

The Exponentiated Kumaraswamy Linear Exponential Distribution: Theory and Application

Suleman Nasiru^{1,*}, Albert Luguterah¹ and Kwara Nantomah²

¹ Department of Statistics, Faculty of Mathematical Sciences, University for Development Studies, P. O. Box 24, Navrongo, Upper East Region, Ghana, West Africa.

² Department of Mathematics, Faculty of Mathematical Sciences, University for Development Studies, P. O. Box 24, Navrongo, Upper East Region, Ghana, West Africa.

Received: 19 Jan. 2017, Revised: 16 Aug. 2017, Accepted: 28 Aug. 2017

Published online: 1 Jan. 2018

Abstract: In this paper, a new class of distribution called the exponentiated Kumaraswamy linear exponential distribution has been proposed. This class of distribution contains several distributions such as linear exponential, Kumaraswamy linear exponential and exponentiated linear exponential as sub-models. The usefulness of the new distribution was illustrated using real data set.

Keywords: Linear exponential, Kumaraswamy, moment, hazard, reverse hazard, quantile

1 Introduction

Myriad of existing statistical distributions have been modified in the literature in order to make them more flexible in modeling lifetime data with both monotonic and non-monotonic failure rates. The linear exponential (LE) distribution is one of such distributions that have been modified by a number of researchers in the statistical literature. The LE distribution has two parameters which are both scale parameters. However, in order to model data that exhibit different degrees of skewness and kurtosis, there is a need for the distribution to have shape parameters. Thus, most modifications of the LE distribution are carried out by adding extra shape parameters to the distribution to make it more suitable for modeling phenomenon with decreasing, increasing, non-monotonic failure rates such as the bathtub, unimodal or modified unimodal, which are common in firmware reliability modeling and biological studies (see [2], [9]).

[7] developed a three parameter modification of the LE distribution and called it the generalized linear failure rate distribution. [3] proposed another generalized linear exponential distribution. [6] also developed a new generalization of the LE distribution and called it exponentiated generalized linear exponential distribution. [8] developed a new four parameter generalization of the linear exponential distribution. [4] developed another

generalization of the LE distribution and called it the Kumaraswamy linear exponential distribution.

In this paper, a new five-parameter distribution function called the exponentiated Kumaraswamy linear exponential (EKLE) distribution have been developed.

2 Exponentiated Kumaraswamy Linear Exponential Distribution

A non-negative random variable X has EKLE distribution with parameters $\lambda, \theta, a, b > 0$ and $\alpha > 0$ if its cumulative distribution function (CDF) is

$$F_{EKLE}(x; \lambda, \theta, a, b, \alpha) = \left[1 - \left(1 - \left(1 - e^{-(\lambda x + \frac{\theta x^2}{2})} \right)^a \right)^b \right]^\alpha, \quad (1)$$

for $x > 0$. The parameters λ and θ are scale parameters while a, b and α are shape parameters. The corresponding probability density function (PDF) of EKLE distribution is

$$f_{EKLE}(x; \lambda, \theta, a, b, \alpha) = \frac{\alpha ab(\lambda + \theta x) e^{-(\lambda x + \frac{\theta x^2}{2})} \left(1 - e^{-(\lambda x + \frac{\theta x^2}{2})} \right)^{a-1}}{\left(1 - \left(1 - e^{-(\lambda x + \frac{\theta x^2}{2})} \right)^a \right)^{1-b} \left[1 - \left(1 - \left(1 - e^{-(\lambda x + \frac{\theta x^2}{2})} \right)^a \right)^b \right]^{1-\alpha}}, \quad (2)$$

* Corresponding author e-mail: sulemanstat@gmail.com / snasiru@uds.edu.gh

for $x > 0$. One advantage of the EKLE distribution is, it has a closed form CDF, which enables us to easily generate random numbers from it using the relation

$$x = \frac{-\lambda + \sqrt{\lambda^2 - 2\theta \ln \left[1 - \left(1 - \left(1 - U^{\frac{1}{\alpha}} \right)^{\frac{1}{b}} \right)^{\frac{1}{a}} \right]}}{\theta}, \quad (3)$$

where U is uniformly distributed on the interval $(0, 1)$. The relation can be used to generate random samples from EKLE distribution.

The hazard rate function of EKLE distribution is given by

$$h_{EKLE}(x; \lambda, \theta, a, b, \alpha) = \frac{\alpha ab(\lambda + \theta x) e^{-(\lambda x + \frac{\theta x^2}{2})} \left(1 - e^{-(\lambda x + \frac{\theta x^2}{2})} \right)^{a-1} \left(1 - \left(1 - e^{-(\lambda x + \frac{\theta x^2}{2})} \right)^a \right)^{b-1}}{\left[1 - \left(1 - \left(1 - e^{-(\lambda x + \frac{\theta x^2}{2})} \right)^a \right)^b \right]^{1-\alpha} \left\{ 1 - \left[1 - \left(1 - \left(1 - e^{-(\lambda x + \frac{\theta x^2}{2})} \right)^a \right)^b \right]^\alpha \right\}}, \quad (4)$$

for $x > 0$. The reverse hazard rate function has been shown to play a useful role in reliability analysis [1]. It is given by

$$\tau_{EKLE}(x; \lambda, \theta, a, b, \alpha) = \frac{f_{EKLE}(x; \lambda, \theta, a, b, \alpha)}{F_{EKLE}(x; \lambda, \theta, a, b, \alpha)} = \frac{\alpha ab(\lambda + \theta x) e^{-(\lambda x + \frac{\theta x^2}{2})} \left(1 - e^{-(\lambda x + \frac{\theta x^2}{2})} \right)^{a-1}}{\left(1 - \left(1 - e^{-(\lambda x + \frac{\theta x^2}{2})} \right)^a \right)^{1-b} \left\{ 1 - \left(1 - \left(1 - e^{-(\lambda x + \frac{\theta x^2}{2})} \right)^a \right)^b \right\}^\alpha}, \quad (5)$$

for $x > 0$.

The PDF and hazard rate function of EKLE distribution for different parameter values are displayed in Figure 1 and 2 respectively. From the figures, it is clear that the PDF can be decreasing or unimodal and the hazard rate function can be increasing, constant and so on depending on the parameter values. The PDF of the EKLE distribution can be written as a linear combination of the PDF of an exponentiated linear exponential (ELE) distribution. This result is important to provide mathematical properties of the EKLE model directly from those properties of ELE distribution. For $d > 0$, a series expansion for $(1-z)^{d-1}$, for $|z| < 1$ is

$$\begin{aligned} (1-z)^{d-1} &= \sum_{k=1}^{\infty} (-1)^k \binom{d-1}{k} z^k \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k \Gamma(d)}{k! \Gamma(d-k)} z^k, \end{aligned} \quad (6)$$

where $\Gamma(\cdot)$ is the gamma function. Since $0 < e^{-(\lambda x + \frac{\theta x^2}{2})} < 1$, for $x > 0$, then using expansion (6) in (2) yields

$$\begin{aligned} f_{EKLE}(x; \lambda, \theta, a, b, \alpha) &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{j+k} \binom{a-1}{k} \binom{b(k+1)-1}{j} \\ &\times \frac{\alpha b a(j+1)(\lambda + \theta x)}{(j+1)} e^{-(\lambda x + \frac{\theta x^2}{2})} \left(1 - e^{-(\lambda x + \frac{\theta x^2}{2})} \right)^{a(j+1)-1}. \end{aligned} \quad (7)$$

It follows from (7) that

$$\begin{aligned} f_{EKLE}(x; \lambda, \theta, a, b, \alpha) &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{j+k} b \Gamma(\alpha+1) \Gamma(b(k+1))}{k!(j+1)! \Gamma(\alpha-k) \Gamma(b(k+1)-j)k!} \\ &\times f_{ELE}(x; \lambda, \theta, a(j+1)). \end{aligned} \quad (8)$$

When α is an integer, the index k stops at $\alpha - 1$, whereas the index j stops at $b(k+1) - 1$ if $b > 0$ is an integer.

2.1 Sub-models

The EKLE distribution is a very flexible model that converges to different distributions when its parameter values are changed. The EKLE distribution contains as special models the following distributions displayed in Table 1.

Table 1: Sub-models of EKLE distribution

Distribution	α	b	a	θ	λ
Kumaraswamy linear exponential	1	-	-	-	-
Kumaraswamy exponential	1	-	-	0	-
Kumaraswamy Rayleigh	1	-	-	-	0
Exponentiated Kumaraswamy exponential	-	-	-	0	-
Exponentiated Kumaraswamy Rayleigh	-	-	-	-	0
Exponentiated exponential	1	1	-	0	-
Exponentiated Rayleigh	1	1	-	-	0
Exponentiated linear exponential	1	1	-	-	-
Exponential	1	1	1	0	-
Rayleigh	1	1	1	-	0
Linear exponential	1	1	1	-	-

2.2 Quantile, Median and Mode

The EKLE quantile function, say $Q(p) = F^{-1}(p)$, can easily be computed by inverting (1). Thus, we have

$$x_p = \frac{-\lambda + \sqrt{\lambda^2 - 2\theta \ln \left[1 - \left(1 - \left(1 - p^{\frac{1}{\alpha}} \right)^{\frac{1}{b}} \right)^{\frac{1}{a}} \right]}}{\theta}, \quad (9)$$

where $p \in (0, 1)$. Using (9), the median of EKLE can be obtained as

$$x_{0.5} = \frac{-\lambda + \sqrt{\lambda^2 - 2\theta \ln \left[1 - \left(1 - \left(1 - 0.5^{\frac{1}{\alpha}} \right)^{\frac{1}{b}} \right)^{\frac{1}{a}} \right]}}{\theta}. \quad (10)$$

The mode, which is defined as the maximum value of the PDF, denoted by x_0 can be obtained numerically by solving the following non-linear equation (11) since it is not possible to obtain the explicit solution in the general case. For different special cases the explicit form may be obtained.

$$\frac{(a-1)(\theta x_0 + \lambda)e^{-(\lambda x_0 + \frac{\theta x_0^2}{2})} - \theta x_0 - \lambda - \frac{a(b-1)(\theta x_0 + \lambda)e^{-(\lambda x_0 + \frac{\theta x_0^2}{2})} \left(1 - e^{-(\lambda x_0 + \frac{\theta x_0^2}{2})}\right)^{a-1}}{1 - \left(1 - e^{-(\lambda x_0 + \frac{\theta x_0^2}{2})}\right)^a} - \frac{ab(\alpha-1)(\theta x_0 + \lambda)e^{-(\lambda x_0 + \frac{\theta x_0^2}{2})} \left(1 - e^{-(\lambda x_0 + \frac{\theta x_0^2}{2})}\right)^{ab-1}}{1 - \left(1 - e^{-(\lambda x_0 + \frac{\theta x_0^2}{2})}\right)^{ab}} = 0. \quad (11)$$

2.3 Moments

In this subsection, the r^{th} non-central moment for the EKLE distribution was derived. Moments play important role in any statistical analysis, especially in applications. They are used for finding measures of central tendency, dispersion, skewness and kurtosis among others. **Theorem 1.** If X has EKLE distribution, then the r^{th} non-central moment of X is given by the following

$$\mu'_r = \omega_{ijk} \left[\frac{\lambda \Gamma(2\ell + r + 1)}{(\lambda(i+1))^{2\ell+r+1}} + \frac{\theta \Gamma(2\ell + r + 2)}{(\lambda(i+1))^{2\ell+r+2}} \right], r = 1, 2, \dots, \quad (12)$$

where

$$\omega_{ijk} = \sum_{\ell, i, j, k=0}^{\infty} \frac{(-1)^{\ell+i+j+k} b \Gamma(\alpha+1) \Gamma(b(k+1)) \Gamma(a(j+1)+1) (\theta(i+1))^\ell}{i! k! \ell! (j+1)! 2^\ell \Gamma(\alpha-k) \Gamma(b(k+1)-j) \Gamma(a(j+1)-i)}.$$

Proof. Let X be a random variable having the density function (2). The r^{th} non-central moment of EKLE random variable is given by

$$\mu'_r = \int_0^\infty x^r f_{EKLE}(x; \lambda, \theta, a, b, \alpha) dx.$$

Using (7) and (8),

$$\mu'_r = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{j+k} b \Gamma(\alpha+1) \Gamma(b(k+1))}{k! (j+1)! \Gamma(\alpha-k) \Gamma(b(k+1)-j)} \times \int_0^\infty x^r f_{ELE}(x; \lambda, \theta, a(j+1)) dx. \quad (13)$$

Using the fact that

$$\left(1 - e^{-(\lambda x + \frac{\theta x^2}{2})}\right)^{a(j+1)-1} = \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(a(j+1))}{i! \Gamma(a(j+1)-i)} e^{-(\lambda x + \frac{\theta x^2}{2})i},$$

then

$$\mu'_r = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{i+j+k} b \Gamma(\alpha+1) \Gamma(b(k+1)) a \Gamma(a(j+1))}{i! k! j! \Gamma(\alpha-k) \Gamma(b(k+1)-j) \Gamma(a(j+1)-i)} \times \int_0^\infty x^r (\lambda + \theta x) e^{-\lambda(i+1)} e^{-\frac{\theta(i+1)x^2}{2}} dx.$$

Also, by using Taylor series expansion, we have

$$e^{-\frac{\theta(i+1)x^2}{2}} = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell \left(\frac{\theta(i+1)x^2}{2}\right)^\ell}{\ell!}.$$

Thus,

$$\begin{aligned} \mu'_r &= \omega_{ijk} \left[\lambda \int_0^\infty x^{2\ell+r} e^{-\lambda(i+1)x} dx + \theta \int_0^\infty x^{2\ell+r+1} e^{-\lambda(i+1)x} dx \right] \\ \mu'_r &= \omega_{ijk} \left[\frac{\lambda \Gamma(2\ell + r + 1)}{(\lambda(i+1))^{2\ell+r+1}} + \frac{\theta \Gamma(2\ell + r + 2)}{(\lambda(i+1))^{2\ell+r+2}} \right], \\ r &= 1, 2, \dots \end{aligned}$$

This completes the proof.

The mean of the random variable X is obtained by putting $r = 1$ in (12). Hence, the mean is

$$\mu = \omega_{ijk} \left[\frac{\lambda \Gamma(2\ell + 2)}{(\lambda(i+1))^{2\ell+2}} + \frac{\theta \Gamma(2\ell + 3)}{(\lambda(i+1))^{2\ell+3}} \right]. \quad (14)$$

The second non-central moment of the random variable X is obtained by putting $r = 2$ in (12). Hence, the second non-central moment is

$$\mu'_2 = \omega_{ijk} \left[\frac{\lambda \Gamma(2\ell + 3)}{(\lambda(i+1))^{2\ell+3}} + \frac{\theta \Gamma(2\ell + 4)}{(\lambda(i+1))^{2\ell+4}} \right]. \quad (15)$$

The variance of the random variable X is given by

$$\sigma^2 = \omega_{ijk} \left[\frac{\lambda \Gamma(2\ell + 3)}{(\lambda(i+1))^{2\ell+3}} + \frac{\theta \Gamma(2\ell + 4)}{(\lambda(i+1))^{2\ell+4}} \right] - \mu^2. \quad (16)$$

Based on the first four non-central moments of the EKLE distribution, the coefficient of skewness and kurtosis can be obtained as

$$\text{Skewness} = \frac{\mu'_3 - 3\mu'_2\mu + 2\mu^3}{\sigma^2}$$

and

$$\text{Kurtosis} = \frac{\mu'_4 - 4\mu'_3\mu + 6\mu'_2\mu^2 - 3\mu^4}{\sigma^4}$$

respectively.

2.4 Moment Generating Function

In this subsection, the moment generating function of the random variable X having an EKLE distribution was derived.

Theorem 2. If X has EKLE distribution, then the moment generating function $M_X(t)$ has the form

$$M_X(t) = \omega_{s\ell ijk}^* \left[\frac{\lambda \Gamma(2\ell + s + 1)}{(\lambda(i+1))^{2\ell+s+1}} + \frac{\theta \Gamma(2\ell + s + 2)}{(\lambda(i+1))^{2\ell+s+2}} \right] t^s, \quad (17)$$

where

$$\omega_{s\ell ijk}^* = \sum_{s\ell, i, j, k=0}^{\infty} \frac{(-1)^{\ell+i+j+k} b \Gamma(\alpha+1) \Gamma(b(k+1)) \Gamma(a(j+1)+1) (\theta(i+1))^\ell}{i! k! \ell! s! (j+1)! 2^\ell \Gamma(\alpha-k) \Gamma(b(k+1)-j) \Gamma(a(j+1)-i)}.$$

Proof. Let X be a random variable having density function (2), then by definition the moment generating function of X is given by

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} f_{EKLE}(x; \lambda, \theta, a, b, \alpha) dx.$$

Using Taylor series expansion, $e^{tX} = \sum_{s=0}^{\infty} \frac{t^s X^s}{s!}$. Thus,

$$\begin{aligned} M_X(t) &= E(e^{tX}) = \sum_{s=0}^{\infty} \frac{t^s}{s!} \int_0^\infty x^s f_{EKLE}(x; \lambda, \theta, a, b, \alpha) dx \\ &= \omega_{s\ell ijk}^* \left[\frac{\lambda \Gamma(2\ell + s + 1)}{(\lambda(i+1))^{2\ell+s+1}} + \frac{\theta \Gamma(2\ell + s + 2)}{(\lambda(i+1))^{2\ell+s+2}} \right] t^s. \end{aligned} \quad (18)$$

This completes the proof.

2.5 Incomplete Moment

In this subsection, the incomplete moment for a random variable X having an EKLE distribution was derived. The incomplete moment is useful in calculating the mean and median deviations, and measures of inequalities such as the Lorenz and Bonferroni curves.

Theorem 3. If X has EKLE distribution, then the incomplete moment $M_r(z)$ has the form

$$M_r(z) = \omega_{\ell ijk} \left[\frac{\lambda \gamma(2\ell + r + 1, \lambda(i+1)z)}{(\lambda(i+1))^{2\ell+r+1}} + \frac{\theta \gamma(2\ell + r + 2, \lambda(i+1)z)}{(\lambda(i+1))^{2\ell+r+2}} \right], \quad (19)$$

$r = 1, 2, \dots,$

where $\gamma(\vartheta, z) = \int_0^z x^{\vartheta-1} e^{-x} dx$ is the lower incomplete gamma function.

Proof. Let X be a random variable having density function (2). The incomplete moment of EKLE distribution is given by

$$M_r(z) = \int_0^z x^r f_{EKLE}(x; \lambda, \theta, a, b, \alpha) dx$$

Using (7) and (8),

$$\begin{aligned} M_r(z) &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{j+k} b \Gamma(\alpha+1) \Gamma(b(k+1))}{k! (j+1)! \Gamma(\alpha-k) \Gamma(b(k+1)-j)} \times \\ &\quad \int_0^z x^r f_{ELE}(x; \lambda, \theta, a(j+1)) dx \end{aligned} \quad (20)$$

It follows that

$$\begin{aligned} M_r(z) &= \omega_{\ell ijk} \left[\lambda \int_0^{\lambda(i+1)z} x^{2\ell+r} e^{-\lambda(i+1)x} dx + \theta \int_0^{\lambda(i+1)z} x^{2\ell+r+1} e^{-\lambda(i+1)x} dx \right] \\ &= \omega_{\ell ijk} \left[\frac{\lambda \gamma(2\ell + r + 1, \lambda(i+1)z)}{(\lambda(i+1))^{2\ell+r+1}} + \frac{\theta \gamma(2\ell + r + 2, \lambda(i+1)z)}{(\lambda(i+1))^{2\ell+r+2}} \right], \\ r &= 1, 2, \dots \end{aligned}$$

This completes the proof.

2.6 Mean and Median Deviations

Let $X \sim \text{EKLE}(\lambda, \theta, a, b, \alpha)$. The amount of scatter in X is evidently measured to extent by the totality of deviations from the mean and median. They are known as the mean deviation and median deviation: defined by

$\delta_1(x) = \int_0^\infty |x - \mu| f_{EKLE}(x; \lambda, \theta, a, b, \alpha) dx$ and $\delta_2(x) = \int_0^\infty |x - \varphi| f_{EKLE}(x; \lambda, \theta, a, b, \alpha) dx$, respectively, $\mu = E(X)$ and φ is the median of X . The measure $\delta_1(x)$ and $\delta_2(x)$ can be determined by $\delta_1(x) = 2\mu F_{EKLE}(\mu) - 2M_1(\mu)$ and $\delta_2(x) = \mu - 2M_1(\varphi)$. It is easy to compute $M_1(\mu)$ and $M_1(\varphi)$ from (19).

2.7 Mean Residual Lifetime

The mean residual lifetime (MRL) at a given time x measures the expected remaining lifetime of an individual of age x . It is also known as the life expectancy and plays a useful role in reliability analysis.

Theorem 4. If random variable X has EKLE distribution, then the MRL $K(x)$ is given by

$$K(x) = \frac{\left[\mu - \omega_{\ell ijk} \left[\frac{\lambda \gamma(2\ell + 2, \lambda(i+1)x)}{(\lambda(i+1)x)^{2\ell+2}} + \frac{\theta \gamma(2\ell + 3, \lambda(i+1)x)}{(\lambda(i+1))^{2\ell+3}} \right] \right]}{S(x)} - x, \quad (21)$$

where $S(x) = 1 - F_{EKLE}(x; \lambda, \theta, a, b, \alpha)$ is the survival function.

Proof. The MRL is given by

$$K(x) = \frac{1}{S(x)} (\mu - M_1(x)) - x,$$

where $M_1(x)$ is the first incomplete moment, which can easily be obtained from (19).

2.8 Inequality Measures

The Lorenz and Bonferroni curves are the widely used inequality measures in income and wealth distribution. Lorenz curve $L_F(x)$ can be defined as the proportion of total income volume accumulated by those units with

income lower than or equal to the volume, and Bonferroni curve, $B_F(x)$ is the scaled conditional mean curve, that is the ratio of group mean income of the population.

Theorem 5. If the random variable X has EKLE distribution, then the Lorenz curve $L_F(x)$ is given by

$$L_F(x) = \frac{\omega_{\ell ijk} \left[\frac{\lambda \gamma(2\ell+2, \lambda(i+1)x)}{(\lambda(i+1))^{2\ell+2}} + \frac{\theta \gamma(2\ell+3, \lambda(i+1)x)}{(\lambda(i+1))^{2\ell+3}} \right]}{\mu}. \quad (22)$$

Proof. By definition, the Lorenz curve can be obtained using the relationship

$$L_F(x) = \frac{\int_0^x t f(t) dt}{\mu}.$$

The integral $\int_0^x t f(t) dt$ is the first incomplete moment which can be obtained from (19). Thus, this completes the proof.

Theorem 6. If the random variable X has EKLE distribution, then the Bonferroni curve $B_F(x)$ is given by

$$B_F(x) = \frac{\omega_{\ell ijk} \left[\frac{\lambda \gamma(2\ell+2, \lambda(i+1)x)}{(\lambda(i+1))^{2\ell+2}} + \frac{\theta \gamma(2\ell+3, \lambda(i+1)x)}{(\lambda(i+1))^{2\ell+3}} \right]}{\mu F_{EKLE}(x; \lambda, \theta, a, b, \alpha)} \quad (23)$$

Proof. The proof can easily be obtained from the relationship

$$B_F(x) = \frac{L_F(x)}{F(x)}.$$

3 Order Statistics

In this section, the PDF of the r^{th} order statistics of a random sample from EKLE distribution was derived.

Theorem 7. Let $X_1, X_2, \dots, X_n$ be a random sample from EKLE distribution and $X_{1:n} < X_{2:n} < \dots < X_{n:n}$ denote the corresponding order statistics obtained from the sample. Then the PDF, $f_{r:n}(x)$, of the r^{th} order statistic $X_{r:n}$ is given by

$$f_{r:n}(x) = \sum_{i,j=0}^{\infty} \sum_{k=0}^{n-r} \frac{\alpha b \Gamma(n+1) \Gamma(\alpha(r+k)) \Gamma(b(j+1))}{j!(i+1)! \Gamma(\alpha(r+k)-j) \Gamma(b(j+1)-i)} \times f_{ELE}(x; \lambda, \theta, a(i+1)). \quad (24)$$

Proof. The PDF of $X_{r:n}$, $r = 1, 2, \dots, n$ is given by

$$f_{r:n}(x) = \frac{1}{B(r, n-r+1)} [F(x)]^{r-1} [1-F(x)]^{n-r} f(x),$$

where $F(x)$ and $f(x)$ are the CDF and PDF given by (1) and (2), respectively and $B(\cdot, \cdot)$ is the beta function. Since $0 < F(x) < 1$ for $x > 0$, by using the binomial series expansion of $[1-F(x)]^{n-r}$, given by

$$[1-F(x)]^{n-r} = \sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} [F(x)]^k,$$

we have

$$f_{r:n}(x) = \frac{1}{B(r, n-r+1)} \sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} [F(x)]^{r+k-1} f(x). \quad (25)$$

substituting (1) and (2) into (25), we have

$$f_{r:n}(x) = \sum_{i,j=0}^{\infty} \sum_{k=0}^{n-r} \frac{\alpha b \Gamma(n+1) \Gamma(\alpha(r+k)) \Gamma(b(j+1))}{j!(i+1)! \Gamma(\alpha(r+k)-j) \Gamma(b(j+1)-i)} \times f_{ELE}(x; \lambda, \theta, a(i+1)).$$

This completes the proof.

Theorem 8. The p^{th} moment of order statistic $X_{r:n}$ is

$$\mu_{r:n}'^{(p)} = \omega_{s\ell ijk} \left[\frac{\lambda \Gamma(2s+p+1)}{(\lambda(\ell+1))^{2s+p+1}} + \frac{\theta \Gamma(2s+p+2)}{(\lambda(\ell+1))^{2s+p+2}} \right], \quad p = 1, 2, \dots, \quad (26)$$

where

$$\omega_{s\ell ijk} = \sum_{s,\ell,i,j,k=0}^{\infty} \sum_{k=0}^{n-r} \frac{(-1)^{i+j+k+s+\ell} \alpha b \Gamma(n+1) \Gamma(\alpha(r+k)) \Gamma(b(j+1)) \Gamma(a(i+1)+1) (\theta(\ell+1))^s}{j! s! \ell! (i+1)! 2^s \Gamma(\alpha(r+k)-j) \Gamma(b(j+1)-i) \Gamma(a(i+1)-\ell)}$$

Proof. The general definition of the p^{th} moment of order statistic $X_{r:n}$ is

$$\mu_{r:n}'^{(p)} = \int_0^{\infty} x^p f_{r:n}(x; \lambda, \theta, a, b, \alpha) dx \quad (27)$$

Substituting (24) into (27), we have

$$\mu_{r:n}'^{(p)} = \sum_{i,j=0}^{\infty} \sum_{k=0}^{n-r} \frac{(-1)^{i+j+k} \alpha b \Gamma(n+1) \Gamma(\alpha(r+k)) \Gamma(b(j+1))}{j!(i+1)! \Gamma(\alpha(r+k)-j) \Gamma(b(j+1)-i)} \times \int_0^{\infty} x^p f_{ELE}(x; \lambda, \theta, a(i+1)) dx. \quad (28)$$

Since the integral in (28) is the p^{th} non-central moment of the $f_{ELE}(x; \lambda, \theta, a(i+1))$, using similar proof as in theorem 1 we obtain the results in (26) which completes the proof.

4 Maximum Likelihood Estimation

In this section, the maximum likelihood estimators of the parameters of the EKLE distribution were derived. Let $X_1, X_2, \dots, X_n$ be a random sample of size n from EKLE distribution. Let $z_i = -(\lambda x_i + \frac{\theta x_i^2}{2})$, then the log-likelihood function for the vector of parameters $\Theta = (\alpha, a, b, \lambda, \theta)'$ is

$$\ell = n \ln(\alpha b) + \sum_{i=1}^n z_i + \sum_{i=1}^n \ln(\lambda + \theta x_i) + (a-1) \sum_{i=1}^n \ln(1 - e^{z_i}) + (b-1) \sum_{i=1}^n \ln[1 - (1 - e^{z_i})^a] + (\alpha-1) \sum_{i=1}^n \ln[1 - (1 - (1 - e^{z_i})^a)^b]. \quad (29)$$

The log-likelihood can be maximized either directly or by solving the non-linear likelihood equations obtained by differentiating (29). The components of the score vector are given by

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \ln[1 - (1 - (1 - e^{z_i})^a)^b], \quad (30)$$

$$\begin{aligned} \frac{\partial \ell}{\partial a} &= \frac{n}{a} + \sum_{i=1}^n \ln(1 - e^{z_i}) - (b-1) \sum_{i=1}^n \frac{(1 - e^{z_i})^a \ln(1 - e^{z_i})}{1 - (1 - e^{z_i})^a} + \\ &(\alpha - 1) \sum_{i=1}^n \frac{b(1 - e^{z_i})^a (1 - (1 - e^{z_i})^a)^{b-1} \ln(1 - e^{z_i})}{1 - (1 - (1 - e^{z_i})^a)^b}, \end{aligned} \quad (31)$$

$$\begin{aligned} \frac{\partial \ell}{\partial b} &= \frac{n}{b} + \sum_{i=1}^n \ln[1 - (1 - e^{z_i})^a] - \\ &(\alpha - 1) \sum_{i=1}^n \frac{[1 - (1 - (1 - e^{z_i})^a)^b] \ln[1 - (1 - e^{z_i})^a]}{1 - (1 - (1 - e^{z_i})^a)^b}, \end{aligned} \quad (32)$$

$$\begin{aligned} \frac{\partial \ell}{\partial \lambda} &= \sum_{i=1}^n \frac{1}{\lambda + \theta x_i} - \sum_{i=1}^n x_i + (a-1) \sum_{i=1}^n \frac{x_i e^{z_i}}{1 - e^{z_i}} - \\ &(b-1) \sum_{i=1}^n \frac{a e^{z_i} (1 - e^{z_i})^a x_i}{1 - (1 - e^{z_i})^a} + \\ &(\alpha - 1) \sum_{i=1}^n \frac{a b e^{z_i} (1 - e^{z_i})^{a-1} (1 - (1 - e^{z_i})^a)^{b-1} x_i}{1 - (1 - (1 - e^{z_i})^a)^b}, \end{aligned} \quad (33)$$

$$\begin{aligned} \frac{\partial \ell}{\partial \theta} &= \sum_{i=1}^n \frac{x_i}{\lambda + \theta x_i} - \sum_{i=1}^n \frac{x_i^2}{2} + (a-1) \sum_{i=1}^n \frac{x_i^2 e^{z_i}}{2(1 - e^{z_i})} - \\ &(b-1) \sum_{i=1}^n \frac{a x_i^2 e^{z_i} (1 - e^{z_i})^{a-1}}{2(1 - (1 - e^{z_i})^a)} + \\ &(\alpha - 1) \sum_{i=1}^n \frac{a b x_i^2 e^{z_i} (1 - e^{z_i})^{a-1} (1 - (1 - e^{z_i})^a)^{b-1}}{2[1 - (1 - (1 - e^{z_i})^a)^b]}. \end{aligned} \quad (34)$$

Using the Fisher information matrix (FIM), the asymptotic confidence intervals for the parameters of the EKLE distribution can be obtained numerically. Let $\hat{\Theta} = (\hat{\alpha}, \hat{a}, \hat{b}, \hat{\lambda}, \hat{\theta})'$ be the maximum likelihood estimates of $\Theta = (\alpha, a, b, \lambda, \theta)'$, under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we obtain: $\sqrt{n}(\hat{\Theta} - \Theta) \xrightarrow{d} N_5(\mathbf{0}, I^{-1}(\Theta))$, where $I(\Theta)$ is the expected FIM. The asymptotic behavior is still valid if $I(\Theta)$ is replaced by the observed information matrix evaluated at $\hat{\Theta}$, that is $I(\hat{\Theta})$. The approximate $100(1 - \eta)\%$ two-sided confidence intervals for

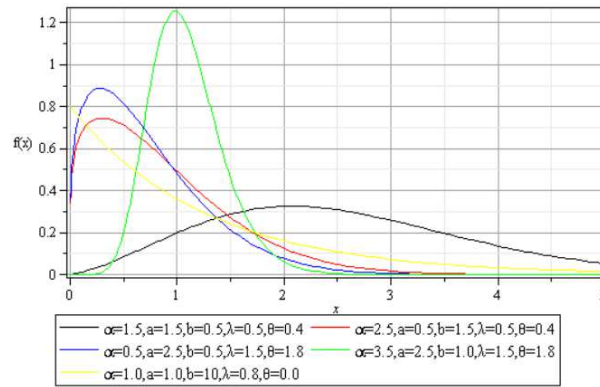


Fig. 1: Plot of density function of the EKLE distribution

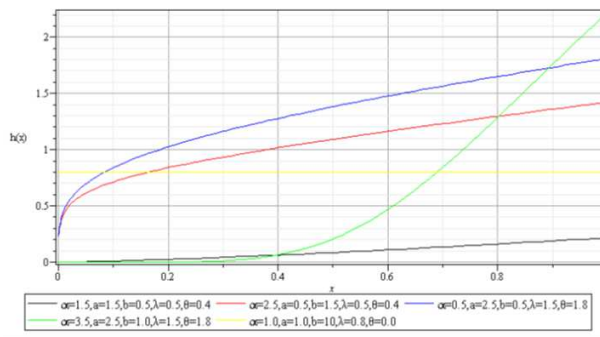


Fig. 2: Plot of hazard rate function of the EKLE distribution

α, a, b, λ and θ are given by: $\hat{\alpha} \pm Z_{\frac{\eta}{2}} \sqrt{I_{\alpha\alpha}^{-1}(\hat{\Theta})}$, $\hat{a} \pm Z_{\frac{\eta}{2}} \sqrt{I_{aa}^{-1}(\hat{\Theta})}$, $\hat{b} \pm Z_{\frac{\eta}{2}} \sqrt{I_{bb}^{-1}(\hat{\Theta})}$, $\hat{\lambda} \pm Z_{\frac{\eta}{2}} \sqrt{I_{\lambda\lambda}^{-1}(\hat{\Theta})}$ and $\hat{\theta} \pm Z_{\frac{\eta}{2}} \sqrt{I_{\theta\theta}^{-1}(\hat{\Theta})}$ respectively, where $Z_{\frac{\eta}{2}}$ is the upper $\frac{\eta}{2}$ percentile of the standard normal distribution. The likelihood ratio (LR) test can be used to compare the fit of the EKLE distribution with that of its sub-models by computing the maximized unrestricted and restricted log-likelihood for a given data set. For example, to test $\alpha = 1$, the LR test statistic is

$$\omega = 2[\ell(\hat{\alpha}, \hat{a}, \hat{b}, \hat{\lambda}, \hat{\theta}) - \ell(1, \hat{a}, \hat{b}, \hat{\lambda}, \hat{\theta})].$$

The test statistic ω is asymptotically ($n \rightarrow \infty$) distributed as χ_d^2 , where d is the degree of freedom. The LR rejects the null hypothesis if $\omega > \chi_{\gamma, d}^2$, where $\chi_{\gamma, d}^2$ denotes the upper $100\gamma\%$ quantile of the chi-square distribution.

5 Application

In this section, a real data set was used to demonstrate the application of the EKLE distribution. The performance of

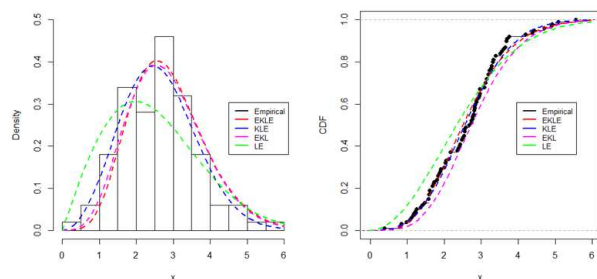


Fig. 3: Empirical, fitted densities and CDF plots

the EKLE was compared with the LE, ELE and Kumaraswamy linear exponential (KLE) distributions. The real data set corresponds to an uncensored data set from [5] on breaking stress of carbon fibres (in GPa): 3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2.00, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65.

Table 2 shows the parameter estimates of the models fitted to the above data. The asymptotic

Table 2: Estimated parameters of fitted models

Model	Parameter Estimates	Standard Error	$-\ell$
LE	$\hat{\lambda} = 1.0 \times 10^{-9}$	0.086	149.501
	$\hat{\theta} = 0.254$	0.048	
ELE	$\hat{a} = 2.198$	1.367	141.533
	$\hat{\lambda} = 0.082$	0.3	
	$\hat{\theta} = 0.344$	0.085	
KLE	$\hat{a} = 2.717$	6.358	141.292
	$\hat{b} = 3.779$	2.646	
	$\hat{\lambda} = 0.166$	61.144	
	$\hat{\theta} = 0.093$	95.487	
EKLE	$\hat{\alpha} = 2.957$	0.478	138.947
	$\hat{a} = 1.311$	0.055	
	$\hat{b} = 0.230$	0.024	
	$\hat{\lambda} = 1.407$	0.019	
	$\hat{\theta} = 1.125$	0.034	

variance-covariance matrix of the maximum likelihood estimates of the EKLE distribution is

$$I^{-1} = \begin{bmatrix} 0.229 & -2.707 \times 10^{-3} & 8.898 \times 10^{-3} & 1.410 \times 10^{-3} & -2.038 \times 10^{-3} \\ -2.707 \times 10^{-3} & 3.024 \times 10^{-3} & -3.096 \times 10^{-5} & 1.427 \times 10^{-5} & -2.261 \times 10^{-5} \\ 8.898 \times 10^{-3} & -3.096 \times 10^{-5} & 5.531 \times 10^{-4} & 1.982 \times 10^{-4} & -9.121 \times 10^{-5} \\ 1.410 \times 10^{-3} & 1.427 \times 10^{-5} & 1.982 \times 10^{-4} & 3.592 \times 10^{-4} & -1.515 \times 10^{-3} \\ -2.038 \times 10^{-3} & -2.261 \times 10^{-5} & -9.121 \times 10^{-5} & -1.515 \times 10^{-3} & 1.161 \times 10^{-3} \end{bmatrix}$$

Therefore, the 95% confidence interval for α , a , b , λ and θ are $[2.020, 3.894]$, $[1.203, 1.419]$, $[0.183, 0.277]$, $[1.370, 1.444]$ and $[1.058, 1.192]$ respectively. The LR test statistic to test the hypothesis $H_0 : \alpha = 1$ versus $H_1 : \alpha \neq 1$ is $\omega = 4.69 > 3.841 = \chi^2_{1,0.05}$, so we reject the null hypothesis. In order to compare the distributions, criteria like the Akaike information criterion (AIC), corrected Akaike information criterion (AICc) and -2ℓ were used. Also, the goodness-of-fit test of the distributions were compared using the Kolmogorov-Smirnov (K-S) statistic. The better distribution corresponds to the smaller AIC, AICc, -2ℓ and K-S values. The values in Table 3 indicates that the EKLE distribution leads to a better fit than the other models. Figure 3 displays the empirical density, the fitted

Table 3: Estimated parameters of fitted models

Model	-2ℓ	AIC	AICc	K-S	P-value
LE	299.002	303.002	303.126	0.139	0.042
ELE	283.066	289.066	289.316	0.076	0.607
KLE	282.584	290.584	291.005	0.081	0.527
EKLE	277.894	287.894	288.532	0.065	0.796

densities, the empirical CDF and the fitted cumulative distribution functions.

6 Conclusion

In this article, a new model has been proposed, the so called exponentiated Kumaraswamy linear exponential distribution which extends the linear exponential distribution in analysis of data. Various statistical properties of the new distribution such as moments, moment generating function and incomplete moment have been derived. The estimation of parameters of this new distribution was approached by the method of maximum likelihood. An application of the exponentiated Kumaraswamy linear exponential distribution to real data set revealed that the new distribution can be used quite effectively to provide better fits than its sub-models.

Acknowledgment

The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

References

- [1] R. C Gupta and R. D Gupta, Proportional reverse hazard model and its applications, *Journal of Statistical Planning and Inference* **137**(1), 3525-3536 (2007).
- [2] C. D Lai, M. Xie and D. N. P Murthy, Bathtub shaped failure rate distributions, in: N. Balakrishnan and C. R Rao (Eds). *Handbook in Reliability* **30**, 69-104 (2001).
- [3] M. A. W Mahmoud and F. M. A Alam, The generalized linear exponential distribution, *Statistics and Probability Letters* **80**, 1005-1014 (2010).
- [4] F. Merovci and I. Elbatal, A new generalization of linear exponential distribution: theory and application, *Journal of Statistics Applications and Probability Letters* **2**(1), 1-14 (2015).
- [5] M. D Nichols and W. J Padgett, A bootstrap control chart for Weibull percentiles, *Quality and Reliability Engineering International* **22**, 141-151 (2006).
- [6] A. M Sarhan, A. E. A Ahmad and A. I Alasbahi, Exponentiated generalized linear exponential distribution, *Applied Mathematical Modeling* **37**, 2838-2849 (2013).
- [7] A. M Sarhan and D. Kundu, Generalized linear failure rate distribution, *Communications in Statistics: Theory and Methods*, **38**(5), 642-660 (2009).
- [8] Y. Tian, M. Tian and Q. Zhu, A new generalized linear exponential distribution and its applications, *Acta Mathematicae Applicatae Sinica, English Series* **30**(4), 1049-1062 (2014).
- [9] T. Zhang, M. Xie, L. C Tang and S. H Ng, Reliability and modeling of systems integrated with firmware and hardware, *International Journal of Reliability and Quality Safety Engineering* **12**(3), 227-239 (2005).

Albert Luguterah (PhD) is currently the General Secretary of the Ghana Statistical Association, as well as a member of several Local and International Statistical organizations. He is widely researched, and his participated, and also presented, in several local and international conferences.



functions.

Kwara Nantomah

(PhD) is a full time Senior Lecturer of the Faculty of Mathematical Sciences at the University for Development Studies, Navrongo, Ghana. His research interest includes: real analysis, functional analysis, theory of inequalities and special



Suleman Nasiru is a Lecturer at the Department of Statistics, University for Development Studies. He is currently pursuing his PhD in Mathematics (Statistics option) at the Pan African University, Institute of Basic Science, Technology and Innovation, Nairobi, Kenya.



Albert Luguterah (PhD) is a full time Senior Lecturer and Dean of the Faculty of Mathematical Sciences at the University for Development Studies, Navrongo, Ghana. He holds a BSc. Statistics with Mathematics from the University of Ghana, MSc. in Statistics from the University

of Cape Coast, and a PhD in Applied Statistics from the University for Development Studies (All in Ghana).