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The Caputo-Fabrizio Fractional Integral to Generate Some New Inequalities

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Abstract: In this paper, we use the Caputo-Fabrizio (CF) fractional integral to establish some new integral inequalities in the case of functions with the same sense of variation.

Keywords: Caputo-Fabrizio fractional operators, Functions with the same sense of variation, Caputo-Fabrizio Fractional Integral Inequalities.

1 Introduction

The Fractional Calculus can be defined as a generalization of the integer-order differentiation. Its history goes back to seventeenth century, when in 1695 the derivative of order $\alpha = 1/2$ was described by Leibnitz in his letter to L'Hospital [25]. The Fractional Calculus has become more popular and useful due to its ability to describe some natural phenomena in numerous fields of engineering [21-24]. Moreover, it is well-known the importance of fractional calculus in obtaining fractional integral inequalities which are often used to prove the existence and uniqueness of fractional differential equations. In this line, there are several known forms of the fractional integrals which have been used to obtain fractional inequalities. The first is the Riemann-Liouville fractional integral [5]:

Theorem 1.*Let f,g and h be positive and continuous functions on $[0,\infty)$, such that*

$$\big(g(\tau)-g(\rho)\big)\left(\frac{f(\rho)}{h(\rho)}-\frac{f(\tau)}{h(\tau)}\right)\geq 0;\quad \tau,\rho\in[0,t],\quad t>0.$$

Then we have

$$\frac{J^{\alpha}\big(f(t)\big)}{J^{\alpha}\big(h(t)\big)}\geq\frac{J^{\alpha}\big(gf(t)\big)}{J^{\alpha}\big(gh(t)\big)},$$

for all $\alpha>0, t>0$.

Theorem 2.*Let f,g and h be positive and continuous functions on $[0,\infty)$, such that*

$$\big(g(\tau)-g(\rho)\big)\left(\frac{f(\rho)}{h(\rho)}-\frac{f(\tau)}{h(\tau)}\right)\geq 0;\quad \tau,\rho\in[0,t],\quad t>0.$$

Then for all $\alpha>0$, $w,t>0$, we have

$$\frac{J^{\alpha}\big(f(t)\big)\cdot J^{w}\big(gh(t)\big)+J^{w}\big(f(t)\big)\cdot J^{\alpha}\big(gh(t)\big)}{J^{\alpha}\big(h(t)\big)\cdot J^{w}\big(gf(t)\big)+J^{w}\big(h(t)\big)\cdot J^{\alpha}\big(gf(t)\big)}\geq 1.$$

Theorem 3.*Let f and h be two positive continuous functions and $f\leq h$ on $[0,\infty)$. If $\frac{f}{h}$ is decreasing and f is increasing on $[0,\infty)$, then for any $p\geq 1,\alpha>0,t>0$, the inequality*

$$\frac{J^{\alpha}\big(f(t)\big)}{J^{\alpha}\big(h(t)\big)}\geq\frac{J^{\alpha}\big(f^{p}(t)\big)}{J^{\alpha}\big(h^{p}(t)\big)},$$

is valid.

Theorem 4.*Let f and h be two positive continuous functions and $f\leq h$ on $[0,\infty)$. If $\frac{f}{h}$ is decreasing and f is increasing on $[0,\infty)$, then for any $p\geq 1,\alpha>0,w>0,t>0$, we have*

$$\frac{J^{\alpha}\big(f(t)\big)\cdot J^{w}\big(h^{p}(t)\big)+J^{w}\big(f(t)\big)\cdot J^{\alpha}\big(h^{p}(t)\big)}{J^{\alpha}\big(h(t)\big)\cdot J^{w}\big(f^{p}(t)\big)+J^{w}\big(h(t)\big)\cdot J^{\alpha}\big(f^{p}(t)\big)}\geq 1.$$

The second is the Hadamard fractional integral [8, 20]:

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Theorem 5. *Let f be an integrable function on $[1,\infty)$ such that there exist two integrable functions ϕ_1,ϕ_2 on $[1,\infty)$ with $\phi_1(t)\le f(t)\le \phi_2(t)$, $\forall t\in[1,\infty)$. Then, for $t>1$, $\alpha,\beta>0$, one has*

$$ {}^H I^{\beta}\phi_1(t)\cdot {}^H I^{\alpha} f(t)+{}^H I^{\alpha}\phi_2(t)\cdot {}^H I^{\beta} f(t) \ge {}^H I^{\alpha}\phi_2(t)\cdot {}^H I^{\beta}\phi_1(t)+{}^H I^{\alpha} f(t)\cdot {}^H I^{\beta} f(t). $$

Theorem 6. *Let f be an integrable function on $[1,\infty)$ and $p,q>0$ satisfying $1/p+1/q=1$. Suppose that there exist two integrable functions ϕ_1,ϕ_2 on $[1,\infty)$ such that $\phi_1(t)\le f(t)\le\phi_2(t)$ for all $t\in[1,\infty)$. Then, for $t>1,\alpha,\beta>0$, one has*

$$ \frac{1}{p}\cdot\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\cdot {}^H I^{\alpha}\left((\phi_2-f)^p\right)(t) + \frac{1}{q}\cdot\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}\cdot {}^H I^{\beta}\left((f-\phi_1)^{p_2}\right)(t) + {}^H I^{\alpha}\phi_2(t)\cdot {}^H I^{\beta}\phi_1(t)+{}^H I^{\alpha} f(t)\cdot {}^H I^{\beta} f(t) \ge {}^H I^{\alpha}\phi_2(t)\cdot {}^H I^{\beta} f(t)+{}^H I^{\alpha} f(t)\cdot {}^H I^{\beta}\phi_1(t). $$

Theorem 7. *Let f be an integrable function on $[1,\infty)$ and $p,q>0$ satisfying $p+q=1$. Suppose that there exist two integrable functions ϕ_1,ϕ_2 on $[1,\infty)$ such that $\phi_1(t)\le f(t)\le\phi_2(t)$ for all $t\in[1,\infty)$. Then, for $t>1,\alpha,\beta>0$, one has*

$$ p\cdot\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\cdot {}^H I^{\alpha}\phi_2(t)+q\cdot\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}\cdot {}^H I^{\beta} f(t) \ge {}^H I^{\alpha}(\phi_2-f)^p(t)\cdot {}^H I^{\beta}(f-\phi_2)^q(t) + p\cdot\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\cdot {}^H I^{\alpha} f(t)+q\cdot\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}\cdot {}^H I^{\beta}\phi_1(t). $$

Another kind of fractional integral that appears in integral inequalities is the Saigo fractional integral [3, 4, 12, 16, **??**]:

Theorem 8. *Let $p\ge 1$ and let f,g be two positive functions on $[0,\infty)$ such that for all $x>0$, $I_{0,x}^{\alpha,\beta,\eta}[f^p(x)]<\infty$, $I_{0,x}^{\alpha,\beta,\eta}[g^q(x)]<\infty$. If $0<m\le\frac{f(\tau)}{g(\tau)}\le M<\infty$, $\tau\in(0,x)$ we have*

$$ \left[I_{0,x}^{\alpha,\beta,\eta}[f^p(x)]\right]^{\frac{1}{p}}+\left[I_{0,x}^{\alpha,\beta,\eta}[g^q(x)]\right]^{\frac{1}{p}} \le \frac{1+M(m+2)}{(m+1)(M+1)}\cdot\left[I_{0,x}^{\alpha,\beta,\eta}[(f+g)^p(x)]\right]^{\frac{1}{p}} $$

for any $\alpha>\max\{0,-\beta\},\beta<1,\beta-1<\eta<0$.

Theorem 9. *Let $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and f,g be two positve functions on $[0,\infty)$ such that $I_{0,x}^{\alpha,\beta,\eta}[f(x)]<\infty$, $I_{0,x}^{\alpha,\beta,\eta}[g(x)]<\infty$. If $0<m\le\frac{f(\tau)}{g(\tau)}\le M<\infty$, $\tau\in[0,x]$, we have*

$$ \left[I_{0,x}^{\alpha,\beta,\eta}[f(x)]\right]^{\frac{1}{p}}+\left[I_{0,x}^{\alpha,\beta,\eta}[g(x)]\right]^{\frac{1}{q}} \le\left(\frac{M}{m}\right)^{\frac{1}{pq}}\cdot\left[I_{0,x}^{\alpha,\beta,\eta}\left[[f(x)]^{\frac{1}{p}}\cdot[g(x)]^{\frac{1}{q}}\right]\right], $$

for all $x>0,\alpha>\max\{0,-\beta\},\beta<1,\beta-1<\eta<0$.

Theorem 10. *Let f,g be two positive function on $[0,\infty)$ such that $I_{0,x}^{\alpha,\beta,\eta}[f^p(x)]<\infty$, $I_{0,x}^{\alpha,\beta,\eta}[g^q(x)]<\infty$, $x>0$. If $0<m\le\frac{[f(\tau)]^p}{[g(\tau)]^q}\le M<\infty$, $\tau\in[0,x]$. Then we have*

$$ \left[I_{0,x}^{\alpha,\beta,\eta}[f^p(x)]\right]^{\frac{1}{p}}+\left[I_{0,x}^{\alpha,\beta,\eta}[g^q(x)]\right]^{\frac{1}{q}} \le\left(\frac{M}{m}\right)^{\frac{1}{pq}}\cdot\left[I_{0,x}^{\alpha,\beta,\eta}[f(x)\cdot g(x)]\right], $$

where $p>1,\frac{1}{p}+\frac{1}{q}=1$, for all $x>0,\alpha>\max\{0,-\beta\},\beta<1,\beta-1<\eta<0$.

Theorem 11. *Let $f\ge 0,g\ge 0$ be two functions defined on $[0,\infty)$ such that g is non-decreasing. If*

$$ I_{0,x}^{\alpha,\beta,\eta}f(x)\ge I_{0,x}^{\alpha,\beta,\eta}g(x),\quad x>0, $$

then for all $\alpha>\max\{0,-\beta\},\beta<1,\beta-1<\eta<0,\gamma>0,\delta>0,\gamma-\delta>0$, we have

$$ I_{0,x}^{\alpha,\beta,\eta}f^{\gamma-\delta}(x)\le I_{0,x}^{\alpha,\beta,\eta}f^{\gamma}(x)g^{-\delta}(x). $$

Theorem 12. *Suppose that f,g and h be positive and continuous functions on $[0,\infty)$ such that*

$$ [g(\tau)-g(\rho)]\left(\frac{f(\rho)}{h(\rho)}-\frac{f(\tau)}{h(\tau)}\right)\ge 0;\quad \tau,\rho\in[0,x),x>0. $$

Then for all $x>0,\alpha>\max\{0,-\beta\},\beta<1,\beta-1<\eta<0$, we have

$$ \frac{I_{0,x}^{\alpha,\beta,\eta}[f(x)]}{I_{0,x}^{\alpha,\beta,\eta}[h(x)]}\ge\frac{I_{0,x}^{\alpha,\beta,\eta}[(gf)(x)]}{I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)]}. $$

Theorem 13. *Suppose that f,g and h be positive and continuous functions on $[0,\infty)$ such that*

$$ [g(\tau)-g(\rho)]\left(\frac{f(\rho)}{h(\rho)}-\frac{f(\tau)}{h(\tau)}\right)\ge 0;\quad \tau,\rho\in(0,t),t>0. $$

Then for all $x>0,\alpha>\max\{0,-\beta\},\psi>\max\{0,-\phi\},\beta<1,\beta-1<\eta<0,\phi<1,\phi-1<\xi<0$, we have

$$ \frac{I_{0,x}^{\alpha,\beta,\eta}[f(x)]I_{0,x}^{\psi,\phi,\xi}[(gh)(x)]+I_{0,x}^{\psi,\phi,\xi}[f(x)]I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)]}{I_{0,x}^{\alpha,\beta,\eta}[h(x)]I_{0,x}^{\psi,\phi,\xi}[(gf)(x)]+I_{0,x}^{\psi,\phi,\xi}[h(x)]I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)]}\ge 1. $$

In 2015, Caputo and Fabrizio introduced a new fractional derivative [13]. The interest for this approach was due to the necessity of using a model describing structures with different scales [13]. In the literature there is no evidence of the use of the Caputo fractional derivative and Caputo-Fabrizio fractional integral to obtain fractional integral inequalities. The propose of this paper is to use the CF fractional integral to establish some new integral inequalities which have been obtained before by using the Riemann-Liouville, Hadamard and Saigo fractional operators [1, 5, 9]. The obtained Caputo-Fabrizio fractional inequalities could be helpful to prove the existence and uniqueness of some ordinary Caputo-Fabrizio fractional differential equations. The paper has been organized as follows, in Section 2, we define basic definitions related to fractional integrals. In Section 3, appears the main results. Finally, conclusion is summarized in Section 4.

2 Preliminaries

Here we give the following definitions:

Definition 1. *Let $\alpha \in \mathbb{R}$ such that $0<\alpha<1$. The Caputo-Fabrizio fractional integral of order α of a function f is defined by [10]*

$$I_{0t}^{\alpha} f(t) = \frac{1}{\alpha}\int_0^t e^{-\frac{1-\alpha}{\alpha}(t-s)} f(s)ds.$$

Definition 2. *Let $\alpha, a \in \mathbb{R}$ such that $0<\alpha<1$. The Caputo-Fabrizio fractional derivative of order α of a function f is defined by*

$$D_{at}^{\alpha} f(t) = \frac{1}{1-\alpha}\int_a^t e^{-\frac{\alpha}{1-\alpha}(t-s)} f'(s)ds.$$

Definition 3. *Let $\alpha>0, \beta, \eta \in \mathbb{R}$, then the Saigo fractional integral $I_{0,t}^{\alpha,\beta,\eta}[f(t)]$ of order α for a real-valued continuous function $f(t)$ is defined by*

$$I_{0,t}^{\alpha,\beta,\eta}\{f(t)\} = \frac{t^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} F_1(\alpha+\beta, -\eta; \alpha; 1-\frac{\tau}{t}) f(\tau)d\tau,$$

where the function $F_1(-)$ is the Gaussian hypergeometric function defined by

$$F_1(a,b;c;t) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{t^n}{n!},$$

and $(a)_n$ is the Pochhammer symbol

$$(a)_n = a(a+1)\cdots(a+n-1), (a)_0 = 1.$$

Definition 4. *The Hadamard fractional integral is defined by*

$${}^H I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} f(\tau)\frac{d\tau}{\tau}$$

for $Re(\alpha)>0$, $t>1$.

Definition 5. *The Riemann-Liouville fractional integral is defined by*

$${}^{RL}I_{0t}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau.$$

Definition 6. *We say that two functions f and g have the same sense of variation (synchronous) on $[0,\infty)$ if [16]*

$$(f(\tau)-f(\rho))(g(\tau)-g(\rho)) \ge 0 \quad \tau,\rho \in (0,t), t>0.$$

For more details, see [1, 6, 7, 8, 11-15, 17-20].

3 Main Results

Here, we give some interesting inequalities concerning the Caputo-Fabrizio fractional integral.

Theorem 14. *Let p be a positive function on $[0,\infty)$ and let f and g be two differentiable functions having the same sense of variation on $[0,\infty)$. If $f', g' \in L_\infty([0,\infty))$, then for all $t>0, \alpha \in (0,1)$, we have*

$$\begin{aligned} 0 &\le I_{0t}^{\alpha} p(t) I_{0t}^{\alpha} pfg(t) - I_{0t}^{\alpha} pf(t)\cdot I_{0t}^{\alpha} pg(t) \\ &\le \|f'\|_{L_\infty}\cdot\|g'\|_{L_\infty}\cdot\left[I_{0t}^{\alpha} p(t)\cdot I_{0t}^{\alpha} t^2 p(t) - \left(I_{0t}^{\alpha} t p(t)\right)^2\right]. \end{aligned} \quad (1)$$

Proof. Let f and g be two functions satisfying the conditions of theorem 14 and let p be a positive function on $[0,\infty)$. Define

$$H(\tau,\rho) := \left(f(\tau)-f(\rho)\right)\left(g(\tau)-g(\rho)\right) \quad (2)$$

for all $\tau,\rho \in (0,t)$. We have

$$H(\tau,\rho) \ge 0. \quad (3)$$

Thanks to (3), we have

$$\begin{aligned} &\frac{1}{2\alpha^2}\int_0^t\int_0^t e^{-\frac{1-\alpha}{\alpha}(t-\tau)}\cdot e^{-\frac{1-\alpha}{\alpha}(t-\rho)}. \\ &\cdot p(\tau)p(\rho)H(\tau,\rho)d\tau d\rho \\ &= I_{0t}^{\alpha} p(t)\cdot I_{0t}^{\alpha} pfg(t) - I_{0t}^{\alpha} pf(t)\cdot I_{0t}^{\alpha} pg(t) \ge 0. \end{aligned} \quad (4)$$

The relation (3) can be written as follows:

$$H(\tau,\rho) = \int_\tau^\rho\int_\tau^\rho f'(y)g'(z)dydz. \quad (5)$$

Hence, we can write

$$H(\tau,\rho) \leq \left| \int_\tau^\rho f'(y)dy \right| \left| \int_\tau^\rho g'(z)dz \right| \leq \|f'\|_{L^\infty} \cdot \|g'\|_{L^\infty} \cdot (\tau-\rho)^2. \tag{6}$$

Therefore

$$\frac{1}{2\alpha^2} \int_0^t \int_0^t e^{-\frac{1-\alpha}{\alpha}(t-\tau)} \cdot e^{-\frac{1-\alpha}{\alpha}(t-\rho)} \cdot p(\tau)p(\rho)H(\tau,\rho)d\tau d\rho \leq \frac{\|f'\|_{L^\infty}\|g'\|_{L^\infty}}{2\alpha^2} \int_0^t \int_0^t e^{-\frac{1-\alpha}{\alpha}(t-\tau)} e^{-\frac{1-\alpha}{\alpha}(t-\rho)} \cdot (\tau^2 - 2\tau\rho + \rho^2)p(\tau)p(\rho)d\tau d\rho. \tag{7}$$

Consequently

$$\frac{1}{2\alpha^2} \int_0^t \int_0^t e^{-\frac{1-\alpha}{\alpha}(t-\tau)} \cdot e^{-\frac{1-\alpha}{\alpha}(t-\rho)} \cdot p(\tau)p(\rho)H(\tau,\rho)d\tau d\rho \leq \|f'\|_{L_\infty} \cdot \|g'\|_{L_\infty} \cdot \left[I_{0t}^\alpha p(t) \cdot I_{0t}^\alpha t^2 p(t) - \left(I_{0t}^\alpha t p(t)\right)^2 \right]. \tag{8}$$

According to (4) and (8), we get (1), and thus theorem 14 is proved.

Now, we are ready to generalize Theorem 14 to the following theorem:

Theorem 15. *Let p be a positive function on $[0,\infty)$ and let f and g be two differentiable functions having the same sense of variation on $[0,\infty)$. If $f', g' \in L_\infty([0,\infty))$, then for all $t>0, \alpha,\beta \in (0,1)$, we have*

$$0 \leq I_{0t}^\alpha pfg(t) \cdot I_{0t}^\beta p(t) + I_{0t}^\alpha p(t) \cdot I_{0t}^\beta pfg(t) - I_{0t}^\alpha pf(t) \cdot I_{0t}^\beta pg(t) - I_{0t}^\alpha pg \cdot I_{0t}^\beta pf \leq \|f'\|_{L^\infty} \cdot \|g'\|_{L^\infty} \{ I_{0t}^\alpha t^2 p(t) \cdot I_{0t}^\beta p(t) + I_{0t}^\alpha p(t) \cdot I_{0t}^\beta t^2 p(t) - 2 I_{0t}^\alpha t p(t) \cdot I_{0t}^\beta t p(t) \} \tag{9}$$

Proof. It is easy to see that

$$\frac{1}{1-\alpha} \cdot \frac{1}{1-\beta} \int_0^t \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} \cdot e^{-\frac{\beta}{1-\beta}(t-\rho)} p(\tau)p(\rho)H(\tau,\rho)d\tau d\rho = I_{0t}^\alpha pfg(t) \cdot I_{0t}^\beta p(t) + I_{0t}^\alpha p(t) \cdot I_{0t}^\beta pfg(t) - I_{0t}^\alpha pf(t) \cdot I_{0t}^\beta pg(t) - I_{0t}^\alpha pg(t) \cdot I_{0t}^\beta pf(t) \geq 0. \tag{10}$$

From the relation (6), we obtain the following estimate

$$\frac{1}{1-\alpha} \frac{1}{1-\beta} \int_0^t \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} \cdot e^{-\frac{\beta}{1-\beta}(t-\rho)} \cdot p(\tau)p(\rho)H(\tau,\rho)d\tau d\rho \leq \|f'\|_{L_\infty} \cdot \|g'\|_{L_\infty} \Big[I_{0t}^\alpha t^2 p(t) \cdot I_{0t}^\beta p(t) + I_{0t}^\alpha p(t) \cdot I_{0t}^\beta t^2 p(t) - 2 I_{0t}^\alpha t p(t) \cdot I_{0t}^\beta t p(t) \Big]. \tag{11}$$

Combining (10) with (11), inequality (9) follows.

Theorem 16. *Let f,g and h be positive and continuous functions on $[0,\infty)$, such that*

$$\Big(g(\tau) - g(\rho)\Big)\left(\frac{f(\rho)}{h(\rho)} - \frac{f(\tau)}{h(\tau)}\right) \geq 0; \quad \tau,\rho \in [0,t], t>0, \tag{12}$$

then we have

$$\frac{I_{0t}^\alpha f(t)}{I_{0t}^\alpha h(t)} \geq \frac{I_{0t}^\alpha gf(t)}{I_{0t}^\alpha gh(t)} \tag{13}$$

for any $\alpha \in (0,1), t>0$.

Proof. Suppose that f,g and h are positive and continuous functions on $[0,\infty)$. Using (12), we can write

$$g(\tau) \cdot \frac{f(\rho)}{h(\rho)} + g(\rho) \cdot \frac{f(\tau)}{h(\tau)} - g(\rho) \cdot \frac{f(\rho)}{h(\rho)} - g(\tau) \cdot \frac{f(\tau)}{h(\tau)} \geq 0$$

for all $\tau,\rho \in [0,t], t>0$. That is

$$g(\tau)f(\rho)h(\tau) + g(\rho)f(\tau)h(\rho) - g(\rho)f(\rho)h(\tau) - g(\tau)f(\tau)h(\rho) \geq 0 \tag{14}$$

for all $\tau,\rho \in [0,t], t>0$.

Now, multiplying both sides of (14) by $\frac{1}{\alpha^2} \cdot e^{-\frac{1-\alpha}{\alpha}(t-\tau)} \cdot e^{-\frac{1-\alpha}{\alpha}(t-\rho)}$, then integrating the resulting inequality with respect to τ and ρ over $(0,t)\times(0,t)$, we get

$$I_{0t}^\alpha gh(t) \cdot I_{0t}^\alpha f(t) - I_{0t}^\alpha gf(t) \cdot I_{0t}^\alpha h(t) \geq 0 \tag{15}$$

From (15), inequality (13) follows.

Theorem 17. *Let f,g and h be positive and continuous functions on $[0,\infty)$, such that*

$$\Big(g(\tau) - g(\rho)\Big)\left(\frac{f(\rho)}{h(\rho)} - \frac{f(\tau)}{h(\tau)}\right) \geq 0; \quad \tau,\rho \in [0,t], t>0, \tag{16}$$

then for all $\alpha,\beta \in (0,1), t>0$, we have

$$\frac{I_{0t}^\alpha gh(t) \cdot I_{0t}^\beta f(t) + I_{0t}^\beta gh(t) \cdot I_{0t}^\alpha f(t)}{I_{0t}^\alpha h(t) \cdot I_{0t}^\beta gf(t) + I_{0t}^\alpha gf(t) \cdot I_{0t}^\beta h(t)} \geq 1 \tag{17}$$

Proof. Suppose that f,g and h are positive and continuous functions on $[0,\infty)$. Now, multiplying both sides of (14) by $\frac{1}{\alpha \cdot \beta} \cdot e^{-\frac{1-\alpha}{\alpha}(t-\tau)} \cdot e^{-\frac{1-\beta}{\beta}(t-\rho)}$, then integrating the resulting inequality with respect to τ and ρ over $(0,t)\times(0,t)$, we get

$$I_{0t}^\alpha gh(t) \cdot I_{0t}^\beta f(t) + I_{0t}^\beta gh(t) \cdot I_{0t}^\alpha f(t) \geq I_{0t}^\alpha h(t) \cdot I_{0t}^\beta gf(t) + I_{0t}^\alpha gf(t) \cdot I_{0t}^\beta h(t).$$

Hence, we obtain (17).

Note 1. It is obviously to see that (16) holds true for f,g,h such that either (i)g is increasing and $\frac{f}{h}$ is decreasing

or

(ii)g is decreasing and $\frac{f}{h}$ is increasing.

Theorem 18. *Let f and h be two positive continuous functions and $f \leq h$ on $[0,\infty)$. If $\frac{f}{h}$ is decreasing and f is increasing on $[0,\infty)$, then for any $p \geq 1$, $\alpha \in (0,1), t > 0$, the inequality*

$$\frac{I_{0t}^{\alpha} f(t)}{I_{0t}^{\alpha} h(t)} \geq \frac{I_{0t}^{\alpha} f^p(t)}{I_{0t}^{\alpha} h^p(t)}, \tag{18}$$

is valid.

Proof. Since $p \geq 1$ and f is increasing, then $g := f^{p-1}$ is also increasing. Then by applying the theorem 16, we get

$$\frac{I_{0t}^{\alpha} f(t)}{I_{0t}^{\alpha} h(t)} \geq \frac{I_{0t}^{\alpha}(f \cdot f^{p-1}(t))}{I_{0t}^{\alpha}(h \cdot f^{p-1}(t))}, \tag{19}$$

now, since $f \leq h$ on $(0,\infty]$, then

$$\frac{1}{\alpha} \cdot e^{-\frac{1-\alpha}{\alpha}(t-\tau)} \cdot h(\tau) \cdot f^{p-1}(\tau) \leq \frac{1}{\alpha} \cdot e^{-\frac{1-\alpha}{\alpha}(t-\tau)} \cdot h^p(\tau) \tag{20}$$

for all $\tau \in [0,t], t > 0$. Integrating both sides of (20) with respect to τ over $(0,t)$, yields

$$I_{0t}^{\alpha}\left(hf^{p-1}(t)\right) \leq I_{0t}^{\alpha}\left(h^p(t)\right). \tag{21}$$

Consequently

$$\frac{I_{0t}^{\alpha}(f \cdot f^{p-1}(t))}{I_{0t}^{\alpha}(h \cdot f^{p-1}(t))} = \frac{I_{0t}^{\alpha} f^p(t)}{I_{0t}^{\alpha}(h \cdot f^{p-1}(t))} \geq \frac{I_{0t}^{\alpha} f^p(t)}{I_{0t}^{\alpha} h^p(t)}. \tag{22}$$

Combining (19) with (22), we obtain (18).

Theorem 19. *Let f and h be two positive and continuous functions and $f \leq h$ on $[0,\infty)$. If $\frac{f}{h}$ is decreasing and f is increasing on $[0,\infty)$, then for any $p \geq 1$, $\alpha, \beta \in (0,1), t > 0$, we have*

$$\frac{I_{0t}^{\beta} f(t) \cdot I_{0t}^{\alpha} h^p(t) + I_{0t}^{\alpha} f(t) \cdot I_{0t}^{\beta} h^p(t)}{I_{0t}^{\alpha} h(t) \cdot I_{0t}^{\beta} f^p(t) + I_{0t}^{\beta} h(t) \cdot I_{0t}^{\alpha} f^p(t)} \geq 1. \tag{23}$$

Proof. Taking $g := f^{p-1}$, then by theorem 17, yields

$$\frac{I_{0t}^{\alpha}[h \cdot f^{p-1}(t)] \cdot I_{0t}^{\beta} f(t) + I_{0t}^{\alpha} f(t) \cdot I_{0t}^{\beta}[h \cdot f^{p-1}(t)]}{I_{0t}^{\alpha} h(t) \cdot I_{0t}^{\beta} f^p(t) + I_{0t}^{\beta} h(t) \cdot I_{0t}^{\alpha} f^p(t)} \geq 1. \tag{24}$$

Using the fact that $f \leq h$ on $[0,\infty)$, we can write

$$\frac{1}{\beta} e^{-\frac{1-\beta}{\beta}(t-\rho)} \cdot h \cdot f^{p-1}(\rho) \leq \frac{1}{\beta} \cdot e^{-\frac{1-\beta}{\beta}(t-\rho)} \cdot h^p(\rho) \tag{25}$$

for all $\rho \in [0,t], t > 0$. Integrating both sides of (25) with respect to ρ over $(0,t)$, we obtain

$$I_{0t}^{\beta} h f^{p-1}(t) \leq I_{0t}^{\beta} h^p(t). \tag{26}$$

Multiplying (21) by $I_{0t}^{\beta} f(t)$ and (26) by $I_{0t}^{\alpha} f(t)$, we can write

$$\begin{aligned} &I_{0t}^{\beta} f(t) \cdot I_{0t}^{\alpha}\left(hf^{p-1}(t)\right) + I_{0t}^{\alpha} f(t) \cdot I_{0t}^{\beta}\left(hf^{p-1}(t)\right) \\ &\leq I_{0t}^{\beta} f(t) \cdot I_{0t}^{\alpha}\left(h^p(t)\right) + I_{0t}^{\alpha} f(t) \cdot I_{0t}^{\beta}\left(hp(t)\right). \end{aligned} \tag{27}$$

Now, using (24) and (27), we deduce (23).

Theorem 20. *Let $\alpha \in (0,1)$, $p \geq 1$ and f, g be two positive continuous functions on $[0,\infty)$. If $0 < m \leq \dfrac{f(\tau)}{g(\tau)} \leq M$, $\tau \in (0,t)$, then we have*

$$\begin{aligned} &\left[I_{0t}^{\alpha} f^p(t)\right]^{1/p} + \left[I_{0t}^{\alpha} g^p(t)\right]^{1/p} \\ &\leq \frac{M(m+2)+1}{(M+1)(m+1)} \left[I_{0t}^{\alpha}(f+g)^p(t)\right]^{1/p}. \end{aligned} \tag{28}$$

Proof. Using the condition $\dfrac{f(\tau)}{g(\tau)} \leq M, \tau \in (0,t), t > 0$, we can write

$$(M+1)^p \cdot f^p(\tau) \leq M^p \cdot (f+g)^p(\tau). \tag{29}$$

Multiplying both sides of (29) by $\frac{1}{\alpha} \cdot e^{-\frac{1-\alpha}{\alpha}(t-\tau)}$, then integrating resulting identity with respect to τ from 0 to x, we get

$$(M+1)^p \cdot I_{0t}^{\alpha} f^p(t) \leq M^p \cdot I_{0t}^{\alpha}(f+g)^p(t)$$

Hence, we can write

$$\left[I_{0t}^{\alpha} f^p(t)\right]^{1/p} \leq \frac{M}{M+1} \cdot \left[I_{0t}^{\alpha}(f+g)^p(t)\right]^{1/p}, \tag{30}$$

on the other hand, using condition $m \leq \frac{f(\tau)}{g(\tau)}$, we obtain

$$\left(1+\frac{1}{m}\right) g(\tau) \leq \frac{1}{m}\left(f(\tau)+g(\tau)\right),$$

therefore

$$\left(1+\frac{1}{m}\right)^p g^p(\tau) \leq \left(\frac{1}{m}\right)^p \left(f(\tau)+g(\tau)\right)^p. \tag{31}$$

Now, multiplying both sides of (31) by $\frac{1}{\alpha} \cdot e^{-\frac{1-\alpha}{\alpha}(t-\tau)}$, then integrating resulting identity with respect to τ from 0 to t, we have

$$\left[I_{0t}^{\alpha} g^p(t)\right]^{1/p} \leq \frac{1}{m+1} \cdot \left[I_{0t}^{\alpha}(f+g)^p(t)\right]^{1/p}. \tag{32}$$

The inequality (28) follows on adding the inequalities (30) and (32)

Theorem 21. *Let $\alpha \in (0,1), p > 1, \frac{1}{p} + \frac{1}{q} = 1$ and f, g be two positive and continuous functions on $[0,\infty)$. If*

$$0 < m \leq \frac{f(\tau)}{g(\tau)} \leq M < \infty, \quad \tau \in [0,t], \tag{33}$$

then the inequality

$$\left[I_{0t}^{\alpha}f(t)\right]^{1/p}\cdot\left[I_{0t}^{\alpha}g(t)\right]^{1/q} \leq \left(\frac{M}{m}\right)^{1/pq}\cdot I_{0t}^{\alpha}\left[f^{1/p}(t)\cdot g^{1/q}(t)\right] \tag{34}$$

holds.

*Proof.*Since $\frac{f(\tau)}{g(\tau)}\leq M, \tau\in[0,t], t>0$, therefore

$$\left[g(\tau)\right]^{1/q}\geq M^{-1/q}\cdot[f(\tau)]^{1/q},$$

and so,

$$\left[f(\tau)\right]^{1/q}\cdot\left[g(\tau)\right]^{1/q}\geq M^{-1/q}\cdot[f(\tau)]^{1/q}\cdot\left[f(\tau)\right]^{1/q} = M^{-1/q}\cdot f(\tau). \tag{35}$$

Multiplying both sides of (35) by $\frac{1}{\alpha}\cdot e^{-\frac{1-\alpha}{\alpha}(t-\tau)}$, then integrating resulting identity with respect to τ from 0 to t, we have

$$I_{0t}^{\alpha}\left[f^{1/p}(t)\cdot g^{1/q}(t)\right]\geq M^{-1/q}\cdot I_{0t}^{\alpha}f(t),$$

and consequently

$$\left(I_{0t}^{\alpha}\left[f^{1/p}(t)\cdot g^{1/q}(t)\right]\right)^{1/p}\geq M^{-1/pq}\cdot\left(I_{0t}^{\alpha}f(t)\right)^{1/p}. \tag{36}$$

On the other hand, since $mg(\tau)\leq f(\tau), \tau\in[0,t], t>0$, then we have

$$\left[f(\tau)\right]^{1/p}\geq m^{1/p}\cdot[g(\tau)]^{1/p},$$

and so,

$$\left[g(\tau)\right]^{1/q}\cdot\left[f(\tau)\right]^{1/p}\geq m^{1/p}\cdot[g(\tau)]^{1/p}\cdot\left[g(\tau)\right]^{1/q} = m^{1/p}\cdot g(\tau). \tag{37}$$

Now, multiplying both sides of (35) by $\frac{1}{\alpha}\cdot e^{-\frac{1-\alpha}{\alpha}(t-\tau)}$, then integrating resulting identity with respect to τ over $(0,t)$, we obtain

$$I_{0t}^{\alpha}\left[g^{1/q}(t)\cdot f^{1/p}(t)\right]\geq m^{1/p}\cdot I_{0t}^{\alpha}g(t).$$

Hence, it follows

$$\left(I_{0t}^{\alpha}\left[g^{1/q}(t)\cdot f^{1/p}(t)\right]\right)^{1/q}\geq m^{1/pq}\cdot\left(I_{0t}^{\alpha}g(t)\right)^{1/q}. \tag{38}$$

Thanks to (36) and (38), we obtain (34).

Theorem 22.*Let* $0<\alpha<1, p>1, \frac{1}{p}+\frac{1}{q}=1$, *f and g be two positive and continuous functions on* $[0,\infty)$. *If*

$$0<m\leq\frac{f^p(\tau)}{g^q(\tau)}\leq M<\infty,\quad \tau\in[0,t],$$

then we have

$$\left[I_{0t}^{\alpha}f^p(t)\right]^{1/p}\cdot\left[I_{0t}^{\alpha}g^q(t)\right]^{1/q}\leq\left(\frac{M}{m}\right)^{1/pq}\cdot I_{0t}^{\alpha}[f(t)\cdot g(t)]. \tag{39}$$

*Proof.*Replacing $f(\tau)$ and $g(\tau)$ respectively by $\left(f(\tau)\right)^p$ and $\left(g(\tau)\right)^q, \tau\in[0,t], t>0$ in theorem 21, we obtain (39).

Theorem 23.*Let* $0<\alpha<1, p>1, \frac{1}{p}+\frac{1}{q}=1$, *f and g be two positive and continuous functions on* $[0,\infty)$. *If*

$$0<m\leq\frac{f(\tau)}{g(\tau)}\leq M<\infty, \tag{40}$$

then we have

$$I_{0t}^{\alpha}\left(\frac{f^p(t)}{g^{p/q}(t)}\right)\leq\left(\frac{M}{m}\right)^{1/q}\cdot\frac{\left(I_{0t}^{\alpha}f(t)\right)^p}{\left(I_{0t}^{\alpha}g(t)\right)^{p/q}}. \tag{41}$$

*Proof.*Using theorem 21, we obtain

$$I_{0t}^{\alpha}f(t)=I_{0t}^{\alpha}\left[\left(\frac{f^p(t)}{g^{p/q}(t)}\right)^{1/p}\cdot\left(g(t)\right)^{1/q}\right] \geq\left(\frac{m}{M}\right)^{1/pq}\cdot\left[I_{0t}^{\alpha}\frac{f^p(t)}{g^{p/q}(t)}\right]^{1/p}\left[I_{0t}^{\alpha}g(t)\right]^{1/q}.$$

Hence, we can write

$$\left[I_{0t}^{\alpha}f(t)\right]^p\geq\left(\frac{m}{M}\right)^{1/q}\cdot\left[I_{0t}^{\alpha}\frac{f^p(t)}{g^{p/q}(t)}\right]\left[I_{0t}^{\alpha}g(t)\right]^{p/q}. \tag{42}$$

Thanks to (42), we obtain (41).

Theorem 24.*Let* $0<\alpha<1, p>1, \frac{1}{p}+\frac{1}{q}=1$ *and f be a positive and continuous function on* $[0,\infty)$. *If* $0<m\leq f(\tau)\leq M<\infty$ *and*

$$I_{0t}^{\alpha}f(t)\geq\left(\frac{1}{1-\alpha}\left[1-e^{-\frac{1-\alpha}{\alpha}\cdot t}\right]\right)^{-p/q}, \tag{43}$$

then the inequality

$$I_{0t}^{\alpha}\left[f^p(t)\right]\leq\left(\frac{M}{m}\right)^{1/q}\cdot\left(I_{0t}^{\alpha}f(t)\right)^{p+1}$$

holds.

*Proof.*Using theorem 23 and condition (43), we obtain

$$\begin{aligned} I_{0t}^{\alpha}\left[f^p(t)\right] &= I_{0t}^{\alpha}\left[\frac{f^p(t)}{(1)^{p/q}}\right] \\ &\leq\left(\frac{M}{m}\right)^{1/q}\cdot\frac{\left(I_{0t}^{\alpha}f(t)\right)^p}{\left(I_{0t}^{\alpha}1\right)^{p/q}} \\ &=\left(\frac{M}{m}\right)^{1/q}\cdot\left(\frac{1}{1-\alpha}\left[1-e^{-\frac{1-\alpha}{\alpha}\cdot t}\right]\right)^{-p/q}\cdot\left(I_{0t}^{\alpha}f(t)\right)^p \\ &\leq\left(\frac{M}{m}\right)^{1/q}\cdot\left(I_{0t}^{\alpha}f(t)\right)^{p+1} \end{aligned}$$

as required.

*Remark.*For any $p > 1$, with f a positive and continuous function, we note that

$$0 < m \leq f^p(\tau) \leq M < \infty \Leftrightarrow 0 < m^{1/p} \leq f(\tau) \leq M^{1/p} < \infty. \tag{44}$$

Theorem 25.*Let $0 < \alpha < 1, p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, and f be a positive and continuous function on $[0,\infty)$. If*

$$0 < m \leq f^p(\tau) \leq M < \infty, \quad \tau \in [0,t],$$

then

$$\left[I_{0t}^{\alpha} f^p(t)\right]^{1/p} \leq \left(\frac{M}{m}\right)^{2/pq} \left(\frac{1}{1-\alpha}\left[1-e^{-\frac{1-\alpha}{\alpha}\cdot t}\right]\right)^{-\frac{p+1}{q}} \left(I_{0t}^{\alpha} f^{1/p}(t)\right)^p. \tag{45}$$

*Proof.*Putting $g(\tau) = 1$ into theorem 22, yields

$$\left[I_{0t}^{\alpha} f^p(t)\right]^{1/p} \cdot \left[I_{0t}^{\alpha}(1)\right]^{1/q} \leq \left(\frac{M}{m}\right)^{1/pq} \cdot I_{0t}^{\alpha}\left[f(t)\right].$$

which is equivalent to

$$\left[I_{0t}^{\alpha} f^p(t)\right]^{1/p} \leq \left(\frac{M}{m}\right)^{1/pq} \cdot \left(\frac{1}{1-\alpha}\left[1-e^{-\frac{1-\alpha}{\alpha}\cdot t}\right]\right)^{-1/q} \cdot I_{0t}^{\alpha}[f(t)]. \tag{46}$$

Substituting $g(\tau) = 1$ into theorem 21 and using (44), we obtain

$$\left[I_{0t}^{\alpha} f(t)\right]^{1/p} \cdot \left[I_{0t}^{\alpha}(1)\right]^{1/q} \leq \left(\frac{M}{m}\right)^{1/p^2 q} \cdot I_{0t}^{\alpha}\left[f^{1/p}(t)\right],$$

that is

$$\left[I_{0t}^{\alpha} f(t)\right]^{1/p} \leq \left(\frac{M}{m}\right)^{1/p^2 q} \cdot \left(\frac{1}{1-\alpha}\left[1-e^{-\frac{1-\alpha}{\alpha}\cdot t}\right]\right)^{-1/q} \cdot I_{0t}^{\alpha}[f^{1/p}(t)].$$

Hence, we can write

$$I_{0t}^{\alpha} f(t) \leq \left(\frac{M}{m}\right)^{1/pq} \left(\frac{1}{1-\alpha}\left[1-e^{-\frac{1-\alpha}{\alpha}\cdot t}\right]\right)^{-p/q} \left(I_{0t}^{\alpha}[f^{1/p}(t)]\right)^p. \tag{47}$$

Combining (46) with (47), inequality (45) follows.

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4 Conclusion

In this paper, we presented some interesting fractional inequalities using the Caputo-Fabrizio fractional integral. The results are concerned with some inequalities using functions with the same sense of variation. As a future work, authors are planning to use these inequalities to prove existence and uniqueness of some Caputo-Fabrizio ordinary fractional differential equations.

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