_1=\tfrac{1}{3}+\tfrac{16}{3}u,\qquad \widetilde{C}_1=-\tfrac{8}{3}.$$

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For Step 5, $P_2 = -1$, $R_3 = 2/x^2 - 1$ and $R_2 + P_3 = -1/x$, so $\Pi_1 = -u$, $\mathscr{R}_1 = 2 - u$ and $\Sigma_1 = -1$. Therefore

$$A_1 = 1 + 6u, \qquad B_1 = -1 + 6u, \qquad C_1 = -2,$$

which reproduces Eq. (58). The same procedure yields A_n, B_n, C_n for $n = 2, 3$ (Section 2.7) and any higher n without further ingenuity.

The construction is purely algebraic: every step is a finite polynomial operation (a Bernoulli polynomial, finite antidifferences, and evaluation at $k = 0, 1, 2$), so no pattern has to be guessed and no conjectured closed formula is required. As a built-in check, the coefficient of the highest power of u in Q_n is $16^n (n!)^2/(2n)!$, Eq. (28). These results supply the odd-moment ingredients needed for the number-operator statistics of the Bessel-type nonlinear coherent states mentioned in the Introduction.

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