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Some Explicit Formulas for the Frobenius-Euler Polynomials of Higher Order

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Abstract: The object of the present paper is to investigate a class of three-term recurrence relations for the Frobenius-Euler polynomials of higher order. By making use of a new explicit formula for the Frobenius-Euler polynomials of higher order in terms of the weighted Stirling numbers of the second kind, we provide an algorithm for calculating these polynomials. Furthermore, as an application of the results derived in this paper, we present an algorithm for computing the Lipschitz-Lerch zeta function at nonnegative integer arguments.

Keywords: Algorithm; Euler polynomials and Euler numbers; Explicit formulas; Frobenius-Euler polynomials; Frobenius-Genocchi polynomials; Genocchi polynomials; Recurrence relations; Stirling numbers of the first and second kind; Whitney numbers of the second kind.

1 Introduction

For $\lambda \in \mathbb{C}$ with $\lambda \neq 1$, the Frobenius-Euler polynomials $H_n^{(\alpha)}(x \mid \lambda)$ of order $\alpha \in \mathbb{C}$, are defined by the following generating function

$$\left(\frac{1-\lambda}{e^z-\lambda}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x \mid \lambda)\frac{z^n}{n!}.$$

When $x = 0$,

$$H_n^{(\alpha)}(0 \mid \lambda) = H_n^{(\alpha)}(\lambda)$$

are called the Frobenius-Euler numbers. Clearly, we have

$$H_n^{(\alpha)}(x \mid \lambda) = \sum_{k=0}^{n}\binom{n}{k} H_n^{(\alpha)}(\lambda)\, x^{n-k}. \qquad (1)$$

The generalized Euler polynomials $E_n^{(\alpha)}(x)$, given by

$$E_n^{(\alpha)}(x) := H_n^{(\alpha)}(x \mid -1)$$

are defined by the following generating function

$$\left(\frac{2}{e^z+1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{z^n}{n!}.$$

In the special case when $\alpha = 1$, the Frobenius-Euler polynomials are reduced to the Eulerian polynomials given by

$$H_n^{(1)}(x \mid \lambda) := H_n(x \mid \lambda),$$

which are widely cited in the literature. For an expository survey on this topic, we refer the reader to [7].

Recently, many research articles were devoted to the study of the Frobenius-Euler polynomials [9,13] and many generalizations were introduced [14,16]. As an example of a recent application of the Frobenius-Euler polynomials, T.-X. He [11] presented a relationship between $H_n(x \mid \lambda)$ and B-splines (see also [1,15]).

In this paper, we propose to investigate several explicit formulas of the Frobenius-Euler polynomials $H_n^{(\alpha)}(x \mid \lambda)$ of order α in terms of the weighted Stirling numbers of the second kind. As a consequence of our investigation,

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we provide an algorithm computing the Frobenius-Euler polynomials $H_n^{(\alpha)}(x \mid \lambda)$.

We first recall some basic definitions and some results that will be useful in the rest of the paper. For $\nu \in \mathbb{C}$, the Pochhammer symbol $(\nu)_n$ is defined by

$$(\nu)_n = \nu(\nu+1)\cdots(\nu+n-1) \qquad \text{and} \qquad (\nu)_0 = 1.$$

The (signed) Stirling numbers $s(n,k)$ of the first kind are the coefficients in the following expansion:

$$x(x-1)\cdots(x-n+1) = \sum_{k=0}^{n} s(n,k)x^k.$$

and satisfy the recurrence relation given by

$$s(n+1,k) = s(n,k-1) - ns(n,k) \qquad (1 \leqq k \leqq n). \quad (2)$$

The Stirling numbers of the second kind, denoted by $S(n,k)$, are the coefficients in the following expansion:

$$x^n = \sum_{k=0}^{n} k! \binom{x}{k} S(n,k).$$

The exponential generating functions for $s(n,k)$ and $S(n,k)$ are given by

$$\sum_{n=k}^{\infty} s(n,k) \frac{z^n}{n!} = \frac{1}{k!}[\ln(1+z)]^k$$

and

$$\sum_{n=k}^{\infty} S(n,k) \frac{z^n}{n!} = \frac{1}{k!}(e^z-1)^k,$$

respectively.

For any nonnegative integer r, the r-Stirling numbers $S_r(n,k)$ of the second kind, which were introduced by Broder [3], are a generalization of the familiar Stirling numbers $S(n,k)$ of the second kind. In fact, the numbers $S_r(n,k)$ count the number of partitions of a set of n objects into exactly k nonempty and disjoint subsets such that the first r elements are in distinct subsets. Furthermore, their exponential generating function is given by

$$\sum_{n=k}^{\infty} S_r(n+r,k+r) \frac{z^n}{n!} = \frac{1}{k!} e^{rz}(e^z-1)^k.$$

For any positive integer m, the r-Whitney numbers $W_{m,r}(n,k)$ of the second kind, which were introduced by Mezö [18], are the coefficients in the following expansion:

$$(mx+r)^n = \sum_{k=0}^{n} m^k W_{m,r}(n,k) x(x-1)\cdots(x-k+1)$$

and are given by their generating function as follows:

$$\sum_{n=k}^{\infty} W_{m,r}(n,k) \frac{z^n}{n!} = \frac{1}{m^k k!} e^{rz}(e^{mz}-1)^k.$$

Clearly, we have

$$W_{1,0}(n,k) = S(n,k) \text{ and } W_{1,r}(n,k) = S_r(n+r,k+r).$$

For more details on these numbers, we refer the reader to [10,19,21,25] and also to the references cited therein.

The weighted Stirling numbers $\mathscr{S}_n^k(x)$ of the second kind are defined by (see [4,5])

$$\begin{aligned}\mathscr{S}_n^k(x) &= \frac{1}{k!}\Delta^k x^n \\ &= \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(x+j)^n,\end{aligned}$$

where Δ denotes the forward difference operator. The exponential generating function of $\mathscr{S}_n^k(x)$ is given by

$$\sum_{n=k}^{\infty} \mathscr{S}_n^k(x) \frac{z^n}{n!} = \frac{1}{k!} e^{xz}(e^z-1)^k \tag{3}$$

and weighted Stirling numbers $\mathscr{S}_n^k(x)$ satisfy the following recurrence relation:

$$\mathscr{S}_{n+1}^k(x) = \mathscr{S}_n^{k-1}(x) + (x+k)\mathscr{S}_n^k(x) \qquad (1 \leqq k \leqq n).$$

As a consequence from the generating function (3), one can deduce the following results:

$$\mathscr{S}_n^k(0) = S(n,k), \tag{4}$$

$$\mathscr{S}_n^k(r) = S_r(n+r,k+r) \tag{5}$$

and

$$m^{n-k}\mathscr{S}_n^k\left(\frac{r}{m}\right) = W_{m,r}(n,k). \tag{6}$$

2 Explicit Formulas for the Frobenius-Euler Polynomials of Order α

An explicit formula for the Frobenius-Euler polynomials $H_n^{(\alpha)}(x \mid \lambda)$ of order $\alpha \in \mathbb{C}$, expressed in terms of the weighted Stirling numbers of the second kind, is given by the following result.

Theorem 1. *The following relationship holds true*:

$$H_n^{(\alpha)}(x \mid \lambda) = \sum_{k=0}^{n} \frac{(\alpha)_k}{(\lambda-1)^k} \mathscr{S}_n^k(x). \tag{7}$$

Proof. From (3), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(\alpha)_k}{(\lambda-1)^k}\mathscr{S}_n^k(x)\right)\frac{z^n}{n!} \\ &= \sum_{k=0}^{\infty}\frac{(\alpha)_k}{(\lambda-1)^k}\left(\sum_{n=k}^{\infty}\mathscr{S}_n^k(x)\frac{z^n}{n!}\right) \\ &= e^{xz}\sum_{k=0}^{\infty}(\alpha)_k\frac{1}{k!}\left(\frac{e^z-1}{\lambda-1}\right)^k.\end{aligned}$$

Since

$$\sum_{n=0}^{\infty}(a)_n\frac{z^n}{n!}=(1-z)^{-a},$$

we get

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(\alpha)_k}{(\lambda-1)^k}\mathscr{S}_n^k(x)\right)\frac{z^n}{n!}=e^{xz}\left(1-\frac{e^z-1}{\lambda-1}\right)^{-a}$$
$$=\sum_{n=0}^{\infty}H_n^{(\alpha)}(x\mid\lambda)\frac{z^n}{n!},$$

which gives, by identification, the desired result.

□

Remark 1. By setting $x=0$ and $\alpha=\mathfrak{s}$ ($\mathfrak{s}$ being a positive integer) in (7), we have

$$H_n^{(\mathfrak{s})}(\lambda)=\sum_{k=0}^{n}\frac{(\mathfrak{s})_k}{(\lambda-1)^k}S(n,k),$$

which is a result due to Carlitz [6].

Remark 2. By substituting $\alpha=1$, Theorem 1 is reduced to the known result given earlier by Chang and Ha [8].

Remark 3. If we set $x=\frac{r}{m}$ in (7) and using (6), we obtain an explicit representation for $H_n^{(\alpha)}(x\mid\lambda)$ at rational arguments involving the r-Whitney numbers $W_{m,r}(n,k)$ of the second kind.

$$H_n^{(\alpha)}\left(\frac{r}{m}|\lambda\right)=\sum_{k=0}^{n}\frac{(\alpha)_k}{(\lambda-1)^k}m^{k-n}W_{m,r}(n,k).$$

In particular, for $m=1$, we have

$$H_n^{(\alpha)}(r\mid\lambda)=\sum_{k=0}^{n}\frac{(\alpha)_k}{(\lambda-1)^k}S_r(n+r,k+r).$$

Remark 4. Setting $\lambda=-1$, we obtain the following explicit formula for the generalized Euler polynomials $E_n^{(\alpha)}(x)$:

$$E_n^{(\alpha)}(x)=\sum_{k=0}^{n}\frac{(-1)^k}{2^k}(\alpha)_k\mathscr{S}_n^k(x),$$

which was given by Boutiche *et al.* [2].

3 Frobenius-Genocchi Polynomials of Higher Order

Recently, Yılmaz Yaşar and Özarslan [26] introduced and studied a new family of polynomials, called Frobenius-Genocchi polynomials, which are defined by the following generating function:

$$\frac{(1-\lambda)z}{e^z-\lambda}e^{xz}=\sum_{n=0}^{\infty}G_n(x\mid\lambda)\frac{z^n}{n!}.$$

It is natural that we define the generalization of the Frobenius-Genocchi polynomials by means of the following generating function:

$$\left(\frac{(1-\lambda)z}{e^z-\lambda}\right)^{\alpha}e^{xz}=\sum_{n=0}^{\infty}G_n^{(\alpha)}(x\mid\lambda)\frac{z^n}{n!}. \tag{8}$$

We call $G_n^{(\alpha)}(x\mid\lambda)$ in (8) the Frobenius-Genocchi polynomials of order α. When $x=0$,

$$G_n^{(\alpha)}(0\mid\lambda)=G_n^{(\alpha)}(\lambda)$$

denote the Frobenius-Genocchi numbers of order α.

The generalized Genocchi polynomials $G_n^{(\alpha)}(x)$, given by (see, for details, [12,23])

$$G_n^{(\alpha)}(x):=G_n^{(\alpha)}(x\mid-1)$$

are defined by the following generating function:

$$\left(\frac{2z}{e^z+1}\right)^{\alpha}e^{xz}=\sum_{n=0}^{\infty}G_n^{(\alpha)}(x)\frac{z^n}{n!}. \tag{9}$$

By setting $\alpha=1$ in (9), we obtain the classical Genocchi polynomials:

$$G_n(x):=G_n^{(1)}(x).$$

It is obvious that $G_n^{(\alpha)}(x\mid\lambda)$ belongs to the class of Appell polynomials. Some of the well-known properties are readily derived from (8). For example, we have

$$\frac{d}{dx}G_n^{(\alpha)}(x\mid\lambda)=nG_{n-1}^{(\alpha)}(x\mid\lambda)$$

and

$$G_n^{(\alpha)}(x\mid\lambda)=\sum_{k=0}^{n}\binom{n}{k}G_n^{(\alpha)}(\lambda)x^{n-k}. \tag{10}$$

Theorem 2. *The following explicit formulas hold true*:

$$G_n^{(l)}(x\mid\lambda)=\frac{n!}{(n-l)!}H_{n-l}^{(l)}(x\mid\lambda) \tag{11}$$
$$=\frac{n!}{(n-l)!}\sum_{k=0}^{n-l}\frac{(l)_k}{(\lambda-1)^k}\mathscr{S}_{n-l}^k(x). \tag{12}$$

The following corollary is derivable easily from Theorem 2.

Corollary. *The Frobenius-Genocchi polynomials $G_n^{(l)}(x\mid\lambda)$ of order l at rational arguments are given by*

$$G_n^{(l)}\left(\frac{r}{m}\mid\lambda\right)=\frac{n!}{m^{n-l}(n-l)!}\sum_{k=0}^{n-l}\frac{(l)_k}{(\lambda-1)^k}m^kW_{m,r}(n-l,k)$$

in terms of the Whitney numbers of the second kind.

By setting $x = 0$ in Theorem 2, we obtain the following explicit formula for the Frobenius-Genocchi numbers of order α :

$$G_n^{(l)}(\lambda) = \frac{n!}{(n-l)!}\sum_{k=0}^{n-l}\frac{(l)_k}{(\lambda-1)^k}\, S(n-l,k)$$

By setting $\lambda = -1$ in (12), we obtain an explicit formula for the Genocchi polynomials $G_n^{(l)}(x)$ of order l as follows:

$$G_n^{(l)}(x) = \frac{n!}{(n-l)!}\sum_{k=0}^{n-l}\frac{(-1)^k}{2^k}(l)_k\,\mathscr{S}_{n-l}^k(x). \tag{13}$$

If we set $x = 0$ in (13), we obtain an explicit formula for the generalized Genocchi numbers of order l, which is given by Luo and Srivastava [17, p. 5709, Eq. (53)].

4 Recurrence Relations for the Frobenius-Euler Polynomials of Higher Order

In this section, we propose an algorithm, which is based on a three-term recurrence relation, for calculating the Frobenius-Euler polynomials $H_n^{(\alpha)}(x\mid\lambda)$ of order α. First, by setting $x = 0$ in Theorem 1, we obtain the following explicit formula for the Frobenius-Euler numbers:

$$H_n^{(\alpha)}(\lambda) = \sum_{k=0}^{n}\frac{(\alpha)_k}{(\lambda-1)^k}\, S(n,k).$$

Now, by means of the Stirling transform (see, for details, [20]), we obtain

$$\frac{(\alpha)_n}{(\lambda-1)^n} = \sum_{k=0}^{n} s(n,k)\, H_k^{(\alpha)}(\lambda).$$

Next, it is convenient to introduce the sequence $\mathscr{A}_{n,m}^{(\alpha)}(\lambda)$ with two indices as follows:

$$\mathscr{A}_{n,m} := \mathscr{A}_{n,m}^{(\alpha)}(\lambda) = \frac{(\lambda-1)^m}{(\alpha)_m}\sum_{k=0}^{m} s(m,k)\, H_{n+k}^{(\alpha)}(\lambda) \tag{14}$$

together with

$$\mathscr{A}_{0,m} = 1 \quad \text{and} \quad \mathscr{A}_{n,0} = H_n^{(\alpha)}(\lambda).$$

Theorem 3. *The $\mathscr{A}_{n,m}^{(\alpha)}(\lambda)$ satisfies the following three-term recurrence relation*:

$$\mathscr{A}_{n+1,m} = \frac{m+\alpha}{\lambda-1}\mathscr{A}_{n,m+1} + m\mathscr{A}_{n,m} \tag{15}$$

with the initial sequence given by

$$\mathscr{A}_{0,m} = 1.$$

Proof. From (14) and (2), we have

$$\mathscr{A}_{n,m+1} = \frac{(\lambda-1)^{m+1}}{(\alpha)_{m+1}}\sum_{k=0}^{m+1}\left[s(m,k-1) - ms(m,k)\right] H_{n+k}^{(\alpha)}(\lambda).$$

After some rearrangement, we find that

$$\mathscr{A}_{n,m+1} = \frac{(\alpha)_m}{(\alpha)_{m+1}}(\lambda-1)\mathscr{A}_{n+1,m} - \frac{m(\lambda-1)(\alpha)_m}{(\alpha)_{m+1}}\mathscr{A}_{n,m}.$$

This evidently completes the proof of Theorem 3. □

Finally, we consider the polynomials $\mathscr{A}_{n,m}^{(\alpha)}(x,\lambda)$ defined by

$$\mathscr{A}_{n,m}(x) := \mathscr{A}_{n,m}^{(\alpha)}(x,\lambda) = \sum_{k=0}^{n}\binom{n}{k}\mathscr{A}_{k,m}^{(\alpha)}(\lambda)\, x^{n-k}.$$

It obviously follows from (1) that

$$\mathscr{A}_{0,m}^{(\alpha)}(x,\lambda) = 1$$

and that

$$\mathscr{A}_{n,0}^{(\alpha)}(x,\lambda) = H_n^{(\alpha)}(x\mid\lambda).$$

Theorem 4. *The polynomials $\mathscr{A}_{n,m}^{(\alpha)}(x,\lambda)$ satisfy the following three-term recurrence relation*:

$$\mathscr{A}_{n+1,m}(x) = (x+m)\mathscr{A}_{n,m}(x) + \frac{m+\alpha}{\lambda-1}\mathscr{A}_{n,m+1}(x) \tag{16}$$

with the initial sequence given by

$$\mathscr{A}_{0,m}(x) = 1.$$

Proof. It is readily seen that

$$x\frac{d}{dx}\mathscr{A}_{n,m}(x) = n\sum_{k=0}^{n}\binom{n}{k}\mathscr{A}_{k,m}^{(\alpha)}(\lambda)\, x^{n-k} - n\sum_{k=0}^{n-1}\binom{n-1}{k}\mathscr{A}_{k+1,m}^{(\alpha)}(\lambda)\, x^{n-k-1}.$$

By using (15), we obtain

$$\begin{aligned} x\frac{d}{dx}\mathscr{A}_{n,m}(x) &= n\mathscr{A}_{n,m}(x) \\ &\quad - n\sum_{k=0}^{n-1}\binom{n-1}{k}\left(\frac{m+\alpha}{\lambda-1}\mathscr{A}_{k,m+1} + m\mathscr{A}_{k,m}\right)x^{n-k-1} \\ &= n\mathscr{A}_{n,m}(x) - n\frac{m+\alpha}{\lambda-1}\sum_{k=0}^{n-1}\binom{n-1}{k}\mathscr{A}_{k,m+1}x^{n-k-1} \\ &\quad - nm\sum_{k=0}^{n-1}\binom{n-1}{k}\mathscr{A}_{k,m}x^{n-k-1} \end{aligned}$$

After some manipulations, we thus find that

$$xn\mathscr{A}_{n-1,m}(x) = n\mathscr{A}_{n,m}(x) - n\frac{m+\alpha}{\lambda-1}\mathscr{A}_{n-1,m+1}(x) - nm\mathscr{A}_{n-1,m}(x),$$

which is obviously equivalent to (16).

□

As an immediate application of (16), let us consider the Lipschitz-Lerch zeta function defined by (see [22,24])

$$\phi(s,a,\xi) := \sum_{k=0}^{\infty} \frac{e^{2\pi i k\xi}}{(a+k)^s} \tag{17}$$

$\left(a>0;\ \Re(s)>0 \text{ when } \xi\in\mathbb{R}\setminus\mathbb{Z};\ \Re(s)>1 \text{ when } \xi\in\mathbb{Z}\right)$.

It is well-known that, if ξ is not an integer, the Lipschitz-Lerch zeta function $\phi(s,a,\xi)$ is an entire function in $s\in\mathbb{C}$ and also that the values of $\phi(s,a,\xi)$ when $s:=-n$ is nonnegative integer are given in terms of Eulerian polynomials $H_n(x\mid\lambda)$ by (see [8])

$$\phi(-n,a,\xi) = \left(1-e^{2\pi i\xi}\right)^{-1} H_n\left(a\middle|e^{-2\pi i\xi}\right).$$

We now present the following algorithm for $\phi(-n,a,\xi)$. We start with the sequence $\mathscr{R}_{0,m} := 1$ as the first row of the matrix $(\mathscr{R}_{n,m})_{n,m\geqq 0}$. Each entry is determined recursively by

$$\mathscr{R}_{n+1,m} = (a+m)\mathscr{R}_{n,m} + \frac{m+1}{e^{-2\pi i\xi}-1}\mathscr{R}_{n,m+1}.$$

Then

$$\phi(-n,a,\xi) := \frac{\mathscr{R}_{n,0}}{1-e^{2\pi i\xi}}, \tag{18}$$

where $\mathscr{R}_{n,0}$ are the first column of the matrix $(\mathscr{R}_{n,m})_{n,m}$.

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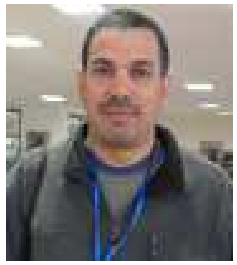

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