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An efficient approximate-analytical method to solve time-fractional KdV and KdVB equations

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Abstract: In this article, we present the modified generalized Mittag-Leffler function method (MGMLFM) as an approximate-analytical method to give a proper solution of time-fractional Korteweg-de Vries (KdV) and Korteweg-de Vries-Burger's (KdVB) equations, which have various applications in physics and applied mathematics. The time-fractional partial derivatives are based on Caputo sense. The obtained solution is constructed in a rapidly convergent power series. By comparing the approximate MGMLFM solutions when the fractional operator equal one with known exact solutions we have an appropriate agreement. The advantage of the article is to apply the suggested method to solve linear and nonlinear time-fractional partial differential equations, where it needs less computational effort which saves time and effort. The convergence of absolute error be controlled on by the parameters in the time-fractional KdV and KdVB equations were found. The simulation of the obtained results is presented in the forms of graphs to illustrate the reliability and efficiency of our method.

Keywords: Applications in the physical sciences; Time-fractional partial differential equations; Mittag-Leffler function; Series solutions; KdV and KdVB equations.

1 Introduction

Korteweg de Vries (KdV) equation was first proposed by Korteweg and de Vries [1]. After that, the KdV equation has been contributing to describe various nonlinear phenomena in physics and applied mathematics such as solid-state physics, particle acoustic waves, stratified internal waves, plasma physics, particle acoustic waves, fluid mechanics, quantum field and so on [2,3,4,5]. The standard form of the KdV equation is

$$U_t + \lambda U U_x + \mu U_{xxx} = 0, \tag{1}$$

where λ and μ are real constants not equal to zero.

The Korteweg de Vries-Burgers (KdVB) equation was introduced by Su and Gardner [6], which considered as a combination of the KdV equation (1) and the following Burgers equation

$$U_t + \lambda U U_x + \epsilon U_{xx} = 0, \tag{2}$$

where ϵ is real constant not equal to zero. So, the standard form of the KdVB equation is

$$U_t + \lambda U U_x + \epsilon U_{xx} + \mu U_{xxx} = 0. \tag{3}$$

The KdVB equation has been used as a nonlinear model to describe physical phenomena of interest such as plasma waves, the flow of liquids containing gas bubbles, the propagation of waves on an elastic tube filled with a viscous fluid, the propagation of undular bores in shallow water, and turbulence, etc (see e.g., [7,8,9,10,11,12,13,14]).

There are many methods used to provide numerical and analytical solutions for KdV and KdVB equations in the literature such as decomposition and Adomian decomposition method [15,16], Radial basis function [17,18], tanh method [19,20], Homotopy and Homotopy perturbation method [21,22], Finite difference method [23,24], Ansatz and cubic B-spline Galerkin method [25,26], and for other different methods (see e,g. [27,28,29,30,31]).

Recently, fractional partial differential equations (FPDEs) have appeared in numerous research fields of physics, finance, engineering and many other applied science (see e.g., [32,33,34,35,36,37,38,39,40,41,42]) and some references cited therein. This is due to the accuracy and realistic of FPDEs than others in describing mathematical models that describe these phenomena in

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these various fields, where depend on upon all the historical states not only its current state. There are many researchers have devoted their efforts to develop methods for solving linear and nonlinear FPDEs and it is noted that there is difficulty in finding approximate and analytical methods of solution, therefore, to find accurate methods to solve these FPDEs are still under consideration. For instance, some of the methods solving FPDEs as homotopy analysis method (HAM) [43,44,45,46,47], homotopy perturbation method (HPM)[48,49,50,51], generalized Mittag-Leffler function method (GMLFM) [54,55,56,57,58], homotopy analysis transform method (HATM) [52,53] and so on.

The main goal and motivation of this research is to investigate MGMLFM as an approximate-analytical method to solve the time-fractional KdV and KdVB equations in the following form, respectively,

$$ {}_0^CD_t^\alpha U(x,t) + U(x,t)U_x(x,t) + \frac{1}{2}U_{xxx}(x,t) = 0, \quad (4)$$

and

$$ {}_0^CD_t^\alpha U(x,t) + U(x,t)U_x(x,t) - \delta U_{xx}(x,t) + \frac{1}{2}U_{xxx}(x,t) = 0, \quad (5)$$

where ${}_0^CD_t^\alpha$ is a Caputo fractional partial derivative of order α with respect to time, δ is real constant not equal to zero, U_{xx} represents a viscous loss, U_{xxx} is dispersion and UU_x is convective nonlinearity.

The most advantages of this method is an easy and simple technique to solve linear and nonlinear FPDEs and achieves the work with small efforts of computational comparing with other methods. We compare the solutions obtained by MGMLFM with the exact solutions and extract the absolute error to prove the efficiency of the method. Moreover, Eqs. (4) and (5) have never been studied before by the proposed method which indicates the novelty of results.

This paper is organized as follows. In Section 2, we give some basic definitions and preliminaries facts which are required to reach our main results. In Section 3, we introduce the analysis of the MGMLFM to solve a general form of time-FPDEs. Section 4 is divided into two subsections and its devoted to applying the MGMLFM to solve time-fractional KdV and KdVB equations. In Section 5, we offered some numerical simulations to compare our results with the exact solution in order to prove the accuracy and efficacy of our methodology. Finally, Section 6 presents conclusion of this research.

2 Some Basic Definitions and Preliminaries Facts

This section presents a review of some basic concepts which are essentially relevant to the results of this manuscript (see e.g. [59,60]).

Definition 1. *Let $\mathcal{F}(t)$ be an integrable function on the interval $[t_0,T]$, $t \in [t_0,T]$. Then, the Riemann-Liouville fractional integral of order $\alpha > 0$ defined by*

$$ {}_{t_0}I_t^\alpha \mathcal{F}(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-\eta)^{\alpha-1}\mathcal{F}(\eta)d\eta, \quad t_0 \geq 0, \quad t > t_0,$$
$$ {}_{t_0}I_t^0 \mathcal{F}(t) = \mathcal{F}(t),$$

where $\Gamma(\cdot)$ is the Euler gamma function which defined as follows

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1}e^{-t}dt, \quad (Re(\xi) > 0).$$

Definition 2. *Let $\mathcal{F}(x,t)$ be absolutely continuous functions on the interval $[t_0,T]$, $t \in [t_0,T]$. and $n < \alpha \leq n-1$, where $n \in \mathbb{N}$. Then, the Caputo fractional partial derivative is defined as*

$$ {}_{t_0}^CD_t^\alpha \mathcal{F}(x,t) = \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^t (t-\eta)^{n-\alpha-1}\frac{\partial^n \mathcal{F}(x,\eta)}{\partial \eta^n}d\eta, \quad t_0 \geq 0, \ t > t_0,$$

In particularly, for $0 < \alpha < 1$, the Caputo fractional partial derivative becomes

$$ {}_{t_0}^CD_t^\alpha \mathcal{F}(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_{t_0}^t (t-\eta)^{-\alpha}\frac{\partial \mathcal{F}(x,\eta)}{\partial \eta}d\eta, \quad t_0 \geq 0, \quad t > t_0.$$

Theorem 1. *let $\mathcal{F}(x,t)$ be a differentiable function in the interval $[t_0,T]$, $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, and $\gamma > -1$, then*

$$ {}_{t_0}^CD_t^\alpha \ {}_{t_0}I_t^\alpha \mathcal{F}(x,t) = \mathcal{F}(x,t),$$
$$ {}_{t_0}I_t^\alpha \ {}_{t_0}^CD_t^\alpha \mathcal{F}(x,t) = \mathcal{F}(x,t) - \sum_{k=0}^{n-1}\frac{\partial^k \mathcal{F}(x,t_0)}{\partial t^k}\frac{(t-t_0)^k}{k!}.$$

In addition, the fractional operator satisfies the following properties:

$$ {}_0^CD_t^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}t^{\gamma-\alpha},$$
$$ {}_0I_t^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}t^{\gamma+\alpha}.$$

Definition 3. *The two-parameter Mittag-Leffler function defined by the power series in the form*

$$E_{\alpha,\beta}(x) = \sum_{n=0}^\infty \frac{x^n}{\Gamma(n\alpha+\beta)}, \quad \alpha,\beta > 0,$$

if $\beta = 1$, this function is denoted by $E_\alpha(\cdot)$, and if $\alpha = \beta = 1$ this function represent the exponential function.

Lemma 1. *The Caputo fractional derivative of generalized Mittag-Leffler function is given by*

$$ {}_0^C D_t^\alpha E_\alpha(\lambda t^\alpha) = {}_0^C D_t^\alpha (\sum_{n=0}^{\infty} \frac{\lambda^n t^{n\alpha}}{\Gamma(n\alpha+1)}) = \sum_{n=1}^{\infty} \frac{\lambda^n t^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)} $$
$$ = \sum_{n=0}^{\infty} \frac{\lambda^{n+1} t^{n\alpha}}{\Gamma(n\alpha+1)} = \lambda E_\alpha(\lambda t^\alpha). $$

Theorem 2.[61] *Let a function $u(x,t) = \sum_{k=0}^{\infty} \xi^k u_k(x,t)$, then a nonlinear operator $\mathcal{N}(u)$ satisfies the following*

$$ \frac{\partial^n}{\partial \xi^n} \mathcal{N}(u)_{\xi=0} = \frac{\partial^n}{\partial \xi^n} \mathcal{N}\left(\sum_{k=0}^{\infty} \xi^k u_k\right)_{\xi=0} = \frac{\partial^n}{\partial \xi^n} \mathcal{N}\left(\sum_{k=0}^{n} \xi^k u_k\right)_{\xi=0}. $$

3 Description of the MGMLFM

In this section, we demonstrate the fundamental idea of MGMLFM to solve nonlinear FPDEs with initial conditions of the following general form:

$$ {}_{t_0}^C D_t^\alpha u(x,t) = \mathcal{L}(\, u(x,t)) + \mathcal{N}(\, u(x,t)) \,, \tag{6} $$

with the initial condition

$$ u(x,0) = \mathcal{G}(x), \tag{7} $$

where $\mathcal{L}$ and $\mathcal{N}$ are constituted the general linear and nonlinear differential operator for the function $u(x,t)$, respectively, and $\mathcal{G}(x)$ is a know function of variable x.

The MGMLFM suggested that the solution of Eq.(6) can be written as the following

$$ u(x,t) = G(x) E_\alpha(A t^\alpha) = \sum_{j=0}^{\infty} G(x) A^j \frac{t^{j\alpha}}{\Gamma(j\alpha+1)}, \tag{8} $$

where A is undetermined coefficient, from the initial condition (7), we obtain $G(x) = \mathcal{G}(x)$. Furthermore, by using Lemma 1 the nonlinear FPDE (6) becomes

$$ \sum_{j=0}^{\infty} \mathcal{G}(x) A^{j+1} \frac{t^{j\alpha}}{\Gamma(j\alpha+1)} = L(\sum_{j=0}^{\infty} \mathcal{G}(x) A^j \frac{t^{j\alpha}}{\Gamma(j\alpha+1)}) + N(\sum_{j=0}^{\infty} \mathcal{G}(x) A^j \frac{t^{j\alpha}}{\Gamma(j\alpha+1)}) \tag{9} $$

Therefore, the linear part can be decomposed as

$$ \mathcal{L}(u(x,t)) = \mathcal{L}(\sum_{j=0}^{\infty} \mathcal{G}(x) A^j \frac{t^{j\alpha}}{\Gamma(j\alpha+1)}) $$
$$ = \mathcal{L}(\, \mathcal{G}(x)) \sum_{j=0}^{\infty} A^j \frac{t^{j\alpha}}{\Gamma(j\alpha+1)} $$
$$ = h \mathcal{G}(x) \sum_{j=0}^{\infty} A^j \frac{t^{j\alpha}}{\Gamma(j\alpha+1)}, \tag{10} $$

where h is a constant. By helping of the Theorem 2 and He's polynomials [61,62,63] the nonlinear part can be written as follows

$$ \mathcal{N}(u(x,t)) = \mathcal{N}(\sum_{j=0}^{\infty} \mathcal{G}(x) A^j \frac{t^{j\alpha}}{\Gamma(j\alpha+1)}) $$
$$ = \mathcal{N}(\sum_{j=0}^{\infty} \mathcal{G}(x) u_j(t)) $$
$$ = \mathcal{N}(\, \mathcal{G}(x)) \, (\mathcal{N}(u_0(t)) + \sum_{j=1}^{\infty} (\mathcal{N}(\sum_{k=0}^{j} u_k(t)) $$
$$ - \mathcal{N}(\sum_{k=0}^{j-1} u_k(t))\,)) \tag{11} $$

By replacing linear and nonlinear parts from Eq.(10) and Eq.(11) in Eq.(9), this leading to identify the recurrence relation and obtain the coefficient A and then obtained the general solution of the nonlinear FPDE.

4 Implementation of the MGMLFM and Results

Here, we apply the MGMLFM on KdV and KdVB equations involving on the time-fractional derivative. Furthermore, we present some numerical simulations to illustrate the advantages and accuracy of the proposed method.

4.1 Time-Fractional KdV Equation

We consider the time-fractional KdV equation as follows

$$ {}_0^C D_t^\alpha U(x,t) + U(x,t) U_x(x,t) + \frac{1}{2} U_{xxx}(x,t) = 0, $$
$$ 0 < \alpha \leq 1, \tag{12} $$

with initial condition

$$ U(x,0) = 6\gamma^2 \operatorname{sech}^2(\gamma x). \tag{13} $$

Note that, this KdV equation have an exact solution when $\alpha = 1$ [52] as follows

$$ U(x,t) = 6\gamma^2 \operatorname{sech}^2(\gamma x - 2\gamma^3 t). \tag{14} $$

Applying the MGMLFM to Eq.(12), and using Eq.(8), we assume

$$ U(x,t) = F(x) E_\alpha(A t^\alpha) = \sum_{n=0}^{\infty} F(x) A^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \tag{15} $$

where A is undetermined coefficient. From the initial condition (13), we have $F(x) = 6\gamma^2 \operatorname{sech}^2(\gamma x)$. By using Eq.(10), we obtain the linear part of Eq.(12) as the following

$$\mathcal{L}(U(x,t)) = \frac{1}{2}U_{xxx}(x,t) = \frac{1}{2}\frac{\partial^3(6\gamma^2\operatorname{sech}^2(\gamma x))}{\partial x^3}$$
$$= 4\gamma^3 U(x,0)(2\tanh(\gamma x)\operatorname{sech}^2(\gamma x) - \tanh^3(\gamma x)).$$

Similarly, the nonlinear part of Eq.(12) satisfies

$$\mathcal{N}(U(x,t)) = U(x,t)U_x(x,t)$$
$$= (6\gamma^2\operatorname{sech}^2(\gamma x))\frac{\partial(6\gamma^2\operatorname{sech}^2(\gamma x))}{\partial x}$$
$$= U(x,0)(-12\gamma^3\tanh(\gamma x)\operatorname{sech}^4(\gamma x)).$$

By using Eq.(9) we get

$$U(x,0)\sum_{n=0}^{\infty}(A^{n+1} - 12\gamma^3\tanh(\gamma x)\operatorname{sech}^2(\gamma x)C^n\Gamma(n\alpha+1) + 4\gamma^3(2\tanh(\gamma x)\operatorname{sech}^2(\gamma x) - \tanh^3(\gamma x))A^n)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)} = 0, \quad (16)$$

where

$$C^n = \sum_{k=0}^{n}\frac{A^k A^{n-k}}{\Gamma(k\alpha+1)\Gamma((n-k)\alpha+1)}$$

Therefore, the recurrence relation will be as follows

$$A^{n+1} = 12\gamma^3\tanh(\gamma x)\operatorname{sech}^2(\gamma x)C^n\Gamma(n\alpha+1) - 4\gamma^3(2\tanh(\gamma x)\operatorname{sech}^2(\gamma x) - \tanh^3(\gamma x))A^n. \quad (17)$$

By substitute different values of n and doing some important calculation, we have

$$A^0 = 1,$$
$$A^1 = 4\gamma^3\tanh(\gamma x),$$
$$A^2 = 16\gamma^6\tanh^2(\gamma x)(3\operatorname{sech}^2(\gamma x)+1),$$
$$A^3 = 64\gamma^9\tanh^3(\gamma x)(9\operatorname{sech}^4(\gamma x) + 6\operatorname{sech}^2(\gamma x) + 3\operatorname{sech}^2(\gamma x)\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+)^2} + 1)\vdots$$

From Eq.(15) , we obtain approximate solution in a series form as the following

$$U(x,t) = 6\gamma^2\operatorname{sech}^2(\gamma x)(1 + 4\gamma^3\tanh(\gamma x)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + 16\gamma^6\tanh^2(\gamma x)(3\operatorname{sech}^2(\gamma x)+1)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 64\gamma^9\tanh^3(\gamma x)(9\operatorname{sech}^4(\gamma x) + 6\operatorname{sech}^2(\gamma x) + 3\operatorname{sech}^2(\gamma x)\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+)^2} + 1)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots). \quad (18)$$

In fact, these results satisfy the exact solution shown earlier in Eq.(14) when $\alpha = 1$, which means that the approximate solutions obtained by MGMLFM are rapidly convergent to the exact solutions and this is illustrated by the simulation presented in Section 5.

4.2 Time-Fractional KdVB Equation

We consider the time-fractional KdVB equation as follows

$$ {}_0^C D_t^{\alpha}U(x,t) + U(x,t)U_x(x,t) - \delta U_{xx}(x,t) + \frac{1}{2}U_{xxx}(x,t) = 0, \quad 0 < \alpha \le 1, (19)$$

with initial condition

$$U(x,0) = a\delta^2(2 - 2\tanh(\phi) + \operatorname{sech}^2(\phi)), \quad (20)$$

where

$$a = \frac{6}{25} \quad \text{and} \quad \phi = \frac{\delta x}{5}.$$

This KdVB equation is exactly solvable when $\alpha = 1$ [52] and introduced as follows

$$U(x,t) = 2a\delta^2 - 2a\delta^2\tanh(\omega) + 2a\delta^2\operatorname{sech}^2(\omega), \quad (21)$$

where $\omega = (\frac{-\delta x}{5} - \frac{2a\delta^3 t}{5})$.

Applying the MGMLFM to Eq.(19), and using Eq.(8), we assume

$$U(x,t) = \mathcal{W}(x)E_{\alpha}(At^{\alpha}) = \sum_{n=0}^{\infty}\mathcal{W}(x)A^n\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \quad (22)$$

From Eq.(20), we have $\mathcal{W}(x) = a\delta^2(2 - 2\tanh(\phi) + \operatorname{sech}^2(\phi))$. By using Eq.(10), we obtain the linear part of Eq.(19) as the following

$$\mathcal{L}(U(x,t)) = \frac{1}{2}U_{xxx}(x,t) - \delta U_{xx}(x,t)$$
$$= \frac{1}{2}\frac{\partial^3(a\delta^2(2-2\tanh(\phi)+\operatorname{sech}^2(\phi)))}{\partial x^3} - \delta\frac{\partial^2(a\delta^2(2-2\tanh(\phi)+\operatorname{sech}^2(\phi)))}{\partial x^2}$$
$$= \frac{12a\delta^5}{125}\operatorname{sech}^2(\phi)(3\operatorname{sech}^2(\phi) - 2\tanh(\phi) - 2 + \tanh(\phi)\operatorname{sech}^2(\phi)).$$

Similarly, the nonlinear part of Eq.(19) satisfies

$$\mathcal{N}(U(x,t)) = U(x,t)U_x(x,t)$$
$$= (a\delta^2(2-2\tanh(\phi)+\operatorname{sech}^2(\phi)))\frac{\partial(a\delta^2(2-2\tanh(\phi)+\operatorname{sech}^2(\phi)))}{\partial x}$$
$$= \frac{12a\delta^5}{125}\operatorname{sech}^2(\phi)(-3\operatorname{sech}^2(\phi) - \tanh(\phi)\operatorname{sech}^2(\phi)).$$

By using Eq.(9) we get

$$\sum_{n=0}^{\infty}\Big(a\delta^2(2-2\tanh(\phi)+\operatorname{sech}^2(\phi))A^{n+1}+\frac{12a\delta^5}{125}\operatorname{sech}^2(\phi)[(-3\operatorname{sech}^2(\phi)-\tanh(\phi)\operatorname{sech}^2(\phi))\,C^n\Gamma(n\alpha+1)$$
$$+(3\operatorname{sech}^2(\phi)-2\tanh(\phi)-2+\tanh(\phi)\operatorname{sech}^2(\phi))\,A^n]\Big)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}=0,$$

where

$$C^n=\sum_{k=0}^{n}\frac{A^kA^{n-k}}{\Gamma(k\alpha+1)\Gamma((n-k)\alpha+1)}.$$

Therefore, the recurrence relation will be as follows

$$A^{n+1}=\frac{12\delta^3}{125}\operatorname{sech}^2(\phi)\,\frac{(3\operatorname{sech}^2(\phi)+\tanh(\phi)\operatorname{sech}^2(\phi))\,C^n\Gamma(n\alpha+1)-(3\operatorname{sech}^2(\phi)-2\tanh(\phi)-2+\tanh(\phi)\operatorname{sech}^2(\phi))\,A^n}{2-2\tanh(\phi)+\operatorname{sech}^2(\phi)}.$$

By substitute different values of n and doing some important calculation, we have:

$$A^0=1,$$
$$A^1=\frac{24\delta^3}{125}\operatorname{sech}^2(\phi)\Big(\frac{\tanh(\phi)+1}{2-2\tanh(\phi)+\operatorname{sech}^2(\phi)}\Big),$$
$$A^2=\frac{12\delta^3}{125}\operatorname{sech}^2(\phi)\frac{(3\operatorname{sech}^2(\phi)+\tanh(\phi)\operatorname{sech}^2(\phi))\,2A^0A^1-(3\operatorname{sech}^2(\phi)-2\tanh(\phi)-2+\tanh(\phi)\operatorname{sech}^2(\phi))\,A^1}{2-2\tanh(\phi)+\operatorname{sech}^2(\phi)},$$
$$A^3=\frac{12\delta^3}{125}\operatorname{sech}^2(\phi)\frac{(3\operatorname{sech}^2(\phi)+\tanh(\phi)\operatorname{sech}^2(\phi))\,C^2-(3\operatorname{sech}^2(\phi)-2\tanh(\phi)-2+\tanh(\phi)\operatorname{sech}^2(\phi))\,A^2}{2-2\tanh(\phi)+\operatorname{sech}^2(\phi)},$$
$$\vdots$$

where

$$C^2=(2A^0A^2+(\frac{A^1}{\Gamma(\alpha+1)})^2\Gamma(2\alpha+1))$$

From Eq.(22) , we obtain approximate solution as the following

$$U(x,t)=a\delta^2(2-2\tanh(\phi)+\operatorname{sech}^2(\phi))\Big(A^0+A^1\frac{t^\alpha}{\Gamma(\alpha+1)}+A^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+A^3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\cdots\Big). \tag{23}$$

This approximate solution obtained by MGMLFM coincides with the exact solution given by Eq.(21) when $\alpha = 1$. This confirms that the approximate solution obtained by the proposed method is rapidly convergent to the exact solutions and this is explained from the simulations presented in the following section.

5 Results and discussion

In this section, we introduce the simulation of our results. In Figure 1, we present the MGMLFM solution (18) of time-fractional KdV equation (12) when $\gamma = 0.1$, with different values of $\alpha = 1, 0.9, 0.8$ and comparing with the exact solution given by Eq.(14). Figure 2 shows the absolute errors for time-fractional KdV when $\alpha = 1$ among different values of γ and it becomes clear that whenever decreases the value of γ decreases the absolute errors. In Figure 3, we show the MGMLFM solution (23) of time-fractional KdVB equation (19) when $\delta = 0.1$, with different values of $\alpha = 1, 0.9, 0.8$ and comparing with the exact solution given by Eq.(21). Figure 4 shows the absolute errors for the time-fractional KdVB when $\alpha = 1$ with different values of δ and it becomes clear that whenever δ very small the absolute errors decrease, hence we obtain an accurate approximation. These graphs prove that the solution obtained by the proposed method approach the exact solution when $\alpha \to 1$.

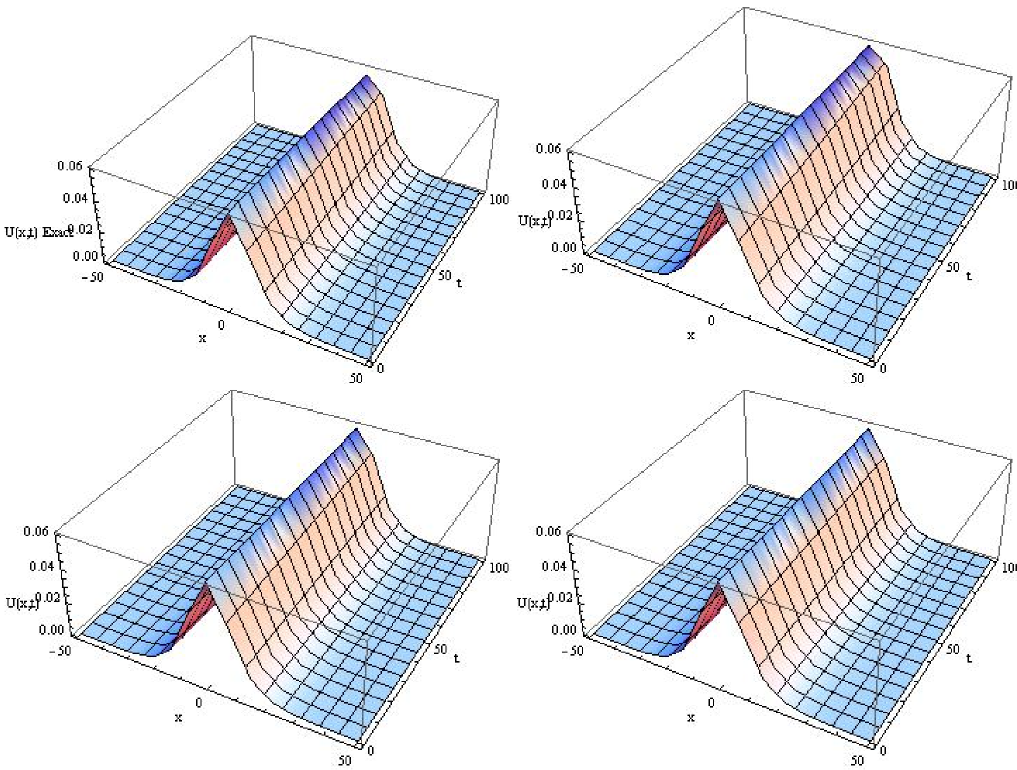

Fig. 1: The exact and approximate solutions obtained by MGMLFM of Eq.(12) for $\gamma = 0.1$ with $\alpha = 1$, $\alpha = 0.9$ and $\alpha = 0.8$, respectively.

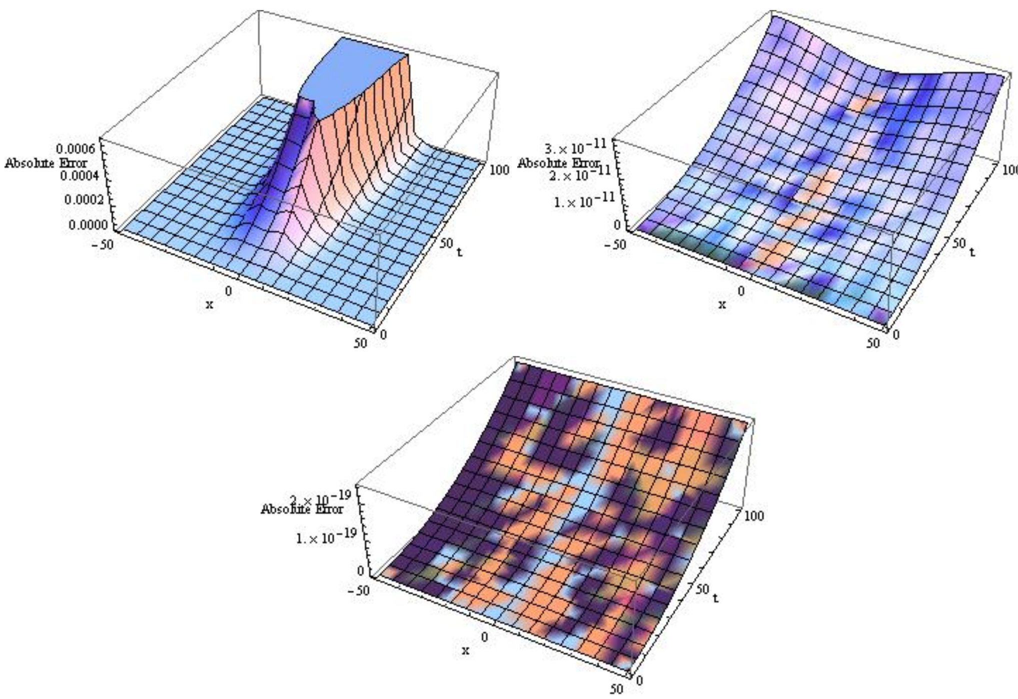

Fig. 2: The absolute error between the exact solution and MGMLF solution when $\alpha = 1$ for KdV equation (12) when $\gamma = 0.1, 0.01$ and 0.001, respectively.

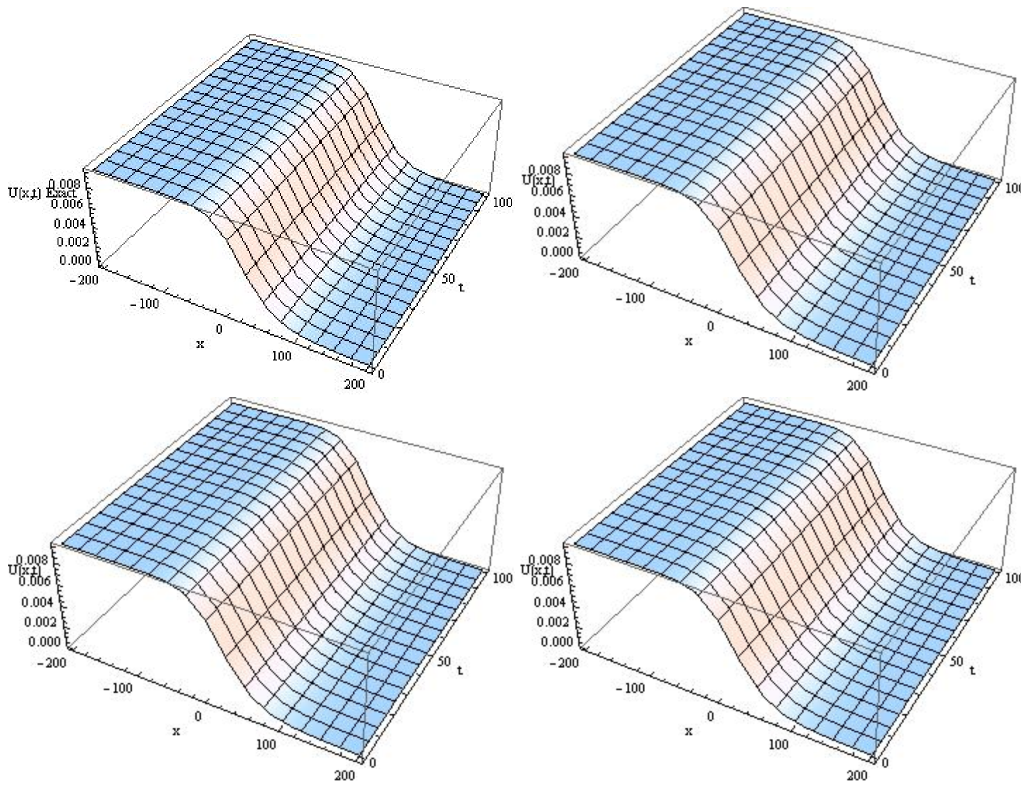

Fig. 3: The exact and approximate solutions obtained by MGMLFM of Eq.(19) for $\delta = 0.1$ with $\alpha = 1$, $\alpha = 0.9$ and $\alpha = 0.8$, respectively.

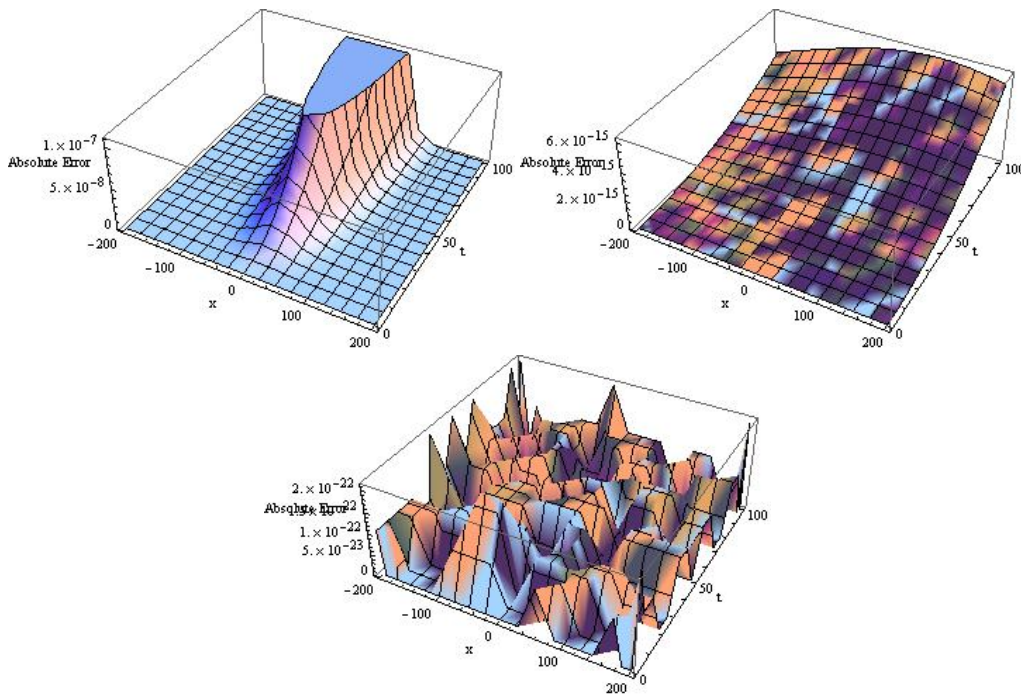

Fig. 4: The absolute error between the exact solution and MGMLF solution when $\alpha = 1$ for KdVB equation (19) when $\delta = 0.1, 0.01$ and 0.001, respectively.

Table 1: Value of solutions obtained by MGMLFM, exact solution, and absolute errors of KdV equation (12) with different values of t and γ when $\alpha = 1$.

γ	x	t	MGMLFM solution	Exact solution	Absolute error
0.1	30	5	0.000603864	0.00060386	4.15823×10^{-9}
		15	0.000628414	0.000628375	3.91059×10^{-8}
		25	0.000653992	0.000653879	1.12544×10^{-7}
0.01	30	5	0.000549085	0.000549085	7.58348×10^{-14}
		15	0.000549092	0.000549092	6.82526×10^{-13}
		25	0.000549098	0.000549098	1.89594×10^{-12}
0.001	30	5	5.9946×10^{-6}	5.9946×10^{-6}	8.47033×10^{-22}
		15	5.9946×10^{-6}	5.9946×10^{-6}	4.23516×10^{-21}
		25	5.9946×10^{-6}	5.9946×10^{-6}	1.52466×10^{-20}

Table 2: Value of solutions obtained by MGMLFM, exact solution, and absolute errors of KdVB equation (19) with different values of t and δ when $\alpha = 1$.

δ	x	t	MGMLFM solution	Exact solution	Absolute error
0.1	30	5	0.00393247	0.00393247	1.1008×10^{-9}
		15	0.00393752	0.00393751	9.92569×10^{-9}
		25	0.00394259	0.00394256	2.76226×10^{-8}
0.01	30	5	0.0000690373	0.0000690373	1.52466×10^{-17}
		15	0.0000690373	0.0000690373	1.37219×10^{-16}
		25	0.0000690374	0.0000690374	3.81165×10^{-16}
0.001	30	5	7.17111×10^{-7}	7.17111×10^{-7}	1.05879×10^{-22}
		15	7.17111×10^{-7}	7.17111×10^{-7}	1.05879×10^{-22}
		25	7.17111×10^{-7}	7.17111×10^{-7}	0

6 Conclusion

In this article, the MGMLFM has been successfully employed to obtain approximate-analytical solutions of time-fractional KdV and KdVB equations, Eq.(12) and Eq.(19), respectively. The MGMLFM gives series solutions as in Eq.(18) and Eq.(23), which converge rapidly, and require less computational work, and provide highly accurate results when comparing these approximate solutions when $\alpha = 1$ with recognized exact solutions which are shown in Eq.(14) and Eq.(21). Some simulation of the obtained results was presented in the forms of graphs and tables. As it turns out that the absolute error can be controlled through the parameters in the time-fractional KdV and KdVB equations. From the previous arguments, It becomes clear to us that placing into the practice of the MGMLFM method is very reliable, well-organized, easily and it relevant to solve further nonlinear FPDEs that have various applications in different areas of applied sciences. We used Mathematica software for computations and plotting the figures.

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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