Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/090311

Existence of Extremal Solutions for Caputo Fractional Differential Equations with Bounded Delay

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Received: 2 Sep. 2021, Revised: 18 Dec. 2021, Accepted: 8 Jan. 2022
Published online: 1 Jul. 2023

Abstract: In this paper we study Caputo fractional differential equations with bounded delay. By finding lower and upper solutions of the problem, we develop two generalized monotone methods in which we construct a decreasing sequence that converges uniformly and monotonically to a minimal solution, and a decreasing sequence that converges uniformly and monotonically to a maximal solution. We finish this work by illustrating our results with an example.

Keywords: Caputo fractional derivatives, differential equations with bounded delay, monotone iterative techniques.

1 Introduction

The idea of a fractional derivative was first introduced in a letter from L'Hôpital to Leibniz in 1695, and definitions of fractional derivatives and fractional integrals were first given by Riemann and Liouville in the ninetieth century. However, fractional calculus has gained popularity since the second half of the twentieth century when scientists discovered their applicability as models with higher accuracy than integer-order differential equations, see the books [1, 2,3,4,5,6] for further details. In recent years researchers have studied fractional differential equations via upper and lower solutions and monotone iterative techniques, which was established for ordinary differential equations in [7], and have been implemented in the book [3], as well as the papers [8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25, 26,27,28,29], to name a few.

One of the simplest types of functional differential equations is a differential equation with delay, and the basic theory, several methods and some applications to delay differential equations are presented in the book [30]. Important work on fractional differential equations with delay can be found in papers such as [31,32,33,34,35,36,37,38]. Recently, a monotone method for neutral fractional differential equation with infinite delay was developed in [39], and the result was extended in [40] for nonlocal impulsive finite delay fractional differential equations. In both [39,40] the authors study the existence of mild solutions in an ordered Banach Space.

In this work we consider a differential equation with Caputo fractional derivative of order q, $0 < q < 1$, and bounded delay. In order to develop a generalized monotone method, we first prove a comparison theorem. Next, by finding coupled lower and upper solutions, we construct sequences in which the iterates are obtained by computing Volterra fractional integral equations, and converge to minimal and maximal solutions uniformly and monotonically. Sequences may be natural or intertwined.

2 Preliminaries

We begin this section by stating the definitions of the Riemann–Liouville fractional derivative and the Caputo fractional derivative, followed by the results that will be needed to obtain the main theorems.

An important function in fractional calculus is the Mittag–Leffler function.

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Definition 1. *The Mittag–Leffler function is defined as*

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+\beta)},$$

and when $\beta = 1$, the Mittag–Leffler function is denoted by

$$E_{\alpha}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+1)}.$$

When $\alpha = \beta = 1$ the function becomes the natural exponential function, and for this reason $E_{\alpha,\beta}(t)$ is also known as the generalized exponential function.

Let $J = [t_0, t_0 + a]$ be a real interval. The definitions of the Riemann–Liouville and Caputo fractional derivatives, respectively, are stated in [1,2,3,6].

Definition 2. *The Riemann-Liouville fractional derivative of order α, where $n-1 \leq \alpha < n$ and $n \in \mathbb{N}$, is denoted by D^{α} and defined by*

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_{t_0}^{t} (t-s)^{n-\alpha-1} f(s)ds.$$

Definition 3. *The Caputo fractional derivative of order α, $n-1 \leq \alpha < n$ for $t \in J$, written ${}^cD^{\alpha}$, is defined as*

$${}^cD^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t-s)^{n-\alpha-1} f^{(n)}(s)ds.$$

In the remainder of this paper we focus on order q, where $0 < q < 1$.

The definition of C_p continuity is important in some of the results of this section.

Definition 4. *If $0 < q < 1$, $p = 1-q$, and G is an open set of $\mathbb{R}$, then $C_p(J,G)$ is the function space*

$$C_p(J,G) = \left\{u \in C((t_0, t_0+a], G) \,\middle|\, (t-t_0)^p u(t) \in C(J,G)\right\},$$

and $u \in C_p(J,G)$ is called a C_p continuous function in J.

Remark. Consider the fractional IVP

$$\begin{aligned} {}^cD^q u(t) &= f(t, u(t)), \\ u(t_0) &= u_0, \end{aligned} \tag{1}$$

where $f : (t_0, t_0+a] \times G \to \mathbb{R}$.

In [2] the authors show that if $q \in (0,1)$, $G \in \mathbb{R}$ is open, and $u \in C([t_0, t_0+a], G)$ is such that f is C_p continuous in J, then u is a solution (1) if and only if it is a solution of the Volterra fractional integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, u(s))ds. \tag{2}$$

In particular, the relationship is true when f is continuous.

In [3] the authors proved that the solution of the non-homogeneous linear equation

$$\begin{aligned} {}^cD^q u(t) &= Mu(t) + f(t), \\ u(t_0) &= u_0, \end{aligned} \tag{3}$$

where $M \in \mathbb{R}$ and f is continuous in J is

$$u(t) = u_0 E_q\left(M(t-t_0)^q\right) + \int_{t_0}^{t} (t-s)^{q-1} E_{q,q}\left(M(t-s)^q\right) f(s)ds \quad t \in J. \tag{4}$$

Let τ be a positive real number. For $u \in C[J, \mathbb{R}]$ and $t \geq t_0$ with $t \in J$, let u_t denote the transformation of u to the interval $[t_0 - \tau, t + a - \tau]$, that is

$$u_t(s) = u(t+s), \quad -\tau \leq s \leq 0,$$

if $\phi \in C[[-\tau, 0], \mathbb{R}]$, then we can generalize (1)-(2) to differential equations with bounded delay.

Remark. The function u is a solution of the differential equation with bounded delay given by

$$\begin{aligned} {}^cD^q u &= f(t, u(t), u_t(s)), \\ u_{t_0}(s) &= \phi(s), \end{aligned} \tag{5}$$

where $f \in C[J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}]$, if and only if u satisfies the Volterra integral equation

$$u(t) = \begin{cases} \phi(t - t_0), & t_0 - \tau \le t < t_0, \\ \phi(0) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - x)^{q-1} f(t, u(s), u_t(s)) dx, & t_0 \le t \le t_0 + a. \end{cases} \tag{6}$$

We finish the section by stating a comparison theorem relative to fractional differential equations with bounded delay, but first we state a lemma from [13] for Riemann–Liouville derivatives.

Lemma 1. *Let $m \in C_p([t_0, t_0 + a], \mathbb{R})$ and for any $t_1 \in (t_0, t_0 + a]$ we have that on (t_0, t_1), $m(t) \le 0$, $m(t_1) = 0$ and $m(t)(t - t_0)^{1-q}|_{t=t_0} \le 0$. Then $D^q m(t_1) \ge 0$.*

With the above lemma we can prove an equivalent result for Caputo derivatives.

Lemma 2. *Suppose that $m(t) \in C([t_0, t_0 + a], \mathbb{R})$. If $t_1 \in [t_0, t_0 + a]$ is such that $m(t_1) = 0$ and $m(t) < 0$ in $[t_0, t_1)$, then*

$${}^cD^q m(t_1) \ge 0.$$

Proof. The relationship between the Caputo and Riemann-Liouville derivatives for $q \in (0, 1)$ is given in [2] by

$${}^cD^q m(t_1) = D^q [m(t_1) - m(t_0)] = D^q m(t_1) - \frac{m(t_0)}{\Gamma(1 - q)} (t - t_0)^{-q} \ge D^q m(t_1).$$

Assume that $t_1 \in [t_0, t_0 + a]$,
It follows from Lemma 1 that $D^q m(t_1) \ge 0$, therefore ${}^cD^q m(t_1) \ge 0$ by the relationship above.

We can finally state the following comparison theorem.

Theorem 1. *Assume $J_0 = [t_0 - \tau, t_0 + a]$, and assume that there are functions $v^0(t), w^0(t) \in C[J_0, \mathbb{R}]$ with $v^0(t) < w^0(t)$ such that the following are true:*
(a) $f(t, u(t), u_t(s))$ and $g(t, u, u_t(s)$ are continuous for $t \ge t_0$ and $-\tau \le s \le 0$,
(b) f is an increasing function in u and u_t, g is a decreasing function in u and u_t, and
(c) If $v_0(t) \le v(t)$ and $w(t) \le w_0(t)$, then $v(t), w(t) \in C[J_0, \mathbb{R}]$ satisfy the following inequalities for $t \in J = [t_0, t_0 + a]$, $s \in [-\tau, 0]$,

$$\begin{aligned} {}^cD^q v(t) &\le f(t, v(t), v_t(s)) + g(t, w(t), w_t(s)),\ v_{t_0}(s) \le \phi(s),\ \text{and} \\ {}^cD^q w(t) &\ge f(t, w(t), w_t(s)) + g(t, v(t), v_t(s)),\ w_{t_0}(s) \ge \phi(s). \end{aligned} \tag{7}$$

Suppose further that f and g satisfy one-sided Lipschitz conditions for $L_1 > 0$, $L_2 > 0$, $M_1 \ge 0$, $M_2 \ge 0$, $x \ge y$, and $\varphi_1(s) \ge \varphi_2(s)$,

$$\begin{aligned} f(t, x, \varphi_1(s)) - f(t, y, \varphi_2(s)) &\le L_1(x - y) \\ &\quad + M_1 \left(\sup_{s \in [-\tau, 0]} \{\varphi_1(s) - \varphi_2(s)\} \right) \\ g(t, x, \varphi_1(s)) - g(t, y, \varphi_2(s)) &\ge -L_2(x - y) \\ &\quad - M_2 \left(\sup_{s \in [-\tau, 0]} \{\varphi_1(s) - \varphi_2(s)\} \right) \end{aligned} \tag{8}$$
,

then $v_{t_0}(s) \le w_{t_0}(s)$ implies that

$$v(t) \le w(t),\ \text{for } t_0 \le t \le t_0 + a.$$

Proof. We will first assume that one inequality in (7) is strict, say, without loss of generality, ${}^cD^q v(t) < f(t, v(t), v_t(s)) + g(t, w(t), w_t(s))$, and $v_{t_0}(0) < w_{t_0}(0)$.
If the conclusion $v(t) < w(t)$ is false, there exists $t_1 \in (t_0, t_0 + a]$ for which

$$v(t_1) = w(t_1),\ \text{and } v(t) < w(t),\ \text{for } t_0 < t < t_1.$$

Note that the previous statement implies that $v_{t_1}(s) \le w_{t_1}(s)$. In fact, if $t_0 - \tau \le t_1 - s \le t_0$ then it follows by hypothesis that $v_{t_1}(s) \le w_{t_1}(s)$. And if $t_0 < t_1 - s < t_1$, then it follows immediately that $v_{t_1}(s) < w_{t_1}(s)$.

Define a new function $m(t) = v(t) - w(t)$. Thus $m(t_1) = 0$ and $m(t) < 0$ for $t_0 < t < t_1$. By Lemma 2, ${}^cD^q m(t_1) \ge 0$. Thus

$$\begin{aligned} f(t_1, v(t_1), v_{t_1}(s)) + g(t_1, w(t_1), w_{t_1}(s)) &> {}^cDv(t_1) \\ &\ge {}^cDw(t_1) \ge f(t_1, w(t_1), w_{t_1}(s)) + g(t_1, v(t_1), v_{t_1}(s)), \end{aligned}$$

which is a contradiction to the assumption $v(t_1) = w(t_1)$ as well as $v_{t_1}(s) \le w_{t_1}(s)$, and we conclude that $v(t) < w(t)$ for $t > t_0$

Suppose the assumption (7) is true.

Define new functions $v^\varepsilon(t) = v(t) - \varepsilon\mu(t)$ and $w^\varepsilon(t) = w(t) + \varepsilon\mu(t)$, where

$$\mu(t) = \begin{cases} 1, & t < t_0, \\ E_q(\lambda(t-t_0)^q) & t \ge t_0, \end{cases}$$

and $\lambda > 1$ is a constant whose value will be determined. Clearly $\mu(t)$ is continuous on J_0, with $\mu(t) > 0$ and $\mu_t(s) > 0$. Thus v^ε and w^ε are continuous on J_0 with $v^\varepsilon(t_0) = v(t_0) - \varepsilon < v(t_0)$, $w^\varepsilon(t_0) = w(t_0) + \varepsilon > w(t_0)$, $v^\varepsilon(t) < v(t)$, $w^\varepsilon(t) > w(t)$, $v_t^\varepsilon(s) < v_t(s)$ and $w_t^\varepsilon(s) > w_t(s)$.

Using (4) with $f \equiv 0$, and (7)-(8), we obtain for $t > t_0$ that

$$\begin{aligned} {}^cD^q v^\varepsilon(t) &= {}^cD^q v(t) - \varepsilon\lambda E_q(\lambda(t-t_0)^q) \\ &\le f(t, v(t), v_t(s)) + g(t, w(t), w_t(s)) - \varepsilon\lambda E_q(\lambda(t-t_0)^q) \\ &= f(t, v(t), v_t(s)) + g(t, w(t), w_t(s)) - f(t, v^\varepsilon(t), v_t^\varepsilon(s)) \\ &\quad - g(t, w^\varepsilon(t), w_t^\varepsilon(s)) + f(t, v^\varepsilon(t), v_t^\varepsilon(s)) + g(t, w^\varepsilon(t), w_t^\varepsilon(s)) \\ &\quad - \varepsilon\lambda E_q(\lambda(t-t_0)^q) \\ &\le L_1(v(t) - v^\varepsilon(t)) + M_1\left(\sup_{s\in[-\tau,0]}\{v_t(s) - v_t^\varepsilon(s)\}\right) \\ &\quad + L_2(w^\varepsilon(t) - w(t)) + M_2\left(\sup_{s\in[-\tau,0]}\{w_t^\varepsilon(s) - w_t(s)\}\right) \\ &\quad + f(t, v^\varepsilon(t), v_t^\varepsilon(s)) + g(t, w^\varepsilon(t), w_t^\varepsilon(s)) - \varepsilon\lambda E_q(\lambda(t-t_0)^q) \\ &= \varepsilon L_1(E_q(\lambda(t-t_0)^q)) + \varepsilon M_1 \sup_{s\in[-\tau,0]}\{\mu(t)\} \\ &\quad + \varepsilon L_2(E_q(\lambda(t-t_0)^q)) + \varepsilon M_2 \sup_{s\in[-\tau,0]}\{\mu(t)\} \\ &\quad + f(t, v^\varepsilon(t), v_t^\varepsilon(s)) + g(t, w^\varepsilon(t), w_t^\varepsilon(s)) - \varepsilon\lambda E_q(\lambda(t-t_0)^q) \\ &\le \varepsilon L_1(E_q(\lambda(t-t_0)^q)) + \varepsilon M_1(E_q(\lambda(t-t_0)^q)) \\ &\quad + \varepsilon L_2(E_q(\lambda(t-t_0)^q)) + \varepsilon M_2(E_q(\lambda(t-t_0)^q)) \\ &\quad + f(t, v^\varepsilon(t), v_t^\varepsilon(s)) + g(t, w^\varepsilon(t), w_t^\varepsilon(s)) - \varepsilon\lambda E_q(\lambda(t-t_0)^q) \\ &= \varepsilon(L_1 + M_1 + L_1 + L_2 - \lambda)(E_q(\lambda(t-t_0)^q)) \\ &\quad + f(t, v^\varepsilon(t), v_t^\varepsilon(s)) + g(t, w^\varepsilon(t), w_t^\varepsilon(s)). \end{aligned}$$

Let $\lambda = L_1 + M_1 + L_2 + M_2 + 1$, then

$${}^cD^q v^\varepsilon(t) < f(t, v^\varepsilon(t), v_t^\varepsilon(s)) + g(t, w^\varepsilon(t), w_t^\varepsilon(s)).$$

By a similar procedure, we can show that

$${}^cD^q w^\varepsilon(t) > f(t, w^\varepsilon(t), w_t^\varepsilon(s)) + g(t, v^\varepsilon(t), v_t^\varepsilon(s)).$$

From the initial result for strict inequalities, we conclude that $v^\varepsilon(t) < w^\varepsilon(t)$ for $t \in J$ and for every $\varepsilon > 0$. We just proved that $v(t) - \varepsilon E_q(\lambda(t-t_0)^q) < w(t) + \varepsilon E_q(\lambda(t-t_0)^q)$, or $v(t) < w(t) + 2\varepsilon E_q(\lambda(t-t_0)^q)$.

Letting $\varepsilon \to 0$, we get that $v(t) \le w(t)$ for $t \in J$.

Theorem 1 has important consequences that will be needed in the development of our main result.

Corollary 1. *Let $m \in C[J_0, \mathbb{R}]$ be such that*

$$\begin{aligned} &{}^cD^q m(t) \le Lm(t) + M\left(\sup_{s\in[-\tau,0]}\{m_t(s)\}\right), \quad t > t_0 \\ &m_{t_0}(s) \le 0, \quad -\tau \le s \le 0, \end{aligned}$$

where $L > 0, M \geq 0$. Then we have from Theorem 1 that

$$m(t) \leq 0, \quad for\, t \in J.$$

Similarly, if $m \in C[J_0,\mathbb{R}]$ is such that

$$\begin{aligned} {}^cD^q m(t) &\geq -Lm(t) - M\left(\sup_{s\in[-\tau,0]}\{m_t(s)\}\right), \quad t > t_0 \\ m_{t_0}(s) &\geq 0, \quad -\tau \leq s \leq 0, \end{aligned}$$

where $L > 0$ and $M \geq 0$, it follows from Theorem 1 that

$$m(t) \geq 0, \quad for\, t \in J.$$

If $L = M = 0$, Corollary 1 is also true. We state it as a separate result.

Corollary 2.*If $m \in C[J_0,\mathbb{R}]$ is such that ${}^cD^q m(t) \leq 0$ for $t \in J$ and $m_{t_0}(s) \leq 0$ for $s \in [-\tau,0]$, then $m(t) \leq 0$ for $t \in J$.*

3 Monotone iterative techniques

In this section we develop two generalized monotone iterative techniques for the nonlinear delay differential equation (9), given below.

Consider the equation

$$\begin{aligned} {}^cD^q u(t) &= f(t,u(t),u_t(s)) + g(t,u(t),u_t(s)), \\ u_{t_0}(s) &= \phi(s), \end{aligned} \tag{9}$$

where $J = [t_0, t_0 + a]$, $J_0 = [t_0 - \tau, t_0 + a]$, $u \in C[J_0,\mathbb{R}]$, f and g are continuous real functions, $t \in J$, $s \in [-\tau,0]$, and $\phi \in C[[-\tau,0],\mathbb{R}]$.

If $u \in C[J,\mathbb{R}]$ satisfies the fractional differential equation with bounded delay (9), then we say that u is a solution of (9).

Furthermore, we assume that f is an increasing function in u and u_t, and g is a decreasing function in u and u_t for $t \in J$ and $s \in [-\tau,0]$.

We define different types of lower and upper solutions of (9).

Definition 5.*Let $v^0, w^0 \in C[J_0,\mathbb{R}]$. Then v^0 and w^0 are:*

–*Natural lower and upper solutions of (9) if*

$$\begin{aligned} {}^cD^q v^0(t) &\leq f(t,v^0(t),v^0_t(s)) + g(t,v^0(t),v^0_t(s)), \\ v^0_{t_0}(s) &\leq \phi(s), \\ {}^cD^q w^0(t) &\geq f(t,w^0(t),w^0_t(s)) + g(t,w^0(t),w^0_t(s)), \\ w^0_{t_0}(s) &\geq \phi(s). \end{aligned} \tag{10}$$

–*Coupled lower and upper solutions of Type I of (9) if*

$$\begin{aligned} {}^cD^q v^0(t) &\leq f(t,v^0(t),v^0_t(s)) + g(t,w^0(t),w^0_t(s)), \\ v^0_{t_0}(s) &\leq \phi(s), \\ {}^cD^q w^0(t) &\geq f(t,w^0(t),w^0_t(s)) + g(t,v^0(t),v^0_t(s)), \\ w^0_{t_0}(s) &\geq \phi(s). \end{aligned} \tag{11}$$

–*Coupled lower and upper solutions of Type II of (9) if*

$$\begin{aligned} {}^cD^q v^0(t) &\leq f(t,w^0(t),w^0_t(s)) + g(t,v^0(t),v^0_t(s)), \\ v^0_{t_0}(s) &\leq \phi(s), \\ {}^cD^q w^0(t) &\geq f(t,v^0(t),v^0_t(s)) + g(t,w^0(t),w^0_t(s)), \\ w^0_{t_0}(s) &\geq \phi(s). \end{aligned} \tag{12}$$

–*Coupled lower and upper solutions of Type III of (9) if,*

$$\begin{aligned} {}^cD^q v^0(t) &\leq f(t,w^0(t),w_t^0(s)) + g(t,w^0(t),w_t^0(s)), \\ v_{t_0}^0(s) &\leq \phi(s), \\ {}^cD^q w^0(t) &\geq f(t,v^0(t),v_t^0(s)) + g(t,v^0(t),v_t^0(s)), \\ w_{t_0}^0(s) &\geq \phi(s). \end{aligned} \tag{13}$$

The following theorem requires the existence of solutions of the form (11), with which we will develop a monotone method for the differential equation with bounded delay (9). Finally, we prove the existence of natural sequences that converge uniformly and monotonically to coupled minimal and maximal solutions of (9).

Theorem 2.*Assume that*
(A1) v^0, w^0 are coupled lower and upper solutions of type I for (9) with $v^0(t) \leq w^0(t)$ in J_0; and
(A2) $f, g \in C\left(J \times [v^0(t), w^0(t)] \times [v_t^0(s), w_t^0(s)], \mathbb{R}\right)$, where $t \in J$, $s \in [-\tau, 0]$, $f(t, u(t), u_t(s))$ is increasing in u and u_t and $g(t, u(t), u_t(s))$ is decreasing in u and in u_t.
If $u(t)$ is a solution of (9) such that $v^0(t) \leq u(t) \leq w^0(t)$ for all $t \in J$, then the sequences given by

$$\begin{aligned} {}^cD^q v^{n+1}(t) &= f(t,v^n(t),v_t^n(s)) + g(t,w^n(t),w_t^n(s)), \\ v_{t_0}^{n+1}(s) &= \phi(s), \end{aligned} \tag{14}$$

and

$$\begin{aligned} {}^cD^q w^{n+1}(t) &= f\left(t,w^n(t),w_t^n(s)\right) + g\left(t,v^n(t),v_t^n(s)\right), \\ w_{t_0}^{n+1}(s) &= \phi(s), \end{aligned} \tag{15}$$

are such that

$$v^0 \leq v^1 \leq v^2 \leq \cdots \leq v^{n+1} \leq u \leq w^{n+1} \leq w^n \leq \cdots \leq w^0$$

on J, where $v^n(t) \to \rho(t)$ and $w^n(t) \to r(t)$ in $C[J,\mathbb{R}]$ uniformly and monotonically, and ρ and r are such that

$$\begin{aligned} {}^cD^q \rho(t) &= f(t,\rho(t),\rho_t(s)) + g(t,r(t),r_t(s)), \\ \rho_{t_0}(s) &= \phi(s), \end{aligned}$$

and

$$\begin{aligned} {}^cD^q r(t) &= f(t,r(t),r_t(s)) + g(t,\rho(t),\rho_t(s)), \\ r_{t_0}(s) &= \phi(s), \end{aligned}$$

with $\rho \leq u \leq r$. That is, ρ, r are coupled minimal and maximal solutions of (9), respectively.

*Proof.*We have by hypothesis that $v^0 \leq u \leq w^0$, and will show that $v^0 \leq v^1 \leq u \leq w^1 \leq w^0$.
From (11) it follows that

$$\begin{aligned} {}^cD^q v^0(t) &\leq f(t,v^0(t),v_t^0(s)) + g(t,w^0(t),w_t^0(s)),\ v_{t_0}^0(s) \leq \phi(s), \\ {}^cD^q w^0(t) &\geq f(t,w^0(t),w_t^0(s)) + g(t,v^0(t),v_t^0(s)),\ w_{t_0}^0(s) \geq \phi(s), \end{aligned}$$

and from (14), we obtain

$$\begin{aligned} {}^cD^q v^1(t) &= f\left(t,v^0(t),v_t^0(s)\right) + g\left(t,w^0(t),w_t^0(s)\right), \\ v_{t_0}^1(s) &= \phi(s). \end{aligned}$$

Therefore, $v_{t_0}^0(s) \leq \phi(s) = v_{t_0}^1(s)$. Let $p(t) = v^0(t) - v^1(t)$, $t \in J$, then for $s \in [-\tau, 0]$, $p_{t_0}(s) \leq 0$. Also,

$$\begin{aligned} {}^cD^q p(t) &= {}^cD^q v_0(t) - {}^cD^q v_1(t) \\ &\leq f\left(t,v^0(t),v_t^0(s)\right) + g\left(t,w^0(t),w_t^0(s)\right) - f\left(t,v^0(t),v_t^0(s)\right) - g\left(t,w^0(t),w_t^0(s)\right) \\ &= 0. \end{aligned}$$

Since ${}^cD^q p(t) \leq 0$ and $p_{t_0}(s) \leq 0$, it follows from Corollary 2 that $p(t) \leq 0$ for $t \in J$. Hence, $v^0(t) \leq v^1(t)$ in J.
Assume that a solution u of (9) satisfies $v^0(t) \leq u(t) \leq w^0(t)$ in J. Since $v_{t_0}^0(s) \leq \phi(s)$, $w_{t_0}^0(s) \geq \phi(s)$ and $u_{t_0}(s) = \phi(s)$, $s \in [-\tau, 0]$, we have that $v_{t_0}^0(s) \leq u_{t_0}(s) \leq w_{t_0}^0(s)$. Thus $v_t^0(s) \leq u_t(s) \leq w_t^0(s)$ for $t \in J$.

Setting $p(t) = v^1(t) - u(t)$, we obtain $p_{t_0}(s) = v^1_{t_0}(s) - u_{t_0}(s) = \phi(s) - \phi(s) = 0$. Furthermore, from the facts that f is increasing and g is decreasing we have that

$$\begin{aligned} {}^cD^q p(t) &= {}^cD^q v^1(t) - {}^cD^q u(t) \\ &= f\left(t, v^0(t), v^0_t(s)\right) + g\left(t, w^0(t), w^0_t(s)\right) \\ &\quad - f\left(t, u(t), u_t(s)\right) - g\left(t, u(t), u_t(s)\right) \\ &\le 0, \end{aligned}$$

and by Corollary 2 it follows that $v^1(t) \le u(t)$ in J. We can use similar arguments to show that $u(t) \le w^1(t)$ and $w^1(t) \le w^0(t)$. Thus, $v^0(t) \le v^1(t) \le u(t) \le w^1(t) \le w^0(t)$ in J.

Next, we prove that $v^k(t) \le v^{k+1}(t)$ for $k \ge 1$ and $t \in J$.

Suppose that

$$v^{k-1}(t) \le v^k(t) \le u(t) \le w^k(t) \le w^{k-1}(t),$$

for $k > 1$. Since $v^k_{t_0}(s) = w^k_{t_0}(s) = \phi(s)$, then $v^{k-1}_{t_0}(s) = v^k_{t_0}(s) = u_{t_0}(s) = w^k_{t_0}(s) = v^{k-1}_{t_0}(s) = \phi(s)$ for $k > 1$, and consequently

$$v^{k-1}_t(s) \le v^k_t(s) \le u_t(s) \le w^k_t(s) \le w^{k-1}_t(s),$$

Let $p = v^k - v^{k+1}$. Then

$$v^k_{t_0}(s) = \phi(s) = v^{k+1}_{t_0}(s),$$

so $p_{t_0}(s) = 0$. From the hypotheses that f is increasin and g is decreasing we have that

$$\begin{aligned} {}^cD^q p(t) &= {}^cD^q v^k(t) - {}^cD^q v^{k+1}(t) \\ &= f\left(t, v^{k-1}(t), v^{k-1}_t(s)\right) + g\left(t, w^{k-1}(t), w^{k-1}_t(s)\right) \\ &\quad - f\left(t, v^k(t), v^k_t(s)\right) - g\left(t, w^k(t), w^k_t(s)\right) \\ &\le 0. \end{aligned}$$

Using again Corollary 2, it follows that $p(t) \le 0$. Therefore, $v^k(t) \le v^{k+1}(t)$.

Similarly, we can prove the inequalities $w^{k+1}(t) \le w^k(t)$, $v^{k+1}(t) \le u(t)$, and $u(t) \le w^{k+1}(t)$ on J. Thus, for $n > 0$ and $t \in J$,

$$v^0 \le v^1 \le \cdots \le v^n \le u \le w^n \le \cdots \le w^1 \le w^0.$$

We are ready to prove that v_n and w_n converge uniformly in J. The proof of uniform convergence requires the Arzela–Ascoli Theorem, for which we need to show uniform boundedness and equicontinuity.

We first show that the sequences are uniformly bounded. Since both $v^0(t)$ and $w^0(t)$ are continuous on J, they are bounded on J, and there is a real number $M > 0$ such that $|v^0(t)|, |w^0(t)| \le M$ for any $t \in J$. Since $v^0(t) \le v^k(t) \le w^k(t) \le w^0(t)$ for each $n > 0$, it follows that $\{v^n(t)\}$ and $\{w^n(t)\}$ are uniformly bounded on J. Moreover, they are uniformly bounded on J_0 because $v^0_{t_0}(s) \le v^n_{t_0}(s) = \phi(s) = w^n_{t_0}(s) \le w^0_{t_0}(s)$ and all these functions are continuous on $[-\tau, 0]$.

Now we prove equicontinuity of $\{v^n(t)\}$. Let $t_0 \le t_1 \le t_2 \le t_0 + a$. Then from (6), we obtain for that

$$\begin{aligned} |v^n(t_1) - v^n(t_2)| &= \Bigg| \phi(0) + \frac{1}{\Gamma(q)} \int_{t_0}^{t_1} (t_1 - x)^{q-1} \left[f\left(x, v_{n-1}(x), v^{n-1}_x(s)\right) + g\left(x, w^{n-1}(x), w^{n-1}_x(s)\right) \right] dx \\ &\quad - \phi(0) - \frac{1}{\Gamma(q)} \int_{t_0}^{t_2} (t_2 - x)^{q-1} \left[f\left(x, v_{n-1}(x), v^{n-1}_x(s)\right) + g\left(x, w^{n-1}(x), w^{n-1}_x(s)\right) \right] dx \Bigg| \\ &= \Bigg| \frac{1}{\Gamma(q)} \int_{t_0}^{t_1} [(t_1 - x)^{q-1} - (t_2 - x)^{q-1}] \left[f\left(x, v^{n-1}(x), v^{n-1}_x(s)\right) + g\left(x, w^{n-1}(x), w^{n-1}_x(s)\right) \right] dx \\ &\quad - \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - x)^{q-1} \left[f\left(x, v^{n-1}(x), v^{n-1}_x(s)\right) + g\left(x, w^{n-1}(x), w^{n-1}_x(s)\right) \right] dx \Bigg| \\ &\le \frac{1}{\Gamma(q)} \int_{t_0}^{t_1} [(t_1 - x)^{q-1} - (t_2 - x)^{q-1}] \left| f\left(x, v^{n-1}(x), v^{n-1}_x(s)\right) + g\left(x, w^{n-1}(x), w^{n-1}_x(s)\right) \right| dx \\ &\quad + \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - x)^{q-1} \left| \left[f\left(x, v^{n-1}(x), v^{n-1}_x(s)\right) + g\left(x, w^{n-1}(x), w^{n-1}_x(s)\right) \right] \right| dx. \end{aligned}$$

Since $\{v^n(t)\}$ and $\{w^n(t)\}$ are uniformly bounded on J_0 and $f(t,u(t),u_t(s))$ and $g(t,u(t),u_t(s))$ are in $C\left[t\times\left[v^0(t),w^0(t)\right]\times\left[v_t^0(s),w_t^0(s)\right],\mathbb{R}\right]$, there exists a real numnber $\bar{M}>)$ that does not depend on n such that

$$\begin{aligned}|f(t,v^n(t),v_t^n(s))|&\le\bar{M},\\|f(t,w^n(t),w_t^n(s))|&\le\bar{M},\\|g(t,v^n(t),v_t^n(s))|&\le\bar{M},\text{ and}\\|g(t,w^n(t),w_t^n(s))|&\le\bar{M}.\end{aligned}$$

Then, from the last expression we obtain that

$$\begin{aligned}&\tfrac{1}{\Gamma(q)}\int_{t_0}^{t_1}[(t_1-x)^{q-1}-(t_2-x)^{q-1}]\left|f\left(x,v^{n-1}(x),v_x^{n-1}(s)\right)+g\left(x,w^{n-1}(x),w_x^{n-1}(s)\right)\right|dx\\&+\tfrac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-x)^{q-1}\left|\left[f\left(x,v^{n-1}(x),v_x^{n-1}(s)\right)+g\left(x,w^{n-1}(x),w_x^{n-1}(s)\right)\right]\right|dx\\\le&\tfrac{\bar{M}}{\Gamma(q)}\int_{t_0}^{t_1}[(t_1-x)^{q-1}-(t_2-x)^{q-1}]dx+\tfrac{\bar{M}}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-x)^{q-1}\\=&-\tfrac{\bar{M}}{q\Gamma(q)}(t_1-x)^q\Big|_{t_0}^{t_1}+\tfrac{\bar{M}}{q\Gamma(q)}(t_2-x)^q\Big|_{t_0}^{t_1}-\tfrac{\bar{M}}{q\Gamma(q)}(t_2-x)^q\Big|_{t_1}^{t_2}\\=&\tfrac{\bar{M}}{\Gamma(q+1)}(t_1-x)^q+\tfrac{\bar{M}}{\Gamma(q+1)}(t_2-t_1)^q-\tfrac{\bar{M}}{\Gamma(q+1)}(t_2-x)^q+\tfrac{\bar{M}}{\Gamma(q+1)}(t_2-t_1)^q\\\le&\tfrac{2\bar{M}}{\Gamma(q+1)}(t_2-t_1)^q\\=&\tfrac{2\bar{M}}{\Gamma(q+1)}|t_1-t_2|^q.\end{aligned}$$

Let $\varepsilon>0$, then there is $\delta=\dfrac{\Gamma(q+1)}{2\bar{M}}\varepsilon^{1/q}>0$ independent of n such that for each n,

$$|v_n(t_1)-v_n(t_2)|<\varepsilon,$$

provided that $|t_1-t_2|<\delta$. This completes the proof of equicontinuity of $\{v^n(t)\}$ on J.

By a similar argument we can prove that $\{w^n(t)\}$ is equicontinuous on J.

Since $\{v^n(t)\}$ and $\{w^n(t)\}$ are both uniformly bounded and equicontinuous on J, we have by the Arzela–Ascoli Theorem that there are subsequences of $\{v^n(t)\}$ and $\{w^n(t)\}$ that are uniformly uniformly convergent to $\rho(t)$ and $r(t)$, respectively. But $\{v^n(t)\}$ and $\{w^n(t)\}$ are monotone, therefore they are uniformly convergent on J.

Furthermore, since $v_{t_0}^n(s)=w_{t_0}^n(s)=\phi(s)$, $s\in[-\tau,0]$, we have that $\{v_t^n(s)\}$ and $\{w_t^n(s)\}$ are also uniformly bounded and equicontinuous on J. Hence, by the same argument, $\{v_t^n(s)\}$ and $\{w_t^n(s)\}$ converge uniformly and monotonically to $\rho_t(s)$ and $r_t(s)$, respectively.

In order to show that ρ is a minimal solution of (9) and r is a maximal solution of (9), observe that since each v^n is constructed as follows

$$\begin{aligned}{}^cD^qv^n(t)&=f(t,v^{n-1}(t),v_t^{n-1}(s))+g(t,w^{n-1}(t),w_t^{n-1}(s)),\\v_{t_0}^n(s)&=\phi(s),\end{aligned}$$

and

$$v^n(t)=\begin{cases}\phi(t-t_0), & t_0-\tau\le t<t_0,\\ \phi(0)+\frac{1}{\Gamma(q)}\int_{t_0}^t(t-x)^{q-1}\big[f(x,v^{n-1}(x),v_x^{n-1}(s)) & \\ +g(x,w^{n-1}(x),w_x^{n-1}(s))\big]dx, & t_0\le t\le t_0+a.\end{cases}$$

it follows from the Lebesgue Dominated Convergence Theorem that the limit as $n\to\infty$ is

$$\rho(t)=\begin{cases}\phi(t-t_0), & t_0-\tau\le t<t_0,\\ \phi(0)+\frac{1}{\Gamma(q)}\int_{t_0}^t(t-x)^{q-1}\big[f(x,\rho(x),\rho_x(s)) & \\ +g(x,r(x),r_x(s))\big]dx, & t_0\le t\le t_0+a.\end{cases}$$

Thus, ρ satisfies the system

$$\begin{aligned}{}^cD^q\rho(t)&=f(t,\rho(t),\rho_t(s))+g(t,r(t),r_t(s))\text{ on }J,\\\rho_{t_0}(s)&=\phi(s).\end{aligned}$$

Similarly we can prove that r satisfies

$$\begin{aligned} {}^cD^q r(t) &= f(t,r(t),r_t(s)) + g(t,\rho(t),\rho_t(s)) \text{ on } J, \\ r_{t_0}(s) &= \phi(s). \end{aligned}$$

We have shown that if u is a solution, then $u(t) \in [v^n(t), w^n(t)]$ on J for any n. Therefore, $u(t) \in [\rho(t), r(t)]$ on J. This proves that ρ is a minimal solution of (9) and r is a maximal solution of (9), and the proof is complete.

Finding coupled lower and upper solutions of Type II is easier than finding coupled lower and upper solutions of Type I. However, the existence of coupled lower and upper solutions of the form (12) does not guarantee the existence of minimal and maximal solutions of (9). For this reason, we need to make an additional assumption on v^1 and w^1 in order to construct intertwined sequences that converge to minimal and maximal solutions of (9) uniformly and monotonically . The proof follows by arguments similar to the ones used in Theorem 2, and we state the result without proof.

Theorem 3. *Assume that*

(B1) v^0 and w^0 are coupled lower and upper solutions of type II for (9) with $v^0(t) \leq w^0(t)$ in J_0; and

(B2) $f, g \in C\left(J \times [v^0(t), w^0(t)] \times [v_t^0(s), w_t^0(s)], \mathbb{R}\right)$, where $f(t,u(t),u_t(s))$ is increasing in u and u_t and $g(t,u(t),u_t(s))$ is decreasing in u and u_t.

Construct two sequences as follows,

$$\begin{aligned} {}^cD^q v^{n+1}(t) &= f\left(t, w^n(t), w_t^n(s)\right) + g\left(t, v^n(t), v_t^n(s)\right), \\ v_{t_0}^{n+1}(s) &= \phi(s), \end{aligned} \tag{16}$$

and

$$\begin{aligned} {}^cD^q w^{n+1}(t) &= f\left(t, v^n(t), v_t^n(s)\right) + g\left(t, w^n(t), w_t^n(s)\right), \\ w_{t_0}^{n+1}(s) &= \phi(s), \end{aligned} \tag{17}$$

If there exists a solution $u(t)$ of (9) for which $v^0(t) \leq w^1(t) \leq u(t) \leq v^1(t) \leq w^0(t)$, it follows from (16) and (17) that there exist intertwined sequences such that

$$v^0 \leq w^1 \leq v^2 \leq \cdots \leq v^{2n} \leq w^{2n+1} \leq u \leq v^{2n+1} \leq w^{2n} \leq \cdots \leq w^2 \leq v^1 \leq w^0,$$

where $\left\{v^{2n}(t), w^{2n+1}(t)\right\}$ converges to $\rho(t)$ and $\left\{w^{2n}(t), v^{2n+1}(t)\right\}$ converges to $r(t)$, both uniformly and monotonically in $C[J,\mathbb{R}]$, and ρ, r are such that

$$\begin{aligned} {}^cD^q \rho(t) &= f\left(t, \rho(t), \rho_t(s)\right) + g\left(t, r(t), r_t(s)\right), \\ \rho_{t_0}(s) &= \phi(s), \end{aligned}$$

and

$$\begin{aligned} {}^cD^q r(t) &= f\left(t, r(t), r_t(s)\right) + g\left(t, \rho(t), \rho_t(s)\right), \\ r_{t_0}(s) &= \phi(s), \end{aligned}$$

that is, ρ and r are minimal and maximal solutions of (9), with $u \in [\rho, r]$.

Remark. We can establish conditions for the uniqueness of the solution of (9). Besides hypotheses (A1)-(A2) in Theorem 2, hypotheses (B1)-(B2) in Theorem 3, assume that f and g satisfy one–sided Lipschitz conditions as follows,

$$\begin{aligned} f(t,x,\varphi_1(s)) - f(t,y,\varphi_2(s)) &\leq M_1(x-y) \\ &\quad + N_1\left(\sup_{s\in[-\tau,0]}\{\varphi_1(s)-\varphi_2(s)\}\right) \\ g(t,x,\varphi_1(s)) - g(t,y,\varphi_2(s)) &\geq -M_2(x-y) \\ &\quad - N_2\left(\sup_{s\in[-\tau,0]}\{\varphi_1(s)-\varphi_2(s)\}\right) \end{aligned} \tag{18}$$
,

where M_1, M_2 are positive constants, N_1, N_2 are non negative constants, and $x \geq y$ and $\varphi_1(s) \geq \varphi_2(s)$. Then $\rho = u = r$ in J; that is, the solution of (9) is unique.

We found that $\rho \leq r$ and we need to prove the inequality $r \leq \rho$. If $p(t) = r(t) - \rho(t)$, then $p_{t_0}(s) = r_{t_0}(s) - \rho_{t_0}(s) = \phi(s) - \phi(s) = 0$. Given $\rho \leq r$ and $\rho_t \leq r_t$, we have from Theorems 2 and 3, and (18) that

$$\begin{aligned}
{}^cD^q p(t) &= {}^cD^q r(t) - {}^cD^q \rho(t) \\
&= f(t, r(t), r_t(s)) + g(t, \rho(t), \rho_t(s)) - f(t, \rho(t), \rho_t(s)) - g(t, r(t), r_t(s)) \\
&\leq M_1(r(t) - \rho(t)) + N_1 \left(\sup_{s \in [-\tau, 0]} \{r_t(s) - \rho_t(s)\} \right) + M_2(r(t) - \rho(t)) + N_2 \left(\sup_{s \in [-\tau, 0]} \{r_t(s) - \rho_t(s)\} \right) \\
&= (M_1 + M_2)(r(t) - \rho(t)) + (N_1 + N_2) \left(\sup_{s \in [-\tau, 0]} \{r_t(s) - \rho_t(s)\} \right) \\
&= (M_1 + M_2) p(t) + (N_1 + N_2) \left(\sup_{s \in [-\tau, 0]} \{p_t(s)\} \right).
\end{aligned}$$

From Corollary 1, we obtain that $p(t) \leq 0$ and, as a consequence, $r(t) \leq \rho(t)$. Hence, $\rho(t) = u(t) = r(t)$, and the $\rho, r \to u$.

4 Numerical result

We finish the paper with an example illustrating Theorem 3.

Example 1. We consider a fractional differential equation with bounded delay, order $q = \frac{1}{2}$, $J = [0, 1]$ and $J_0 = [-1, 1]$,

$$\begin{aligned}
{}^cD^{1/2} u(t) &= \tfrac{1}{10} u(t) + \tfrac{1}{6} u^2(t-1) - \tfrac{1}{11} u^2(t) - \tfrac{1}{4} u(t-1), \\
u(s) &= 1, \quad s \in [-1, 0].
\end{aligned} \tag{19}$$

Observe that

$$f(t, u(t), u_t(1)) = \frac{1}{10} u(t) + \frac{1}{6} u^2(t-1)$$

is increasing in u and u_t, and

$$g(t, u(t), u_t(1)) = -\frac{1}{11} u^2(t) - \frac{1}{4} u(t-1)$$

is decreasing in u and u_t, for $t \in J$. Since $v^0(s) = 0 \leq 1 = u(s)$ and $w^0(s) = 2 \geq 1 = u(s)$ for s in $[-1, 0]$, $v^0 \equiv 0, w^0 \equiv 2$ satisfy (12), so they are lower and upper solutions of Type II for (9).

Observe also that

$$\begin{aligned}
0 = {}^cD^{1/2} v^0(t) &\leq f\left(t, w^0(t), w_t^0(1)\right) + g\left(t, v^0(t), v_t^0(1)\right) \\
&= \tfrac{1}{10}(2)^2 + \tfrac{1}{6}(2)^2 - \tfrac{1}{11}(0)^2 - \tfrac{1}{4}(0) = \tfrac{13}{15},
\end{aligned}$$

and

$$\begin{aligned}
0 = {}^cD^{1/2} w^0(t) &\geq f\left(t, v^0(t), v_t^0(1)\right) + g\left(t, w^0(t), w_t^0(1)\right) \\
&= \tfrac{1}{10}(0)^2 + \tfrac{1}{6}(0)^2 - \tfrac{1}{11}(2)^2 - \tfrac{1}{4}(2) = -\tfrac{19}{22}.
\end{aligned}$$

After constructing the sequences based on Theorem 3, in Figure 1 we show seven terms of $\{v^n\}$ and seven terms of $\{w^n\}$ on $[-1, 1]$.

Finally, we show a table with approximate values of the last two iterates on $[0, 1]$

We have used Sympy to integrate the functions, compute the iterates, plot the graphs and calculate the values on Table 1.

5 Conclusion

We developed a monotone iterative technique for a Caputo differential equation of order $q \in (0, 1)$ with bounded delay. Moreover, in (14)-(15) and (16)-(17) the forcing functions of ${}^cD^q v_{n+1}$ and ${}^cD^q w_{n+1}$ depend on v_n and w_n only, and by using the equivalent Volterra integral equation (6) we do not need to find a formula for the explicit solution of a linear differential equation where ${}^cD^q v_{n+1}$ and ${}^cD^q w_{n+1}$ depend on v_{n+1} and w_{n+1}, respectively, as it has been done in the past with monotone iterative techniques. In our example it took seven iterates to make it look clearer that the sequences were converging to a unique solution. In the near future we plan to develop methods for accelerating the rate of convergence.

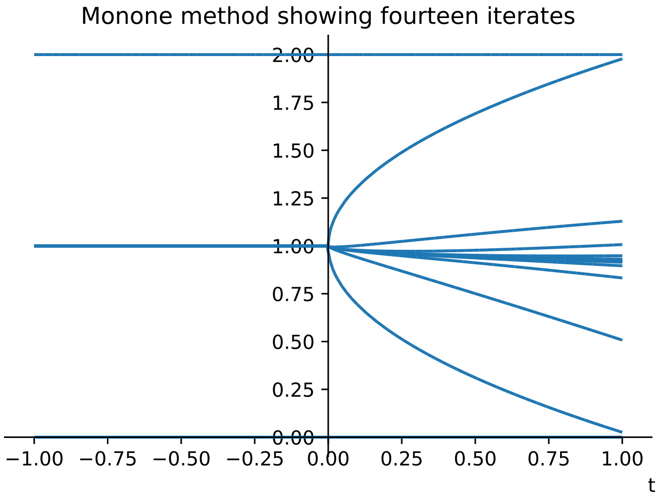

Fig. 1: The graph shows v^n and w^n, $0 \leq n \leq 7$.

Table 1: Approximate values of $t \in [0,1]$ for $v^7(t)$ and $w^7(t)$ in equation (19).

t	$v_7(t)$	$w_7(t)$
0.0	1.000000	1.000000
0.1	0.974239	0.974239
0.2	0.963967	0.963960
0.3	0.956234	0.956209
0.4	0.949819	0.949751
0.5	0.944254	0.944109
0.6	0.939299	0.939032
0.7	0.934819	0.934371
0.8	0.930724	0.930026
0.9	0.926955	0.925927
1.0	0.923471	0.922021

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