Progressively Censored Data from The Weibull Gamma Distribution Moments and Estimation

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Received: 2 Feb. 2013, Revised: 25 Sep. 2013, Accepted: 12 Oct. 2013
Published online: 1 Mar. 2014

Abstract: In this paper, we establish new recurrence relations satisfied by the single and product moments of the progressively type-II right censored order statistics from non truncated and truncated Weibull gamma distribution (WGD), and derive approximate moments of progressively type-II right censored order statistics from this distribution. Using these moments to derive the best linear unbiased estimates (BLUE’s) and maximum likelihood estimates (MLE’s) of the location and scale parameters from the Weibull gamma distribution. In addition, we use Monte-Carlo simulation method to obtain the \( \text{MSE} \) of (BLUE’s) and (MLE’s) and make comparison between them. Finally, a numerical examples based on simulation and real data to illustrate the inference procedures developed in this distribution are presented.

Keywords: Recurrence relations, Single moments, Product moments, Truncated form, Best linear unbiased estimates (BLUE’s), Maximum likelihood estimates (MLE’s), Monte-Carlo Method, Numerical examples.

1 Introduction

Progressive type-II censored sampling is an important method of obtaining data in lifetime studies. Live units removed early on can be readily used in other tests, thereby saving cost to the experimenter, and a compromise can be achieved between time consumption and the observation of some extreme values.

Let us consider the following type-II right censoring scheme: Suppose \( N \) units are placed on test at time zero. Immediately following the first failure, \( R_1 \) surviving items are removed from the test at random. Then, immediately following the second observed failure, \( R_2 \) surviving items are removed from the test at random. This process continues until, at the time of the \( m^{th} \) observed failure, the remaining \( R_m = n - R_1 - R_2 - \cdots - R_{m-1} - m \) items are all removed from the experiment.

In this scheme, \( R_1, R_2, \cdots, R_m \) are pre-determined. Thus, here the censoring times \( (T_i) \) are random, but the numbers of items to fail before each censoring time are fixed. The resulting \( m \) ordered values which are obtained are referred to as progressively type-II right censored order statistics. [6, Balakrishnan and Aggarwala (2000)]

If the failure times are based on an absolutely continuous distribution function \( F(x) \) with probability density function \( f(x) \), the joint probability density function of progressively censored failure times \( X_{1:n:n}, X_{2:n:n}, \cdots, X_{m:n:n} \) is given by:

\[
f(x_{1:n:n}, x_{2:n:n}, \cdots, x_{m:n:n}) = A_{n,R_m-1} \prod_{i=1}^{m} f(x_i) \left[ 1 - F(x_i) \right]^{R_i},
\]

where \( f(.) \) and \( F(.) \) are, respectively, the pdf and the cdf of the random variable \( X \).

\[
A_{n,R_m-1} = n(n - R_1 - 1) \cdots (n - R_1 - R_2 - \cdots - R_{m-1} - m + 1) \text{ and } A_{n,R_0} = n.
\]

[6, Balakrishnan and Aggarwala (2000)].

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In this paper, we derive new recurrence relations satisfied by the single and product moments of the progressively type-II right censored order statistics from the Weibull gamma distribution (WGD) and its truncated form.

Numerous authors have developed new recurrence relations satisfied by the single and product moments of the progressively type-II right censored order statistics for different distributions, see, for example [2, Arnold et al. (1992)], [10, Balakrishnan and Sandhu (1995)], [5, Balakrishnan and Aggarwala (1998)], [6, Balakrishnan and Aggarwala (2000)], [1, Abd El-Baset A. and Mohammed A. (2003)], [14, David and Nagaraja (2003)], [13, Balakrishnan et al. (2004)], [4, Balakrishnan (2007)] and [11, Balakrishnan et al. (2011)]. Approximate moments of progressively type-II right censored order statistics from the Weibull gamma distribution (WGD) are derived. These moments are used to derive the best linear unbiased estimates (BLUE’s) and maximum likelihood estimates (MLE’s) of the location and scale parameters from the Weibull gamma distribution. Several interesting mathematical results for inference procedures have been developed by the authors, see for examples: [18, Lindely (1969)], [6, Balakrishnan and Aggarwala (2000)], [12, Balakrishnan et al. (2002)], [8, Balakrishnan and Basak (2003)], [14, David and Nagaraja (2003)], [9, Balakrishnan and Rao (2003)], [15, Fernandez (2004)], [4, Balakrishnan (2007)], [3, Asgharzadeh (2006)], [19, Mahmoud and Mohie El-Din (2006)] and [21, Raqab et al. (2010)]. In addition, we use Monte-Carlo simulation method to make comparison between the (MSE) of (BLUE’s) and (MLE’s). Finally, a numerical examples based on simulation and real data to illustrate the inference procedures developed in this distribution are presented.

Let \( X_{1:m,n}, X_{2:m,n}, \ldots, X_{m:m,n} \) be the progressively type-II right censored order statistics of size \( m \) from the sample of size \( n \) with censoring scheme \((R_1, R_2, \ldots, R_m)\) be from the Weibull gamma distribution whose probability function is given by:

\[
f(x) = \frac{c}{\delta} \beta x^{\beta - 1} \left(1 + \frac{1}{\delta} x^\beta\right)^{-(\beta + 1)}, x \geq 0, \beta, \delta, c > 0,
\]

and distribution function is given by

\[
F(x) = 1 - \left[1 + \frac{1}{\delta} x^\beta\right]^{-\beta}, x \geq 0, \beta, \delta, c > 0,
\]

also, the characterizing differential equations are given by:

\[
f(x) = \frac{c}{\delta} \beta x^{\beta - 1} \left[1 - F(x)\right]^{(1 + \frac{1}{\beta})},
\]

\[
x \left(1 + \frac{\delta}{\beta}\right) f(x) = c \beta \left[1 - F(x)\right].
\]

Making use of Equations (5) and (6) the following recurrence relations for the single and product moments of progressively type-II censored order statistics from Weibull gamma distribution have been derived.

2 Recurrence Relations for the Single Moments

Let \( X_{1:m,n}, X_{2:m,n}, \ldots, X_{m:m,n} \) be the progressively type-II right censored order statistics of size \( m \) from the sample of size \( n \) with censoring scheme \((R_1, R_2, \ldots, R_m)\) be from the Weibull gamma distribution whose probability function is given by (3) and distribution function is given by (4). The single moments of the progressively type-II can be written as:

\[
\mu_{1,m,n}^{(R_1, \ldots, R_m)} = A_{n,R_{m-1}} \int_{0 < x_1 < x_2 < \cdots < x_{m} < \infty} x f(x_1) [1 - F(x_1)]^{R_1} f(x_2) \cdots [1 - F(x_{m})]^{R_m} dx_1 \cdots dx_m,
\]

where \( A_{n,R_{m-1}} \) is defined in (2). [6, Balakrishnan and Aggarwala (2000)].

The single moments of progressively type-II right censored order statistics given by (7) satisfied the following recurrence relations.

Relation 1 For \( 2 \leq m \leq n, \beta \leq 1, k, c > 0 \) and \( \delta > 0 \)

\[
\mu_{1,m,n}^{(R_1, \ldots, R_m)} = \frac{\beta}{n(R_1 + \frac{1}{\beta} + 1)} \left[ \frac{\delta}{\beta} \left(1 + \frac{1}{\beta}\right)^{R_1, R_2, \ldots, R_m} \right]^{(k+c)}
\]

\[
\left. - \frac{n(n - R_1 - 1)}{(n + \frac{1}{\beta})} \mu_{1,m-1,n+\frac{1}{\beta}}^{(R_1+1, R_2, \ldots, R_m)} \right]^{(k+c)},
\]

\[
(8)
\]
and for \( s = 1, \ n = 1, 2, \cdots, k \geq 0, \ c > 0 \) and \( \beta \leq 1 \),
\[
\mu_{1:1:n+\frac{1}{p}}^{(n+\frac{1}{p})-1} = \frac{\delta (k+c)}{nc\beta} \mu_{1:1:n}^{(n-1)^{(k)}}.
\]

Proof. From (7), we write
\[
\mu_{1,1:n}^{(R_1,\cdots, R_m)} = A_{n,R_{m-1}} \int \cdots \int \times \cdots \times d x_2 \cdots d x_m,
\]
where \( A_{n,R_{m-1}} \) is defined in (1.2) and
\[
I(x) = \int_{0}^{x_1} [1 - F(x_1)]^{R_1} f(x_1) dx_1,
\]
which upon using (5) and integrating by parts, we get
\[
I(x_2) = \frac{c\beta}{\delta} \int_{0}^{x_1} \frac{x_1^{k+c} - 1}{k+c} \left[ 1 - F(x_1) \right]^{R_1+\frac{1}{p}+1} dx_1
\]
\[
= \frac{c\beta}{\delta} \left[ \frac{x_2^{k+c}}{k+c} - 1 \left[ 1 - F(x_2) \right]^{R_1+\frac{1}{p}+1} + \left( \frac{R_1 + \frac{1}{p} + 1}{k+c} \right) \int_{0}^{x_1} \left[ 1 - F(x_1) \right]^{R_1+\frac{1}{p}+1} f(x_1) dx_1 \right],
\]
using (12) in (10), and simplifying the resulting equation, we get
\[
\mu_{1,1:n}^{(R_1,\cdots, R_m)} = \frac{c\beta}{\delta (k+c)} \left[ \frac{n(n-R_1-1)}{(n+\frac{1}{p})} \mu_{1,m-1:n+\frac{1}{p}}^{(R_1+\frac{1}{p}+1, R_2, \cdots, R_m)^{(k+c)}} + \frac{n(R_1+\frac{1}{p}+1)}{(n+\frac{1}{p})} \mu_{1,m-1:n+\frac{1}{p}}^{(R_1+\frac{1}{p}, R_2, \cdots, R_m)^{(k+c)}} \right],
\]
thus, we get (8).

Now for \( m = 1, n = 1, 2, \cdots \) and \( k \geq 0, c > 0 \) using (7), we have
\[
\mu_{1:1:n}^{(R_1)^{(k)}} = A_{n,R_0} \int_{0}^{x_1} f(x_1) \left[ 1 - F(x_1) \right]^{R_1} dx_1
\]
\[
= \frac{nc\beta}{\delta} \int_{0}^{x_1} \frac{x_1^{k+c} - 1}{k+c} \left[ 1 - F(x_1) \right]^{R_1+\frac{1}{p}+1} dx_1
\]
\[
= \frac{nc\beta}{\delta (k+c)} \mu_{1,m+\frac{1}{p}}^{(R_1+\frac{1}{p})},
\]
and hence (9) is proved.

Relation 2 For \( 2 \leq i \leq m - 1, \ k \geq 0, c > 0, \beta \leq 1 \) and \( m \leq n + \frac{1}{p} + 1 \),
\[
\mu_{1,m+\frac{1}{p}}^{(R_1,\cdots, R_i+\frac{1}{p}+1, \cdots, R_m)^{(k+c)}} = \frac{A_{n+\frac{1}{p}, R_{i-1}}}{A_{n,R_{i-1}}} \left[ \frac{1}{R_i + \frac{1}{p} + 1} \right] \left[ \frac{\delta (k+c)}{c\beta} \mu_{1,m+\frac{1}{p}}^{(R_1,\cdots, R_i+\frac{1}{p}+1, \cdots, R_m)^{(k+c)}} \right]
\]
\[
- \frac{A_{n,R_i}}{A_{n+\frac{1}{p}, R_{i-1}}} \mu_{1,m-1:n+\frac{1}{p}}^{(R_1,\cdots, R_i+R_{i-1}+\frac{1}{p}+1, \cdots, R_m)^{(k+c)}}
\]
\[
+ \frac{A_{n,R_{i-1}}}{A_{n+\frac{1}{p}, R_{i-2}}} \mu_{1,m-1:n+\frac{1}{p}}^{(R_1,\cdots, R_{i-1}+R_{i-2}+\frac{1}{p}+1, \cdots, R_m)^{(k+c)}}.
\]
Relation 3 For \( 2 \leq m \leq n \), \( k \geq 0 \), \( c > 0 \) and \( \beta \leq 1 \),
\[
\mu^{(k+1)}_{m,m,n+\frac{1}{\beta}} = \frac{A_{n+\frac{1}{\beta}R_{m-1}}}{(R_{m} + \frac{1}{\beta} + 1)} + \frac{1}{A_{n+\frac{1}{\beta}R_{m-2}}} \left[ \frac{n}{c} \left( R_{1}, R_{2}, \ldots, R_{m-1} + c \right) + \frac{n}{c} \left( R_{1}, R_{2}, \ldots, R_{m-1} + c \right) \right]^{(k+1)}.
\]

Relation 4 For \( 1 \leq m \leq n \), \( R_i > -1 \) and \( k \geq c \),
\[
\mu^{(k)}_{1,m,n} = \frac{c \beta (R_1 + 1)}{c \beta (R_1 + 1) - k} \left[ \frac{n}{c} \left( R_{1}, R_{2}, \ldots, R_{m-1} + c \right) \right]^{(k+1)} \left( \frac{n}{c} \left( R_{1}, R_{2}, \ldots, R_{m-1} + c \right) \right]^{(k)}.
\]

Relation 5 For \( 2 \leq i \leq m-1 \), \( m \leq n \) and \( k > c \),
\[
\mu^{(k)}_{i,m,n} = \frac{c \beta (R_1 + 1)}{c \beta (R_1 + 1) - k} \left[ \frac{n}{c} \left( R_{1}, R_{2}, \ldots, R_{m-1} + c \right) \right]^{(k+1)} \left( \frac{n}{c} \left( R_{1}, R_{2}, \ldots, R_{m-1} + c \right) \right]^{(k)}.
\]

Relation 6 For \( 2 \leq m \leq n \), \( R_m > -1 \) and \( k \geq c \),
\[
\mu^{(k)}_{m,m,n} = \frac{c \beta (R_1 + 1)}{c \beta (R_1 + 1) - k} \left[ \frac{n}{c} \left( R_{1}, R_{2}, \ldots, R_{m-1} + c \right) \right]^{(k+1)} \left( \frac{n}{c} \left( R_{1}, R_{2}, \ldots, R_{m-1} + c \right) \right]^{(k)}.
\]

Relation 7 For \( 2 \leq m \leq n \), \( R_i > -1 \), \( k > -1 \) and \( k \geq c - 1 \),
\[
\mu^{(k+1)}_{1,m,n} = \frac{(k + 1)}{(k + 1) - c \beta (R_1 + 1)} \left[ \frac{n}{c} \left( R_{1}, R_{2}, \ldots, R_{m-1} + c \right) \right]^{(k+1)} \left( \frac{n}{c} \left( R_{1}, R_{2}, \ldots, R_{m-1} + c \right) \right]^{(k+1)}.
\]

Relation 8 For \( 2 \leq i \leq m-1 \), \( m \leq n \), \( R_i > -1 \) and \( k > c - 1 \),
\[
\mu^{(k+1)}_{i,m,n} = \frac{(k + 1)}{(k + 1) - c \beta (R_1 + 1)} \left[ \frac{n}{c} \left( R_{1}, R_{2}, \ldots, R_{m-1} + c \right) \right]^{(k+1)} \left( \frac{n}{c} \left( R_{1}, R_{2}, \ldots, R_{m-1} + c \right) \right]^{(k+1)}.
\]

Relation 9 For \( 2 \leq m \leq n \), \( R_m > -1 \) and \( k \geq 1 \),
\[
\mu^{(k+1)}_{m,m,n} = \frac{k + 1}{(k + 1) - c \beta (R_1 + 1)} \left[ \frac{n}{c} \left( R_{1}, R_{2}, \ldots, R_{m-1} + c \right) \right]^{(k+1)} \left( \frac{n}{c} \left( R_{1}, R_{2}, \ldots, R_{m-1} + c \right) \right]^{(k+1)}.
\]
3 Recurrence Relations for the Product Moments

For any continuous distribution, we can write the \((i, j)\)-th product moment of the progressively type-II right censored order statistics from \(1\) as:

\[
\mu_{i,j;S,n}^{(R_1, R_2, \ldots, R_n)(i,j)} = E \left[ X_{i}^{(R_1, \ldots, R_i)} \left( X_{j}^{(R_1, \ldots, R_j)} \right)^{(i,j)} \right]
\]

\[
= A_{n,R_{i-1}} \int \int \cdots \int x_{i}^{j} x_{j}^{f(x_{1}) [1 - F(x_{1})]^{R_{i}}} \times f(x_{2}) [1 - F(x_{2})]^{R_{2}} \cdots \times f(x_{j}) [1 - F(x_{j})]^{R_{j}} dx_{1} \cdots dx_{j},
\]

[6, Balakrishnan and Aggarwala (2000)], where \(A_{n,R_{i-1}}\) is defined in (2). The product moments defined in (24) satisfied the following recurrence relations.

**Relation 10** For \(1 < i < j \leq m - 1\) and \(m \leq n\)

\[
\mu_{i,j;m,n}^{(R_1, \ldots, R_i + \frac{1}{n} \cdots R_m)} = \delta \left[ \frac{A_{n,R_{i}+1} - A_{n,R_{i}}} {A_{n,R_{i-1}}} \right] \left[ \frac{1}{\beta} \mu_{i-1;m,n}^{(R_1, \ldots, R_i)} \right]
\]

\[
= \frac{1}{\delta} \frac{A_{n,R_{i-1}+1}} {A_{n,R_{i-1}}} \mu_{i-1,m-1,n}^{(R_1, \ldots, R_i)} [1 + 1, \ldots, R_n]^{(1e)}
\]

Using (5) in (27) and integrating by parts, we get

\[
I(x_{j}) = \int_{x_{j-1}}^{x_{j+1}} [1 - F(x_{j})]^{R_{j}} f(x_{j}) dx_{j},
\]

\[
\mu_{i,j;m,n}^{(R_1, \ldots, R_i)} = A_{n,R_{i-1}} \int \int \cdots \int x_{i}^{j} x_{j}^{f(x_{1}) [1 - F(x_{1})]^{R_{i}}} \times f(x_{2}) [1 - F(x_{2})]^{R_{2}} \cdots \times f(x_{j}) [1 - F(x_{j})]^{R_{j}} dx_{1} \cdots dx_{j},
\]

where

\[
I(x_{j}) = \int_{x_{j-1}}^{x_{j+1}} f(x_{j}) dx_{j},
\]

\[
= \frac{c\beta}{\delta} \left[ x_{j+1}^{R_{j}+\frac{1}{n}} - x_{j-1}^{R_{j}+\frac{1}{n}} \right] dx_{j}
\]

\[
= \beta x_{j+1}^{R_{j}+\frac{1}{n}} - \beta x_{j-1}^{R_{j}+\frac{1}{n}} + \beta \left[ R_{j} + \frac{1}{n} + 1 \right] x_{j+1}^{R_{j}+\frac{1}{n}} - x_{j-1}^{R_{j}+\frac{1}{n}}
\]

using (28) in (27), and rearrangement, (25) is obtained.
In this section, we present recurrence relations for the single and product moments of progressively type-II right censored order statistics from the doubly truncated Weibull gamma distribution.

Thus cumulative distribution function of doubly truncated Weibull gamma distribution is given by:

\[
F_t(x) = \frac{1}{P - Q} \left[ 1 + \frac{1}{\delta} Q_1^{-1} \right]^{-\beta} - \left[ 1 + \frac{1}{\delta} x \right]^{-\beta},
\]

where \(0 < Q_1 < x < P_1, \ldots, \alpha, \beta, \delta > 0, \ldots, x \geq 0.\)

4 The Doubly Truncated Weibull Gamma Distribution

In this section, we present recurrence relations for the single and product moments of progressively type-II right censored order statistics from the doubly truncated Weibull gamma distribution.

The probability density function of the doubly truncated Weibull gamma distribution is given by:

\[
f_t(x) = \frac{1}{P - Q} \beta x^{\beta - 1} \left[ 1 + \frac{1}{\delta} x \right]^{-\beta - 1},
\]

The probability density function of the doubly truncated Weibull gamma distribution is given by:

\[
\frac{\mu_{i,j,m,n}}{\mu_{i,j,m,n}} = \frac{1}{c} \left[ \frac{n - R_{1} - \cdots - R_{j - 1} - f + 1}{R_{j} + 1} \right]^{\mu_{i,j-1,m-1,n}} - \frac{n - R_{1} - \cdots - R_{j} - f}{R_{j} + 1} \mu_{i,j,m-1,n} + \frac{1}{c} \beta \left[ \frac{n - R_{1} - \cdots - R_{j} - f}{R_{j} + 1} \right]^{\mu_{i,j,m,n}} \left[ \frac{n - R_{1} - \cdots - R_{j} - f + 1}{R_{j} + 1} \right]^{\mu_{i,j-1,m-1,n}}.
\]
so, the characterizing differential equation for this distribution is given by:

\[
[x + \delta x^{-\xi+1}] f_i(x) = c\beta \left[ \frac{1 - P}{P - Q} \right] + c\beta [1 - F_i(x)].
\]  

(37)

Thus, we can conclude new recurrence relations for the single and product moments of progressively type-II right censored order statistics from the doubly truncated Weibull gamma distribution.

**Relation 15** For \(2 \leq m \leq n - 1, k > -1\) and \(R_1 > -1\),

\[
\mu^{(R_1, \ldots, R_m)}_{i,m,n} = \frac{c\beta}{(k+1) - c\beta(R_i + 1)} \times \left[ \frac{1 - P}{P - Q} \right] \left( (n - R_1 - 1) \mu^{(R_1 + R_2 + \ldots + R_m)}_{1,m-1,n} \right)
\]

\[
\times \left[ \frac{nQ^k + R_1}{n - 1} \mu^{(R_1 - 1, R_2 - R_1, \ldots, R_m)}_{1,m-1,n} \right]
\]

\[
+ (n - R_1 - 1) \mu^{(R_1 + R_2 + \ldots + R_m)}_{1,m-1,n}
\]

\[
- nQ^k - \delta(k + 1) \mu^{(R_1, \ldots, R_m)}_{1,m,n} \left( R_i, \ldots, R_m \right)^{(k-1)}.
\]

(38)

**Relation 16** For \(2 \leq i \leq m, m \leq n - 1, k > -1\) and \(R_i > -1\),

\[
\mu^{(R_1, \ldots, R_m)}_{i,m,n} = \frac{c\beta}{(k+1) - c\beta(R_i + 1)} \times \left[ \frac{1 - P}{P - Q} \right] \left( \frac{A_n R_i}{n - 1, R_{i-1}} \right) \mu^{(R_1, \ldots, R_i + R_{i+1} + \ldots + R_m)}_{1,m-1,n}
\]

\[
+ R_i \frac{A_n R_i}{n - 1, R_{i-1}} \mu^{(R_1, \ldots, R_i - 1, \ldots, R_m)}_{1,m,n}
\]

\[
+ (n - R_1 - \cdots - R_i - i - 1) \mu^{(R_1, \ldots, R_i - 1, R_{i+1} + \ldots + R_m)}_{1,m-1,n}
\]

\[
- (n - R_1 - \cdots - R_i - 1) \mu^{(R_1, \ldots, R_i - 1, R_{i+1} + \ldots + R_m)}_{1,m-1,n}
\]

\[
- nQ^k - \delta(k + 1) \mu^{(R_1, \ldots, R_m)}_{i,m,n} \left( R_i, \ldots, R_m \right)^{(k-1)}.
\]

(39)

**Relation 17** For \(2 \leq m \leq n - 1, k > -1\) and \(R_m > -1\),

\[
\mu^{(R_1, \ldots, R_m)}_{m,n,n} = \frac{c\beta}{(k+1) - c\beta(R_m + 1)} \times \left[ \frac{1 - P}{P - Q} \right] \left( \frac{A_n R_{m-1}}{n - 1, R_{m-2}} \right) \mu^{(R_1, \ldots, R_{m-1} + R_m)}_{1,m-1,n}
\]

\[
- R_m \frac{A_n R_{m-1}}{n - 1, R_{m-2}} \mu^{(R_1, \ldots, R_{m-1} + R_{m+1})}_{m,m-1,n}
\]

\[
+ (n - R_1 - \cdots - R_{m-1} - m + 1) \mu^{(R_1, \ldots, R_{m-1} + R_{m+1})}_{m,m-1,n}
\]

\[
- nQ^k - \delta(k + 1) \mu^{(R_1, \ldots, R_m)}_{m,n,n} \left( R_1, \ldots, R_m \right)^{(k-1)}.
\]

(40)
Relation 18For $1 \leq i < j \leq m - 1, m \leq n$ and $R_j > 1$,
\[
\mu_{i,j;mm}^{(R_1,R_2,\ldots,R_m)} = \frac{c\beta}{1 - c\beta(R_j + 1)} \times \left[ \frac{(1 - P)}{(P - Q)} \frac{A_{n,R_j}}{A_{n-1,R_{j-1}}} \mu_{i,j;mm-1,R_{m}+1}^{(R_1,\ldots,R_j+1,R_{j+1},\ldots,R_n)} - \frac{A_{n,R_{j-1}}}{A_{n-1,R_{j-2}}} \mu_{i,j;mm-1,R_{m}+1}^{(R_1,\ldots,R_{j-1}+R_j+1,\ldots,R_n)} + R_j \frac{A_{n,R_{j-1}}}{A_{n-1,R_{j-1}}} \mu_{i,j;mm-1,R_{m}+1}^{(R_1,\ldots,R_{j-1},R_j+1,\ldots,R_n)} \right] + \left[ (n - R_1 - \cdots - R_j - j) \mu_{i,j;mm-1,R_{m}+1}^{(R_1,\ldots,R_{j-1}+R_j+1,\ldots,R_n)} - (n - R_1 - \cdots - R_{j-1} - j + 1) \mu_{i,j;mm-1,R_{m}+1}^{(R_1,\ldots,R_{j-1}+R_j+1,\ldots,R_n)} \right] - \frac{\delta}{c\beta} \mu_{i,j;mm}^{(1,\ldots,1)} \right] .
\] (41)

Relation 19For $1 \leq i \leq m - 1, m \leq n$ and $R_m > 1$,
\[
\mu_{i,m;mm}^{(R_1,\ldots,R_m)} = \frac{c\beta}{c\beta(R_m + 1) - 1} \times \left[ \frac{(1 - P)}{(P - Q)} \frac{A_{n,R_m}}{A_{n-1,R_{m-1}}} \mu_{i,m;mm-1,R_{m+1}}^{(R_1,\ldots,R_m+1)} + R_m \frac{A_{n,R_{m-1}}}{A_{n-1,R_{m-2}}} \mu_{i,m;mm-1,R_{m+1}}^{(R_1,\ldots,R_{m-1})} \right] - \frac{\delta}{c\beta} \mu_{i,m;mm}^{(1,\ldots,1)} \right] .
\] (42)

Remark. Setting $c = 1$ and $\delta = 1$ in the recurrence relations given in section (2) and (3), we deduce the recurrence relations for the single and product moments from the Lomax distribution. [17, Hassan and Sultan (2005)]

Remark. Setting $p = 1, Q = 0$ in Theorems (16), (17), (18), (19) and (20), we obtain Theorems (7), (8), (9), (14), (15) in our paper.

Remark. Setting $R_1 = R_2 = \cdots = R_m = 0$, so that $m = n$ in which the case of the progressively type-II censored order statistics gives the usual order statistics $X_{1,n}, X_{2,n}, \ldots, X_{n,n}$, the relations established for the Weibull gamma and Lomax distributions reduce to the corresponding recurrence relations based on the usual order statistics. [20, Malik et al. (1998)].

5 Progressively Type-II Right Censored Transformation

By using Equations (3) and (4), we can write the joint density function of $X_{1,n}, X_{2,n}, \ldots, X_{m,n}$ of a progressively type-II right censored sample from the Weibull gamma distribution, with censoring scheme $(R_1, R_2, \ldots, R_m)$, in the form:
\[
f_{X_{1,n},X_{2,n},\ldots,X_{m,n}}(x_1, x_2, \ldots, x_m) = A_{n,R_{m-1}}^{(n)} \prod_{i=1}^{m} a^\alpha x_i^{\alpha-1} \left[ 1 + \frac{1}{\delta} x_i^{\beta} \right]^{-\beta(R_i+1)-1}
\] (43)

where $A_{n,R_{m-1}}$ is defined by (2). [6, Balakrishnan and Aggarwal(2000)].

Notation 20For simplicity, put $c = \alpha$ in Equations (3) and (4).

Since, the joint density function is more complicated, so we try to find relationship between the Weibull gamma distribution and uniform distribution.
Let $U_{1:m,n}, U_{2:m,n}, \ldots, U_{m:m,n}$ be the progressively type-II right censored order statistics of size $m$ from the sample of size $n$ with censoring scheme $(R_1, R_2, \ldots, R_n)$ be from the uniform $(0, 1)$ distribution.

The exact moments of progressively type-II right censored order statistics from the uniform $(0,1)$ distribution can be written in the form:

$$
E(U_{i:m:n}) = 1 - \prod_{j=m-i+1}^{m} \alpha_j, i = 1,2,\cdots,m,
$$

$$
Var(U_{i:m:n}) = \left( \prod_{j=m-i+1}^{m} \alpha_j \right) \left( \prod_{j=m-k+1}^{m} \gamma_j - \prod_{j=m-k+1}^{m} \alpha_j \right),
$$

$$
Cov(U_{i:m:n}, U_{k:m:n}) = \left( \prod_{j=m-i+1}^{m} \alpha_j \right) \left( \prod_{j=m-k+1}^{m} \gamma_j - \prod_{j=m-k+1}^{m} \alpha_j \right), k < i,
$$

where

$$
a_i = i + \sum_{j=m-i+1}^{m} R_j, i = 1,2,\cdots,m,
$$

$$
\alpha_i = \frac{a_i}{a_i + 1}, i = 1,2,\cdots,m,
$$

$$
\beta_i = \frac{1}{(a_i + 1)(a_i + 2)}, i = 1,2,\cdots,m,
$$

$$
\gamma_i = \alpha_i + \beta_i, i = 1,2,\cdots,m.
$$

Theorem 21. If $X$ is a random variable with $E(X) = \mu$, $D^2(X) = \sigma^2$, and $Y = \phi(X)$ then, for sufficiently small $\sigma$, and well-behaved $\phi$

$$
E(Y) \simeq \phi(\mu) + \frac{1}{2} \sigma^2 \phi''(\mu),
$$

and

$$
D^2(Y) \simeq (\phi'(\mu))^2 \sigma^2.
$$

Theorem 22. If $X$ and $Y$ are random variables with $E(X) = \mu$, $E(Y) = \upsilon$, $D^2(X) = \sigma^2$, $D^2(Y) = \tau^2$ and $\rho(X,Y) = \rho$, and $Z = \phi(X,Y)$ then, for sufficiently small $\sigma$ and $\tau$, and well behaved $\phi$

$$
E(Z) \simeq \phi(\mu, \upsilon) + \frac{1}{2} \sigma^2 \phi''(\mu) + \rho \sigma \tau \phi''(\mu) + \frac{1}{2} \tau^2 \phi''(\mu),
$$

and

$$
D^2(Z) \simeq \sigma^2 \left( \frac{\partial \phi}{\partial x} \right)^2 + 2 \rho \sigma \tau \left( \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial y} \right) + \tau^2 \left( \frac{\partial \phi}{\partial y} \right)^2,
$$

where all partial derivatives are evaluated at $x = \mu$, $y = \upsilon$.

6 Deriving Moments Using Transformation

Since, the joint density function is more difficult to use it to find the moments, so we get relationship between the Weibull gamma distribution and uniform distribution, as follows:

$$
U \simeq 1 - \left( 1 + \frac{1}{\delta^{a}} \right)^{-\beta}.
$$
So by using Equations (46), (47) and (48), the approximate moments can be written as follows:

\[ E(X_{i,m,n}) \approx \left[ \delta (1 - \mu)^{1 - \frac{1}{n}} - 1 \right]^{\frac{1}{n}} + \frac{1}{2} \sigma^2 \left\{ \left[ \delta (1 - \mu)^{1 - \frac{1}{n}} - 1 \right] \frac{1}{\alpha} - \frac{1}{\alpha} \left[ \frac{1}{\alpha} - 1 \right] \left[ \left( 1 - \mu \right)^{\frac{1}{n}} - 1 \right] \right\}^{2 - \frac{1}{n}} \]

\[ + \frac{\delta}{\alpha \beta} \left( \frac{1}{\beta} + 1 \right) \left( 1 - \mu \right)^{\frac{1}{n} - 2} \left\{ \left[ \delta \left( 1 - \mu \right)^{\frac{1}{n}} - 1 \right] \right\}^{\frac{1}{n} - 1} \right\}, \quad (51) \]

\[ D^2(X_{i,m,n}) \approx \sigma^2 \left( \frac{\delta}{\alpha \beta} \right)^2 \left[ \left[ \delta \left( 1 - \mu \right)^{\frac{1}{n}} - 1 \right] \left[ \delta \left( 1 - \mu \right)^{\frac{1}{n}} - 1 \right] \right]^2, \quad (52) \]

\[ Z = \left[ \delta \left( 1 - u_i \right)^{\frac{1}{n}} - 1 \right] \left[ \delta \left( 1 - u_i \right)^{\frac{1}{n}} - 1 \right]^{\frac{1}{n}}, \quad (53) \]

and

\[ \text{Cov}(X_{i,m,n}, X_{j,m,n}) \approx E(Z) - E(X_{i,m,n})E(X_{j,m,n}). \]

We use these moments to derive the best linear unbiased estimators for the location (\( \mu \)) and scale (\( \sigma \)) parameters of Weibull gamma distribution.

### 7 Estimation of Parameters

The best linear unbiased and maximum likelihood methods are used to obtain estimators of the location (\( \mu \)) and scale (\( \sigma \)) parameters. Let the probability density function is given by:

\[ f(x) = \frac{1}{\sigma} \frac{\alpha \beta}{\delta} \left( \frac{x - \mu}{\sigma} \right)^{\alpha - 1} \left[ 1 + \frac{1}{\delta} \left( \frac{x - \mu}{\sigma} \right)^{\alpha} \right]^{-\beta - 1}, \quad x > \mu, \quad (54) \]

also the distribution function is given by:

\[ F(x) = 1 - \left( 1 + \frac{1}{\delta} \left( \frac{x - \mu}{\sigma} \right)^{\alpha} \right)^{-\beta}. \quad (55) \]

#### 7.1 Best linear unbiased estimates (BLUEs)

Consider an arbitrary continuous distribution \( F(x) \). Suppose that the progressively censored order statistics can be represented by the linear transformation \( Y = \mu 1 + \sigma X \), where the vector \( X \) represent a vector of progressively type-II censored order statistics from the standard distribution \( F(x) \), then the best linear unbiased estimators of \( \mu \) and \( \sigma \) will be minimizing the generalized variance \( Q(\theta) = (Y - A\theta)^T \Sigma^{-1} (Y - A\theta) \) with respect to \( \theta \) where \( \theta = (\mu, \sigma)^T \), \( A \) is the \( p \times p \) matrix, \( 1 \) is \( p \times 1 \) vector with components all 1’s, \( \mu \) is the mean vector of \( X \) and \( \Sigma \) is the variance-covariance matrix of \( X \). The minimum occurs when

\[ \mu^* = -\mu^T \Gamma Y = \sum_{i=1}^{m} A_i Y_{i,m,n}, \quad (56) \]

and

\[ \sigma^* = 1^T \Gamma Y = \sum_{i=1}^{m} B_i Y_{i,m,n}, \quad (57) \]

where

\[ \Gamma = \sum_{i=1}^{m} \left( 1 \mu^T - \mu 1^T \right) \Sigma^{-1} / \Delta, \quad (58) \]

and

\[ \Delta = \left( 1^T \sum_{i=1}^{m} 1 \right) \left( \mu^T \sum_{i=1}^{m} \mu \right) \left( 1^T \sum_{i=1}^{m} \mu \right)^2. \quad (59) \]
[6, Balakrishnan and Aggarwala(2000)].

From these results, we can get the variances and covariances of the estimators in the form:

\[ \text{Var}(\mu^*) = \left( \sigma^2 \mu^T \sum^{-1} \mu \right) / \Delta, \]

(60)

\[ \text{Var}(\sigma^*) = \left( \sigma^2 \sigma^T \sum^{-1} \sigma \right) / \Delta, \]

(61)

\[ \text{Cov}(\mu^*, \sigma^*) = \left( -\sigma^2 \mu^T \sum^{-1} \sigma \right) / \Delta. \]

(62)

The coefficients \( A_i, B_i, i = 1, 2, \cdots, m \) which represented in Equations (56) and (57) satisfy the relations \( \sum_{i=1}^{m} A_i = 1, \sum_{i=1}^{m} B_i = 0 \).[6, Balakrishnan and Aggarwala(2000)].

### 7.2 Maximum likelihood estimators

Let \( X_{1:n}, X_{2:n}, \cdots, X_{m:n} \) be the progressively type-II right censored order statistics of size \( m \) from the sample of size \( n \) with censoring scheme \( (R_1, R_2, \cdots, R_m) \) taken from the Weibull gamma distribution whose probability function is given by (54) and the cumulative distribution function is given by (55), then likelihood function can be written in the form:

\[ L(\mu, \sigma) = A_{n,R_{m-1}} \prod_{i=1}^{m} f(X_{i:n}) [1 - F(X_{i:n})]^{R_i}, \]

(63)

where \( e \) is normalizing constant, see [6, Balakrishnan and Aggarwala(2000)].

The likelihood function to be maximized for estimators of \( \mu \) and \( \sigma \) is given by:

\[ L(\mu, \sigma) = (\text{constant}) (\alpha \beta)^m \sigma^{\alpha \beta} \delta^{\alpha \beta} \prod_{i=1}^{m} (x_i - \mu)^{\alpha - 1} \left[ \delta \sigma^\alpha + (x_i - \mu)^\alpha \right]^{-\beta(R_i+1)-1}. \]

(64)

The log-likelihood function can be written in form:

\[ \ln L(\mu, \sigma) = \ln \text{constant} + m \ln \alpha \beta + n \alpha \beta \ln \sigma + n \beta \ln \delta + (\alpha - 1) \sum_{i=1}^{m} \ln (x_i - \mu) \]

\[ - \sum_{i=1}^{m} \beta (R_i + 1) \ln \left( \delta \sigma^\alpha + (x_i - \mu)^\alpha \right), \]

(65)

by differentiating the log-likelihood function given by (65) with respect to \( \mu \) and \( \sigma \). The resulting equations to be solved for maximum likelihood estimators \( \mu \) and \( \sigma \) are given by:

\[ \sum_{i=1}^{m} \alpha [\beta (R_i + 1) + 1] \frac{(x_i - \mu)^{\alpha - 1}}{\delta \sigma^\alpha + (x_i - \mu)^\alpha} - (\alpha - 1) \sum_{i=1}^{m} \frac{1}{x_i - \mu} = 0. \]

(66)

\[ \frac{n \beta \alpha}{\sigma} - \sum_{i=1}^{m} \beta (R_i + 1) \frac{c \delta \sigma^{\alpha - 1}}{\delta \sigma^\alpha + (x_i - \mu)^\alpha} = 0. \]

(67)

Since Equations (66) and (67) cannot be solved analytically, so we can use MATLAB program to solve these equations.

### 8 Simulation Study

Let us consider the following table represented different schemes of progressively censored data:

So, by using Equations (56) and (57), the coefficients of the BLUE’s of \( \mu \) and \( \sigma \) from the Weibull gamma distribution using different schemes represented in table (1) about \( \mu = 0 \) and \( \sigma = 1 \) are obtained in the following tables (2,3,4 and 5):
Table 1: sample sizes and censoring schemes from the Weibull gamma distribution.

<table>
<thead>
<tr>
<th>m</th>
<th>n</th>
<th>scheme</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>15</td>
<td>( R_1 = [2 \ 0 \ 4 \ 0.4] )</td>
</tr>
<tr>
<td>6</td>
<td>20</td>
<td>( R_2 = [4 \ 0 \ 4 \ 0.24] )</td>
</tr>
<tr>
<td>7</td>
<td>25</td>
<td>( R_1 = [4 \ 0 \ 4 \ 2.24.2] )</td>
</tr>
<tr>
<td>8</td>
<td>30</td>
<td>( R_1 = [2 \ 2.4 \ 4.2 \ 0.4] )</td>
</tr>
</tbody>
</table>

Table 2: Coefficients of Blues of \( \sigma \) and \( \mu \) from the Weibull gamma distribution using the first scheme.

<table>
<thead>
<tr>
<th>sch1</th>
<th>( \delta = 0.5, \beta = 0.25, \alpha = 0.5 )</th>
<th>( \delta = 1.5, \beta = 1, \alpha = 1.5 )</th>
<th>( \delta = 2, \beta = 1.5, \alpha = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>m</td>
<td>( n )</td>
<td>( A_i )</td>
<td>( B_i )</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td>0.3357</td>
<td>-0.3444</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.4657</td>
<td>-0.2104</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.7172</td>
<td>-0.0381</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.0236</td>
<td>0.2561</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.1967</td>
<td>0.3368</td>
</tr>
<tr>
<td></td>
<td>( \approx 1 )</td>
<td>( \approx 0 )</td>
<td>( \approx 1 )</td>
</tr>
</tbody>
</table>

Table 3: Coefficients of Blues of \( \sigma \) and \( \mu \) from the Weibull gamma distribution using the Second scheme.

<table>
<thead>
<tr>
<th>sch2</th>
<th>( \delta = 0.5, \beta = 0.25, \alpha = 0.5 )</th>
<th>( \delta = 1.5, \beta = 1, \alpha = 1.5 )</th>
<th>( \delta = 2, \beta = 1.5, \alpha = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>m</td>
<td>( n )</td>
<td>( A_i )</td>
<td>( B_i )</td>
</tr>
<tr>
<td>6</td>
<td>20</td>
<td>0.1621</td>
<td>-0.4013</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.4056</td>
<td>-0.2514</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.4024</td>
<td>-0.0648</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.2116</td>
<td>0.2005</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.0210</td>
<td>0.2912</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.2028</td>
<td>0.2259</td>
</tr>
<tr>
<td></td>
<td>( \approx 1 )</td>
<td>( \approx 0 )</td>
<td>( \approx 1 )</td>
</tr>
</tbody>
</table>

Table 4: Coefficients of Blues of \( \sigma \) and \( \mu \) from the Weibull gamma distribution using the Third scheme.

<table>
<thead>
<tr>
<th>sch3</th>
<th>( \delta = 0.5, \beta = 0.25, \alpha = 0.5 )</th>
<th>( \delta = 1.5, \beta = 1, \alpha = 1.5 )</th>
<th>( \delta = 2, \beta = 1.5, \alpha = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>m</td>
<td>( n )</td>
<td>( A_i )</td>
<td>( B_i )</td>
</tr>
<tr>
<td>7</td>
<td>25</td>
<td>-0.8300</td>
<td>-0.9465</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.1277</td>
<td>-0.5728</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.3581</td>
<td>-0.1292</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.7466</td>
<td>0.4003</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.8161</td>
<td>0.7097</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.5380</td>
<td>0.6851</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-0.5109</td>
<td>-0.1466</td>
</tr>
<tr>
<td></td>
<td>( \approx 1 )</td>
<td>( \approx 0 )</td>
<td>( \approx 1 )</td>
</tr>
</tbody>
</table>
Also, the variances and covariances of the estimators $\mu$ and $\sigma$ can be represented in the following table:

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$\beta$</th>
<th>$\alpha$</th>
<th>$m$</th>
<th>$n$</th>
<th>$sch$</th>
<th>$Var(\mu^*)$</th>
<th>$Var(\sigma^*)$</th>
<th>Cov($\mu^<em>, \sigma^</em>$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.25</td>
<td>0.5</td>
<td>5</td>
<td>15</td>
<td>1</td>
<td>5.3559</td>
<td>0.0110</td>
<td>0.0453</td>
</tr>
<tr>
<td>1.5</td>
<td>1.5</td>
<td>5</td>
<td>15</td>
<td>1</td>
<td>0.0010</td>
<td>0.0040</td>
<td>0.0062</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.5</td>
<td>5</td>
<td>15</td>
<td>1</td>
<td>1.8108</td>
<td>0.0013</td>
<td>0.0014</td>
<td></td>
</tr>
</tbody>
</table>

$MSE$ of $\mu$ and $\sigma$ from the Weibull gamma distribution with $\mu = 0$ and $\sigma = 1$ can be represented in the following table:

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$\beta$</th>
<th>$\alpha$</th>
<th>$m$</th>
<th>$n$</th>
<th>$sch$</th>
<th>$MSE(\mu^*)$</th>
<th>$MSE(\sigma^*)$</th>
<th>$MSE(\mu)$</th>
<th>$MSE(\sigma)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.25</td>
<td>0.5</td>
<td>5</td>
<td>15</td>
<td>1</td>
<td>1.3516</td>
<td>1.5327</td>
<td>1.5246e-012</td>
<td>1.8314e-012</td>
</tr>
<tr>
<td>1.5</td>
<td>1.5</td>
<td>5</td>
<td>15</td>
<td>1</td>
<td>0.5138</td>
<td>0.1682</td>
<td>1.7668e-007</td>
<td>1.7662e-008</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.5</td>
<td>5</td>
<td>15</td>
<td>1</td>
<td>0.8050</td>
<td>0.3606</td>
<td>1.1579e-009</td>
<td>1.0979e-009</td>
<td></td>
</tr>
</tbody>
</table>

From the numerical results presented in tables 2, 3, 4, 5 and 7, we can conclude the following:

1. As a check of the entries of tables 2, 3, 4 and 5, we see that $\sum_{i=1}^{n} A_i \approx 1$, $\sum_{i=1}^{n} B_i \approx 0$. 

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2. From Table (7), we see that as \( n \) increases, the mean square error \( MSE(\mu^*) \) and \( MSE(\sigma^*) \) decrease for all censoring schemes and all values of \( \alpha, \delta \) and \( \beta \).

3. From Table (7), we see that as \( n \) increases, the mean square error \( MSE(\hat{\mu}) \) and \( MSE(\hat{\sigma}) \) decrease for all censoring schemes and all values of \( \alpha, \delta \) and \( \beta \).

### 9 Numerical Examples

**Example 1A** progressively type-II censored sample of size \( m = 5 \) from a sample of size \( n = 15 \) from the Weibull gamma distribution with \( \mu = 0, \sigma = 1, \delta = 1.5, \beta = 1, \alpha = 1.5 \) with scheme \( R_i = (2 \ 0 \ 4 \ 0 \ 4) \). was simulated using MATLAB program. The simulated progressively type-II right censored sample is given by:

Table 8: Progressively Type-II right censored sample generated from the weibull gamma distribution.

<table>
<thead>
<tr>
<th>( R_i )</th>
<th>2</th>
<th>0</th>
<th>4</th>
<th>0</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_{i:15} )</td>
<td>0.0503</td>
<td>0.2537</td>
<td>0.2705</td>
<td>0.2935</td>
<td>0.6190</td>
</tr>
</tbody>
</table>

By making use of equation (56) and (57), and using the coefficients \( A_i \) and \( B_i \) given in table (2) for \( n = 15 \) and \( m = 5 \), we get the BLUE's of \( \mu^* \) and \( \sigma^* \) as follows:

\[
\mu^* = (0.5232 \times 0.0503) + (0.5773 \times 0.2537) + (0.3891 \times 0.2705) + (-0.1156 \times 0.2935) + (-0.3740 \times 0.6190) = 0.01259492
\]

and

\[
\sigma^* = (-0.7892 \times 0.0503) + (-0.4841 \times 0.2537) + (-0.0893 \times 0.2705) + (0.5870 \times 0.2935) + (0.7757 \times 0.6190) = 0.46577422
\]

The standard error of the estimates \( \mu^* \) and \( \sigma^* \) are

\[
SE(\mu^*) = \sigma^* (Var(\mu^*))^{1/2} = 0.465774220 \times (0.0010)^{1/2} = 0.01472907
\]

\[
SE(\sigma^*) = \sigma^* (Var(\sigma^*))^{1/2} = 0.465774220 \times (0.0040)^{1/2} = 0.02945148
\]

Using the same data, we can get by simulation

\[
\hat{\mu} = 0.000505005
\]

\[
\hat{\sigma} = 0.686868111
\]

Then, the best linear unbiased prediction for the failure following \( Y_{(2 \ 0 \ 4 \ 0 \ 4)}^{(2\delta:5:15)} \) may be determined simply by equating \( \mu^* \) or \( \sigma^* \) based on the sample of size \( m = 5 \) to \( \mu^* \) and \( \sigma^* \) based on the sample of size \( n = 15 \) with progressive censoring \( (2 \ 0 \ 4 \ 0 \ 3) \) whose coefficients are given in the form:

Table 9: coefficients of the BLUEs of \( \mu \) and \( \sigma \) from the Weibull gamma distribution using different scheme.

| \( A_i \) | 0.3347 | 0.5248 | 0.4522 | 0.0886 | -0.1521 | -0.2481 |
| \( B_i \) | -0.5139 | -0.4075 | -0.1814 | 0.2888 | 0.4517 | 0.3623 |

then

\[
\mu^* = (0.3347 \times 0.0503) + (0.5248 \times 0.2537) + (0.4522 \times 0.2705) + (0.0886 \times 0.2935) + (-0.1521 \times 0.6190) + (-0.2481 \times y_{6:6:15})
\]

Upon equating this with \( \mu^* = 0.012594920 \) and solving, we get the first-order approximation to the BLUE of \( y_{6:6:15}^* \) as

\[
y_{6:6:15}^* = \frac{0.20415147 - 0.01259492}{0.2481} = 0.772094115
\]
Remark. One can see that the values of $\mu^*$ close to 0 and $\sigma^*$ close to 1.

**Example 2** We consider the following set of data reported by Nelson (1982). Nelson presents the results of a life-test experiment in which specimens of a type of electrical insulating fluid were subject to a constant voltage stress (34kv/minutes). In analyzing the complete data, Nelson assumed a Weibull distribution for the times to breakdown.

We have to compute correlation coefficient. Since, the correlation coefficient between two sets of two data is very high to find relationship between the previous data and data generated by Weibull gamma distribution presented in table 11. We have to compute correlation coefficient. Since, the correlation coefficient between two sets of two data is very high to find relationship between the previous data and data generated by Weibull gamma distribution presented in table 11. We have to compute correlation coefficient. Since, the correlation coefficient between two sets of two data is very high to find relationship between the previous data and data generated by Weibull gamma distribution presented in table 11. We have to compute correlation coefficient. Since, the correlation coefficient between two sets of two data is very high to find relationship between the previous data and data generated by Weibull gamma distribution presented in table 11. We have to compute correlation coefficient. Since, the correlation coefficient between two sets of two data is very high.

<p>| Table 10: Progressively type-II right censored sample from the real data set. |</p>
<table>
<thead>
<tr>
<th>$X(j)$</th>
<th>$R(j)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X(j)$</td>
<td>0.8977</td>
</tr>
<tr>
<td>$R(j)$</td>
<td>0</td>
</tr>
</tbody>
</table>

then, Weibull gamma distribution is appropriate for the data (set 1).

From that, we can get:

$$\mu^* = 0.2580, \quad \sigma^* = 0.4454$$

$$\hat{\mu} = 0.2021, \quad \hat{\sigma} = 0.3211$$

and the standard error of $\mu^*$ and $\sigma^*$ are written:

$$SE(\mu^*) = \sigma^* (\text{var}(\mu^*))(1/2)) = 0.4454 ((1.5667)^*(1/2)) = 0.5574.$$  

$$SE(\sigma^*) = \sigma^* (\text{var}(\sigma^*))(1/2)) = 0.4454 ((0.0090)^*(1/2)) = 0.13362.$$  

References


