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Multi-Adjoint Algebras Versus Extended-Order Algebras

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Abstract: Adjoint triples are an interesting generalization of t-norms and their residuated implications, since they increase the flexibility in the framework where they are used. Following the same motivation of adjoint triples, in order to reduce the mathematical requirements for the computation, extended-order algebras are studied. Extended-order algebras are implicative algebras that generalize the integral residuated structures. In this paper, adjoint triples will be related to the operators considered in extended-order algebras. Furthermore, a comparison between adjoint negations and the negations introduced in extended-order algebras is presented.

Keywords: Adjoint triples, extended-order algebras, negation operators

1 Introduction

The use of general aggregation operators in different frameworks is very important obtaining useful consequences in the applications [15,16,32,33,30,31]

Several of these frameworks, such as Fuzzy Logic [26], Fuzzy Relation Equations [18], Rough Set Theory [29] and Formal Concept Analysis [17,30], need to consider algebras with implications. The most usual operators in these frameworks are left-continuous t-norms and their residuated implications and, specifically, residuated lattices.

Adjoint triples are general operators which provide less restrictive settings, since their conjunctors are neither required to be commutative nor associative. Therefore, the use of this kind of operators increases the flexibility and applicability of the frameworks in which they are considered, such as Logic Programming [20], Fuzzy Formal Concept Analysis [19], Fuzzy Relation Equations [11] and Rough Set Theory [7]. This consequence is one of the most important reasons which justifies that these triples have widely been studied in several papers [2,5].

An important generalization of the integral residuated structures are extended-order algebras, which were introduced by C. Guido and P. Toto in [14] and developed in several papers [10,9]. Extended-order algebras are implicative general structures that follows the same motivation of adjoint triples in order to reduce the mathematical requirements of the basic operators for the computation, but based on implications. The main goal of this paper is the comparison of w-eo algebras with the operators mentioned previously and the obtainment of the relationship between them. As this paper will prove, w-eo algebras are more restrictive than multi-adjoint algebras.

In addition, this paper will carry out a study of negation operators. Negation operators are widely studied in [10,12,27,28] and are very useful in fuzzy logic and logic programming. This paper considers adjoint negations obtained from adjoint triples and operators introduced in w-eo algebras corresponding to the negation connectives [10]. The comparison between these two kinds of negations is also shown.

The organization of this paper is as follows: Section 2 recalls the notion of adjoint triple and presents the multi-adjoint algebras. The definitions of the different extended-order algebras and several remarks about the comparison with adjoint triples are included in Section 3. Section 4 presents the corresponding relationship between negation operators. Lastly, the paper finishes with some conclusions and prospects for future work.

2 Adjoint triples and multi-adjoint algebras

This section recalls the definition of adjoint triple, several interesting properties derived from these operators and introduces the definition of multi-adjoint algebra.

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Definition 1. *Let* $(P_1,\le_1)$, $(P_2,\le_2)$, $(P_3,\le_3)$ *be posets and* $\&\colon P_1\times P_2\to P_3$, $\swarrow\colon P_3\times P_2\to P_1$, $\nwarrow\colon P_3\times P_1\to P_2$ *be mappings, then* $(\&,\swarrow,\nwarrow)$ *is an* adjoint triple *with respect to* P_1,P_2,P_3 *if* $\&,\swarrow,\nwarrow$ *satisfy the* adjoint property*:*

$$x\le_1 z\swarrow y \;\text{ iff }\; x\& y\le_3 z \;\text{ iff }\; y\le_2 z\nwarrow x$$

where $x\in P_1$, $y\in P_2$ *and* $z\in P_3$.

The following monotonicity properties are straightforwardly obtained from the adjoint property.

Proposition 1. *If* $(\&,\swarrow,\nwarrow)$ *is an* adjoint triple *w.r.t. the posets* $(P_1,\le_1)$, $(P_2,\le_2)$ *and* $(P_3,\le_3)$, *then*

1. *$\&$ is order-preserving on both arguments.*
2. *$\swarrow$, $\nwarrow$ are order-preserving on the first argument and order-reversing on the second argument.*

The result below states that given the conjunctor of an adjoint triple, its residuated implications are unique.

Proposition 2. *Given a conjunctor* $\&$, *if there exist its residuated implications* $\swarrow$ *and* $\nwarrow$, *they are unique.*

Moreover, these residuated implications conserve the infima on the first argument.

Proposition 3. *Let* $(\&,\swarrow,\nwarrow)$ *be a adjoint triple with respect to the posets* $(P_1,\le_1)$, $(P_2,\le_2)$ *and* $(P_3,\le_3)$, *then the following properties are satisfied:*

1. $\left(\bigwedge_{z_i\in Z} z_i\right)\swarrow y=\bigwedge_{z_i\in Z}(z_i\swarrow y)$, *for any* $Z\subseteq P_3$ *and* $y\in P_2$, *when the infima exist.*
2. $\left(\bigwedge_{z_i\in Z} z_i\right)\nwarrow x=\bigwedge_{z_i\in Z}(z_i\nwarrow x)$, *for any* $Z\subseteq P_3$ *and* $x\in P_1$, *when the infima exist.*

Another notion needed in this paper is associated with a well-know property of implications, which was called *forcing-implication* in [23,24]. Later, in [2], the authors used it in a more general environment and interesting properties were proven. Since this definition and properties will be considered later, these will be recalled next.

Definition 2. *Given two posets* $(Q,\le_Q)$, $(P,\le_P)$, *with a top element* $\top_P$ *in* $(P,\le_P)$. *The operator* $\to\colon Q\times Q\to P$ *which is order-reversing on the first argument and order-preserving on the second argument, satisfying the equivalence*

$$a\to b=\top_P \text{ if and only if } a\le_Q b,\ \text{for all } a,b\in Q \qquad (1)$$

is called forcing-implication on Q.

Before introducing the following result, the definition of the next mappings is required: $\swarrow^{\mathrm{op}}\colon P_2\times P_3\to P_1$, $\nwarrow_{\mathrm{op}}\colon P_1\times P_3\to P_2$, as $y\swarrow^{\mathrm{op}} z=z\swarrow y$ and $x\nwarrow_{\mathrm{op}} z=z\nwarrow x$, for all $x\in P_1$, $y\in P_2$ and $z\in P_3$.

Proposition 4([2]). *Given an adjoint triple* $(\&,\swarrow,\nwarrow)$ *with respect to* P_1,P_2 *and* P_3.

1. *If* $P_2\subseteq P_3$ *and* P_1 *has a maximum* $\top_1$, *the following statements are equivalent.*
   - $\swarrow^{op}$ *is a forcing implication.*
   - $\top_1\& y=y$, *for all* $y\in P_2$.
2. *If* $P_1\subseteq P_3$ *and* P_2 *has a maximum* $\top_2$, *the following statements are equivalent.*
   - $\nwarrow^{op}$ *is a forcing implication.*
   - $x\&\top_2=x$, *for all* $x\in P_1$.

Examples of adjoint triples are the Gödel, product and Łukasiewicz t-norms together with their residuated implications. Note that, these t-norms are commutative, then the residuated implications satisfy that $\swarrow^{\mathrm{G}}=\nwarrow_{\mathrm{G}}$, $\swarrow^{\mathrm{P}}=\nwarrow_{\mathrm{P}}$ and $\swarrow^{\mathrm{L}}=\nwarrow_{\mathrm{L}}$. Specifically, they are defined on $[0,1]$ as:

$$\&_{\mathrm{G}}(x,y)=\min(x,y) \qquad z\nwarrow_{\mathrm{G}} x=\begin{cases}1 & \text{if } x\le z\\ z & \text{otherwise}\end{cases}$$

$$\&_{\mathrm{P}}(x,y)=x\cdot y \qquad z\nwarrow_{\mathrm{P}} x=\min(1,z/x)$$

$$\&_{\mathrm{L}}(x,y)=\max(0,x+y-1) \qquad z\nwarrow_{\mathrm{L}} x=\min(1,1-x+z)$$

Example 1. Given $m\in\mathbb{N}$, the set $[0,1]_m$ is a regular partition of $[0,1]$ in m pieces, for example $[0,1]_2=\{0,0.5,1\}$ divides the unit interval in two pieces.

A discretization of the product t-norm is the operator $\&_{\mathrm{P}}^*\colon [0,1]_{20}\times[0,1]_8\to[0,1]_{100}$ defined, for each $x\in[0,1]_{20}$ and $y\in[0,1]_8$ as:

$$x\&_{\mathrm{P}}^* y=\frac{\lceil 100\cdot x\cdot y\rceil}{100}$$

where $\lceil\_\rceil$ is the ceiling function and whose residuated implications $\swarrow_{\mathrm{P}}^*\colon [0,1]_{100}\times[0,1]_8\to[0,1]_{20}$, $\nwarrow_{\mathrm{P}}^*\colon [0,1]_{100}\times[0,1]_{20}\to[0,1]_8$ are defined as:

$$b\swarrow_{\mathrm{P}}^* a=\frac{\lfloor 20\cdot\min\{1,b/a\}\rfloor}{20} \qquad b\nwarrow_{\mathrm{P}}^* c=\frac{\lfloor 8\cdot\min\{1,b/c\}\rfloor}{8}$$

where $\lfloor\_\rfloor$ is the floor function.

Hence, the triple $(\&_{\mathrm{P}}^*,\swarrow_{\mathrm{P}}^*,\nwarrow_{\mathrm{P}}^*)$ is an adjoint triple and the operator $\&_{\mathrm{P}}^*$ is straightforwardly neither commutative nor associative. Similar adjoint triples can be obtained from the Gödel and Łukasiewicz t-norms. □

The algebraic structure that considers these triples is the biresiduated multi-adjoint algebra. In [21,22], the notion of multi-adjoint lattice was introduced considering only pairs, that is, several conjunctors $\&_i$ and the corresponding residuated implications $\swarrow^i$. Later, in [20] biresiduated multi-adjoint lattices were presented in which adjoint triples $(\&_i,\swarrow^i,\nwarrow_i)$ on lattices were considered. The definition below generalizes this last notion, since posets are only assumed as carriers.

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Definition 3. *Given the posets $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ and a family of adjoint triples $(\&_i, \swarrow^i, \nwarrow_i)$, with $i \in \{1, \dots, n\}$. A (biresiduated) multi-adjoint algebra is the tuple*

$$\mathscr{P} = (P_1, P_2, P_3, \leq_1, \leq_2, \leq_3, \&_1, \swarrow^1, \nwarrow_1, \dots, \&_n, \swarrow^n, \nwarrow_n)$$

From now on, we will denote these algebras as $\mathscr{P} = (P_1, P_2, P_3, \leq_1, \leq_2, \leq_3, \&_1, \dots, \&_n)$, since, by Proposition 2, the residuated implications are unique.

3 The comparison with extended-order algebras

C. Guido and P. Toto [14] introduced extended-order algebras as implicative algebras that generalize the integral residuated structures, Hilbert algebras, BCK algebras, etc. These operators have been used in several frameworks, chasing the same motivation of adjoint triples, that is, introducing a general setting which reduces the mathematical requirements needed to compute in several frameworks, such as in fuzzy logic, fuzzy relation equations, rough sets, etc. Now, a deeper comparison than the one given in [3,4] is introduced and it is proven that extended-order algebras are more restrictive.

3.1 Extended-order algebras with an operator

This section recalls several algebraic structures given in [10,14] with only one operator and compares them with adjoint triples. Firstly, the definition of w-eo algebra is introduced, from which the rest of structures will be presented.

Definition 4([14]). *Let P be a non-empty set, $\rightarrow : P \times P \rightarrow P$ a binary operation and $\top$ a fixed element of P. The triple $(P, \rightarrow, \top)$ is a* w-eo algebra, *if for all $a, b, c \in P$ the following conditions are satisfied*[1]

(o_1) $a \rightarrow \top = \top$ (upper bound condition)
(o_2) $a \rightarrow a = \top$ (reflexivity condition)
(o_3) $a \rightarrow b = \top$ and $b \rightarrow a = \top$ then $a = b$ (antisymmetry condition)
(o_4) $a \rightarrow b = \top$ and $b \rightarrow c = \top$ then $a \rightarrow c = \top$ (weak transitivity condition)

From a w-eo algebra $(P, \rightarrow, \top)$ an ordering can be defined on the set P, which provides P with a poset structure. This relation $\leq$ is defined as follows:

$$a \leq b \text{ if and only if } a \rightarrow b = \top, \text{ for all } a, b \in P \qquad (2)$$

Straightforwardly, $\leq$ is an order relation in P, which was called *the natural ordering in P* [14]. Note that the poset $(P, \leq)$ has a greatest element which coincides with the fixed element $\top$ of P. Other interesting structures introduced in [10,14] arise when the poset $(P, \leq)$ associated with the w-eo algebra $(P, \rightarrow, \top)$ is a complete lattice. In this case, we say that $(P, \rightarrow, \top)$ is a *complete w-eo algebra* $(P, \rightarrow, \top)$, in short, a *w-ceo algebra*. In this case we will write L and $\preceq$ instead of P and $\leq$, respectively.

The notion of right-distributive w-ceo algebra is defined as follows.

Definition 5([10]). *Let L be a non-empty set, $\rightarrow : L \times L \rightarrow L$ a binary operation and $\top$ a fixed element of L. The triple $(L, \rightarrow, \top)$ is a* right-distributive w-ceo algebra, *if it is a w-ceo algebra and satisfies the following condition*

(d_r') for any $a \in L$, $B \subseteq L$: $a \rightarrow \bigwedge_{b \in B} b = \bigwedge_{b \in B} (a \rightarrow b)$

In the remainder of this section, we will present several results relating the latest structures to adjoint triples.

The first one states that Equivalence (2) coincides with the forcing-implication property.

Proposition 5. *Given a poset $(P, \leq)$ with a greatest element $\top$ and $\rightarrow : P \times P \rightarrow P$ a forcing-implication on P, then $(P, \rightarrow, \top)$ is a w-eo algebra.*

Proof. As $\rightarrow : P \times P \rightarrow P$ is a forcing-implication on P, then Equation (2) holds, which clearly provides properties (o1), (o2), (o3) and (o4). □

The second result shows under what conditions should be defined an adjoint triple in order to provide a w-eo algebra.

Proposition 6. *Given a poset $(P, \leq)$, with a maximum element $\top$, and an adjoint triple $(\&, \swarrow, \nwarrow)$ with respect to P. The conjunctor satisfies $\top \& y = y$, for all $y \in P$, if and only if $(P, \swarrow^{\mathrm{op}}, \top)$ is a w-eo algebra and the natural ordering in P is $\leq$.*

Proof. First of all, we will prove that $(P, \swarrow^{\mathrm{op}}, \top)$ is w-eo algebra and the natural ordering in P is $\leq$. By hypothesis, we have that $\top \& y = y$, for all $y \in P$, and applying Proposition 4 we obtain that $\swarrow^{\mathrm{op}}$ is a forcing-implication and $\leq$ is the natural ordering in P. Moreover, by Proposition 5 the triple $(P, \swarrow^{\mathrm{op}}, \top)$ is a w-eo algebra.

The counterpart is obtained since the triple $(P, \swarrow^{\mathrm{op}}, \top)$ is a w-eo algebra and $\leq$ is the natural ordering in P, which is defined by means of the Equivalence (1). Therefore, applying Proposition 4 the boundary condition $\top \& y = y$ is obtained, for all $y \in P$. □

Analogously, the following proposition is obtained.

Proposition 7. *Given a poset $(P, \leq)$, with a maximum element $\top$, and an adjoint triple $(\&, \swarrow, \nwarrow)$ with respect to P. The conjunctor satisfies $x \& \top = x$, for all $x \in P$, if and only if $(P, \nwarrow_{\mathrm{op}}, \top)$ is a w-eo algebra and the natural ordering in P is $\leq$.*

[1] Note that the names of the properties are those in [14].

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Proof. The proof is similar to the previous one. □

As a consequence, adjoint triples are less restrictive operators. Therefore, multi-adjoint algebras are more flexible structures than w-eo algebras. Moreover, different adjoint triples can be considered, which provides an extra useful feature as in the papers [8,11,19,21] was shown.

3.2 Extended-order algebras with two operators

This section begins presenting the residuated operator $\otimes\colon L\times L\to L$, which was defined in [10,14] from right-distributive w-ceo algebra $(L,\to,\top)$ as follows:

$$a\otimes x=\bigwedge\{t\in L\mid x\preceq a\to t\}\qquad(3)$$

Moreover, an additional binary operation was introduced in order to enrich the structures shown in [10]. For this purpose, a right-distributive w-ceo algebra $(L,\to,\top)$ was considered and the presented operator was denoted as $\rightsquigarrow\colon L\times L\to L$ satisfying the equivalence

$$a\preceq b\rightsquigarrow c\quad\text{iff}\quad a\otimes b\preceq c\quad\text{iff}\quad b\preceq a\to c\qquad(4)$$

for all $a,b,c\in L$.

So that, a triple is considered. Consequently, we will compare this triple with adjoint triples in this section.

The flexibility supported by adjoint triples provides that $(\rightsquigarrow,\otimes,\to)$ straightforwardly is an adjoint triple.

Proposition 8. *Given a complete lattice $(L,\preceq)$ and the mappings $\rightsquigarrow$, $\otimes$ and $\to$ defined above, the triple $(\rightsquigarrow,\otimes,\to)$ is an adjoint triple with respect to L.*

Proof. The proof straightforwardly follows from Equivalence (4). □

The symmetrical w-eo algebra was the following structure shown in [10]. In the definition of this structure, the operators $\rightsquigarrow$ and $\to$ are considered.

Definition 6([10]). *A w-eo algebra $(L,\to,\top)$ is called* symmetrical *if there exists a binary operation $\rightsquigarrow\colon L\times L\to L$ such that $(L,\rightsquigarrow,\top)$ is a w-eo algebra, $\to$ and $\rightsquigarrow$ induce the same ordering and*

$$y\leq x\rightsquigarrow b\quad\textit{if and only if}\quad x\leq y\to b$$

holds, for all $b,x,y\in L$.

The w-eo algebras $(L,\to,\top)$, $(L,\rightsquigarrow,\top)$ and their implications $\to$, $\rightsquigarrow$ are said to be dual to each other.

Due to symmetrical character of this notion, $(L,\rightsquigarrow,\top)$ is symmetrical if and only if $(L,\to,\top)$ is symmetrical [10].

From Proposition 8 we assert that every symmetrical right-distributive w-ceo algebra always provides an adjoint triple.

Proposition 9. *Let $(L,\to,\top)$ be a symmetrical right-distributive w-ceo algebra and the operator $\otimes\colon L\times L\to L$ defined by Equation (3), then $(\to,\otimes,\rightsquigarrow)$ is an adjoint triple on L.*

However, the counterpart is not true. The following proposition specifies the properties which must be satisfied by an adjoint triple to obtain a symmetrical right-distributive w-eo algebra.

Proposition 10. *Given a poset $(P,\leq)$, with $\top$ as maximum element, and an adjoint triple $(\&,\swarrow,\nwarrow)$ with respect to P. The conjunctor satisfies $x\,\&\,\top=x$ and $\top\,\&\,y=y$, for all $x,y\in P$, if and only if $(P,\swarrow^{\mathrm{op}},\top)$ is a symmetrical right-distributive w-eo algebra and the natural ordering in P is $\leq$.*

Proof. By Proposition 6, we have that $(P,\swarrow^{\mathrm{op}},\top)$ is a w-eo algebra and the natural ordering in P is $\leq$. Also, as $(\&,\swarrow,\nwarrow)$ is an adjoint triple, by Condition (1) of Proposition 3, we obtain that $(P,\swarrow^{\mathrm{op}},\top)$ is a right-distributive w-eo algebra.

Now, the symmetrical property must be proven. Taking into account Proposition 7, we obtain that $(P,\nwarrow_{\mathrm{op}},\top)$ is a w-eo algebra and the natural ordering in P is $\leq$. Therefore, we have that the implications $\nwarrow_{\mathrm{op}}$ and $\swarrow^{\mathrm{op}}$ induce the same ordering in P. In addition, the equivalence

$$y\leq x\nwarrow_{\mathrm{op}}z\quad\text{if and only if}\quad x\leq y\swarrow^{\mathrm{op}}z$$

holds, for all $x,y,z\in P$, since $(\&,\swarrow,\nwarrow)$ is an adjoint triple. Therefore, $(P,\swarrow^{\mathrm{op}},\top)$ is a symmetrical right-distributive w-eo algebra.

The counterpart is straightforwardly obtained from Proposition 4. □

An analogous result is obtained with respect to $(P,\nwarrow_{\mathrm{op}},\top)$.

In a similar way, these results can be developed with the left-distributive structures which were also introduced in [10]. Therefore, right and left-distributive w-eo algebras are more restrictive settings than multi-adjoint algebras.

4 Adjoint negations and extended-order algebras with negations

In this section, we will show a comparison between the negation operators presented in [10] and adjoint negations [6]. The negation operators introduced by Della Stella and Guido are defined from symmetrical algebras as follows.

Definition 7([10]). *Let $(L,\to,\top)$ a w-ceo algebra. We define the following unary operation*

$$[\cdot]^{-}\colon L\to L,\quad x\mapsto x^{-}=x\to\bot$$

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If $(L, \to, \top)$ is a symmetrical w-ceo algebra, then we can define a further unary operator

$$[\cdot]^{\sim} : L \to L, \quad x \mapsto x^{\sim} = x \rightsquigarrow \bot$$

Both these operations are called negation *and they are said to be dual to each other.*

The negation $[\cdot]^{-}$ ($[\cdot]^{\sim}$, respectively) is involutive *if $x^{--} = x$ ($x^{\sim\sim} = x$, respectively), for every $x \in L$.*

The negations $[\cdot]^{-}$ and $[\cdot]^{\sim}$, and the symmetrical w-ceo algebra as well, are said to be cross-involutive *if $x^{\sim -} = x^{-\sim} = x$, for every $x \in L$.*

From definition above, basic properties of the negations were stated in [10].

The adjoint negations are also residuated negations [1, 13, 25] defined from the implications of an adjoint triple.

Definition 8([6]). *Given two posets $(P_1, \leq_1)$, $(P_2, \leq_2)$, a lower bounded poset $(P_3, \leq_3, \bot_3)$ and an adjoint triple $(\&, \swarrow, \nwarrow)$ with respect to P_1, P_2 and P_3, the mappings $n_n : P_1 \to P_2$ and $n_s : P_2 \to P_1$ defined, for all $x \in P_1$, $y \in P_2$ as*

$$n_n(x) = \bot_3 \nwarrow x \qquad n_s(y) = \bot_3 \swarrow y$$

are called adjoint negations with respect to P_1 and P_2.

The operators n_s and n_n satisfying that $x = n_s(n_n(x))$ and $y = n_n(n_s(y))$, for all $x \in P_1$ and $y \in P_2$, are called strong adjoint negations.

Note that, three different bounded partially ordered sets have been considered in the definition of adjoint negations, which provides a more general definition of negation operator.

The following result fixes the relation between both negation operators.

Proposition 11. *Given a symmetrical w-ceo algebra $(L, \to, \top)$, the unary operations $[\cdot]^{-}$ and $[\cdot]^{\sim}$ are adjoint negations.*

Proof. The proof is straightforwardly obtained from Proposition 9 and Definitions 8 and 7. □

As a consequence, adjoint negations defined from multi-adjoint algebras are more general than the ones given from symmetrical algebras. Moreover, we can conclude that the properties of the adjoint negations are also satisfied by the operators $[\cdot]^{-}$, $[\cdot]^{\sim}$.

Furthermore, almost all these properties given in [10] are satisfied by adjoint negations n_s and n_n and so, less conditions are needed to be satisfied.

One property that the adjoint negations do not verify, in general, is that $n_s(\top) = \bot$ and $n_n(\top) = \bot$, although the negation operators $[\cdot]^{-}$ and $[\cdot]^{\sim}$ defined in [10] always satisfy these conditions $\top^{-} = \bot$ and $\top^{\sim} = \bot$.

Besides this property, for example, Proposition 6.2, 6.3 and 6.4 of [10] show properties that adjoint negations verify. From now on, we will introduce some technical results of adjoint negations which require less hypotheses than the ones given in [10].

Firstly, we will introduce an important result which relates these negations to Galois connections.

Proposition 12. *Let $(P_1, \leq_1)$, $(P_2, \leq_2)$ be two posets, $(P_3, \leq_3, \bot_3)$ a lower bounded poset and n_s, n_n adjoint negations. The pair (n_s, n_n) forms an antitone Galois connection between P_1 and P_2.*

Proof. In order to prove that (n_s, n_n) is an antitone Galois connection between P_1 and P_2, we must to check that n_s, n_n are order-reversing and the inequalities $x \leq_1 n_s n_n(x)$ and $y \leq_2 n_n n_s(y)$ hold, for all $x \in P_1$ and $y \in P_2$.

Firstly, we will prove that if $x_1, x_2 \in P_1$ and $x_1 \leq_1 x_2$, then $n_n(x_2) \leq_2 n_n(x_1)$. We suppose that $x_1 \leq_1 x_2$, by the monotony of the operator $\nwarrow$ we obtain $\bot_3 \nwarrow x_2 \leq_2 \bot_3 \nwarrow x_1$, which is equivalent to $n_n(x_2) \leq_2 n_n(x_1)$. The monotonicity of n_s is proven analogously.

Now, we will check that $x \leq_1 n_s n_n(x)$ holds, for all $x \in P_1$. The adjoint property provides that the inequality $x \leq_1 \bot_3 \swarrow (\bot_3 \nwarrow x)$ is equivalent to the inequality $x \& (\bot_3 \nwarrow x) \leq_3 \bot_3$ and they are true since, by the adjoint property, the trivial inequality $\bot_3 \nwarrow x \leq_2 \bot_3 \nwarrow x$ holds, for all $x \in P_1$. The proof of the inequality $y \leq_2 n_n n_s(y)$ is analogous. □

As a consequence, the properties of Galois connections will be inherited by adjoint negations. The following proposition recalls several of them, which are associated with some of the properties corresponding to Proposition 6.2, 6.3 and 6.4 of [10], avoiding extra restrictions.

Proposition 13. *Let $(P_1, \leq_1)$, $(P_2, \leq_2)$ be posets, $(P_3, \leq_3, \bot_3)$ lower bounded poset and n_s, n_n adjoint negations. The following statements hold:*

1. *If $(P_1, \leq_1, \bot_1, \top_1)$ and $(P_2, \leq_2, \bot_2, \top_2)$ are bounded partially ordered sets, then $n_s(\bot_2) = \top_1$ and $n_n(\bot_1) = \top_2$;*
2. *n_n and n_s are antitone;*
3. *$x \leq_1 n_s n_n(x)$ and $y \leq_2 n_n n_s(y)$;*
4. *$n_s n_n n_s = n_s$ and $n_n n_s n_n = n_n$;*
5. *$n_s n_n$ and $n_n n_s$ are closure operators;*
6. *$x \leq_1 n_s(y)$ iff $y \leq_2 n_n(x)$, for all $x \in P_1, y \in P_2$;*
7. *When the supremum and the infimum exist, for any $X \subseteq P_1$, $Y \subseteq P_2$,*
   (a) $n_s(\bigvee_{y \in Y} y) = \bigwedge_{y \in Y} n_s(y)$,
   (b) $n_n(\bigvee_{x \in X} x) = \bigwedge_{x \in X} n_n(x)$.

The next results are associated with Proposition 6.2 of [10].

Proposition 14. *Let $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to the two posets $(P_1, \leq_1)$, $(P_2, \leq_2)$ and the lower bounded poset $(P_3, \leq_3, \bot_3)$. The adjoint negation n_s and n_n, obtained from the adjoint triple, satisfies that $n_s(y) \leq_1 z \swarrow y$ and $n_n(x) \leq_2 z \nwarrow x$, for all $x \in P_1, y \in P_2, z \in P_3$.*

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Proof. Clearly $\bot_3 \leq_3 z$ holds, for all $z \in P_3$. As $\swarrow$ is order-preserving on the first argument, the inequality $\bot_3 \swarrow y \leq_1 z \swarrow y$ is satisfied, for all $y \in P_2$ and $z \in P_3$, which is equivalent to $n_s(y) \leq_1 z \swarrow y$.

The proof is analogous for $n_n(x) \leq_2 z \nwarrow x$. □

Proposition 15. *The adjoint negation n_s, obtained from the adjoint triple $(\&, \swarrow, \nwarrow)$ with respect to the join-semilattice $(P_1, \leq_1)$, the meet-semilattice $(P_2, \leq_2)$ and the lower bounded poset $(P_3, \leq_3, \bot_3)$, satisfies the following properties for all $\{y_i\}_{i\in I} \subseteq P_2$.*

1. $\bigvee\limits_{i\in I} n_s(y_i) \leq_1 n_s(\bigwedge\limits_{i\in I} y_i)$;
2. *If n_s and n_n are strong adjoint negations, then* $\bigvee\limits_{i\in I} n_s(y_i) = n_s(\bigwedge\limits_{i\in I} y_i)$.

Proof. (1) By the infimum property and the monotonicity of $\swarrow$, the inequality $\bot_3 \swarrow y \leq_1 \bot_3 \swarrow (\bigwedge_{i\in I} y_i)$ is verified, for all $\{y_i\}_{i\in I} \subseteq P_2$. Now, applying the supremum property, it is obtained that $\bigvee_{i\in I}(\bot_3 \swarrow y_i) \leq_1 \bot_3 \swarrow (\bigwedge_{i\in I} y_i)$, i.e., $\bigvee_{i\in I} n_s(y_i) \leq_1 n_s(\bigwedge_{i\in I} y_i)$.

(2) Applying the operator n_s to the Condition 7(b) of Proposition 13, we obtain $n_s(\bigwedge_{i\in I} n_n(x_i)) = n_s(n_n(\bigvee_{i\in I} x_i))$, for all $\{x_i\}_{i\in I} \subseteq P_1$. Since n_s is a strong adjoint negation, the equality $n_s(\bigwedge_{i\in I} n_n(x_i)) = \bigvee_{i\in I} x_i$ holds, for all $\{x_i\}_{i\in I} \subseteq P_1$.

Now, given $\{y_i\}_{i\in I} \subseteq P_2$, applying the previous equality to $x_i = n_s(y_i)$, we obtain $n_s(\bigwedge_{i\in I} n_n(n_s(y_i))) = \bigvee_{i\in I} n_s(y_i)$. Hence, as n_n is a strong adjoint negation, we can conclude that $\bigvee_{i\in I} n_s(y_i) = n_s(\bigwedge_{i\in I} y_i)$, for all $\{y_i\}_{i\in I} \subseteq P_2$. □

An analogous result is obtained considering n_n.

Proposition 16. *Let $(\&, \swarrow, \nwarrow)$ be an adjoint triple with respect to the meet-semilattice $(P_1, \leq_1)$, the join-semilattice $(P_2, \leq_2)$ and the lower bounded poset $(P_3, \leq_3, \bot_3)$. The adjoint negation n_n satisfies the following statements for all $\{x_i\}_{i\in I} \subseteq P_1$.*

1. $\bigvee\limits_{i\in I} n_n(x_i) \leq_2 n_n(\bigwedge\limits_{i\in I} x_i)$;
2. *If n_s and n_n are strong adjoint negations, then* $\bigvee\limits_{i\in I} n_n(x_i) = n_n(\bigwedge\limits_{i\in I} x_i)$.

The following proposition presents a generalization of the properties given in Proposition 6.3 of [10], since they are established in a more general framework.

Proposition 17. *Given an adjoint triple $(\&, \swarrow, \nwarrow)$ with respect to the posets $(P_1, \leq_1)$, $(P_2, \leq_2)$ and the lower bounded poset $(P_3, \leq_3, \bot_3)$. The adjoint negation n_n, obtained from the residuated implication $\swarrow$, satisfies the following properties, for all $x \in P_1, y \in P_2$.*

1. $x \& n_n(x) = \bot_3$
2. $y \leq_2 n_n(x)$ *if and only if* $x \& y = \bot_3$

Proof. (1) The trivial inequality $\bot_3 \leq_3 x\&(\bot_3 \nwarrow x)$ holds, for all $x \in P_1$. Therefore, we only have to prove that $x\&(\bot_3 \nwarrow x) \leq_3 \bot_3$, which follows applying the adjoint property directly to $\bot_3 \nwarrow x \leq_2 \bot_3 \nwarrow x$.

(2) Firstly, we will prove the first implication. The equivalence $y \leq_2 n_n(x)$ if and only if $x \& y \leq_3 \bot_3$ is obtained straightforwardly from adjoint property. Moreover, $\bot_3 \leq_3 x \& y$ holds, for all $x \in P_1, y \in P_2$. Therefore, we obtain that $x \& y = \bot_3$.

In order to prove the counterpart, we suppose that $x \& y = \bot_3$. Clearly, $x \& y \leq_3 \bot_3$ which is equivalent to $y \leq_2 n_n(x)$ by the adjoint property. □

The operator n_s satisfies a similar result.

Therefore, although the definition of adjoint negations is more general, they almost satisfy the same properties that the negation operators from symmetrical w-eo algebras, requiring less conditions in general.

5 Conclusions and future work

Two important structures, multi-adjoint algebras and extended-order algebras, have been taken into account, which were introduced under the same motivation: reducing the mathematical requirements needed to compute in several frameworks, such as in fuzzy logic, fuzzy formal concept analysis, etc.

The formal definition of multi-adjoint algebra has been introduced. Moreover, the main contribution have been the comparison of both algebras in order to know what is the most general one, keeping the needed properties to compute in the applications.

Furthermore, since the use of residuated negations is very useful in fuzzy logic and other frameworks, this paper has considered adjoint negations from multi-adjoint algebras, which are a generalization of the definition of the logic connective. We have presented a comparison between adjoint negations and negation operators introduced in [10], obtaining that adjoint negations are more general operators. Indeed, the properties satisfied by the negation operators defined from symmetrical w-eo algebras have been generalized avoiding extra restrictions in most cases. Consequently, the applications in which adjoint negations can be considered are wider.

As future work, more properties will be studied and the comparison with other general structures will be given.

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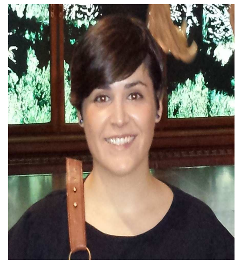

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