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Isomorphism of Integer Array of Staircase Paths

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Abstract: Out of several combinatorial objects like alternating sign matrices; totally symmetric self-complementary plane partitions (TSSCPPs), Catalan objects, tournaments, semi standard young tableaux, and totally symmetric plane partitions in order to put ASMs in larger context, they are viewed as poset. In this paper we present two and three dimensional structure of pentagonal poset and the sub-posets such that their ordered ideals are ordered by inclusion order relation. Moreover isomorphism of integer array of staircase paths and distributive lattice generated by poset of order ideals will also be shown.

Keywords: Alternating sign matrix, admissible subset, integer arrays.

1 Introduction

We begin with the following definitions:

1.1 Alternating sign matrices

Alternating sign matrices (ASMs) are square matrices with the following properties: entries ∈ {0, 1,−1} the entries in each row and column sum to one non-zero entries in each row and column alternate in sign. Mills, Robbins, and Rumsey thought that the total number of n × n alternating sign matrices can be given by the expression [14]

$$\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

And after long time Doron Zeilberger proved it [3]. Later by using a bijection between ASMs and configurations of the statistical physics model of square ice with domain wall boundary conditions [4] Kuperberg proved it by using shorter method. Detailed exposition of the conjecture and proof of the enumeration of ASMs can be seen in [2].By Definition of alternating sign matrices; we see that permutation matrices are none other than ASMs whose entries are non-negative. Thus, permutation matrices and ASMs are strongly connected. There exists a partial ordering on alternating sign matrices that is a distributive lattice. This close relationship between permutations and ASMs gives idea for ASMn analogous theorems. Our object is to investigate some combinatorial problems associated with distributive lattice. Thus, this paper can be regarded as a survey of some combinatorial aspects of distributive lattices. We examined a poset structure on ASMs, which turned out to be a distributive lattice with poset of join irreducible very similar to that of the ASM lattice. We have taken idea to construct pentagonal poset from J. Striker [5].

1.2 Partially Ordered Set

A partially ordered set (poset) is an ordered pair (P,≤) consisting of a set P and relation ≤ on P satisfying the following three properties:

1) $\forall\, x \in P, x \leq x\ (reflexivity)$

2) $\forall\, x, y \in P, if\ x \leq y\ and\ y \leq x\ then\ x = y\quad (anti-symmetry)$

3) $\forall\, x, y, z \in P, x \leq y\ and\ y \leq z, then\ x \leq z\ (transitivity)$

1.3 Sub posets

We say Q is a sub poset of P if the elements of Q are a subset of the elements of P and the partial ordering on Q is such that if $x \leq y$ in Q then $x \leq y$ in P.

1.4 Order ideal

A set I is an order ideal of a poset P if I ⊆ P and for all x ∈ I and y ∈ P, if $x \leq y$, then y ∈ I. The collection of all order ideals of P is denoted by *J (P)*, ordered by inclusion order is lattice.

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1.5 Comparable elements

Given a poset, P and two element $x, y \in P$ *we say* x *is comparable to* y *if* $x \le y$ *or* $y \le x$ otherwise we say that x and y are incomparable.

1.6 Cover of elements

An element y covers an element x in a poset, P if $x \le y$ but $x \ne y$ and there is no other element $z \in P$ where $x < z < y$.

1.7 Maximum chain

Let P be a poset. A chain, C ∈ P is a maximum chain if no other chainof P has greater length. A chain, C ∈ P is a maximal chain if it cannot be extended, i.e.the addition of any other element to the chain would result in it containing at least one pair of incomparable elements.

1.8 Ranked poset

A finite poset, P, is ranked (or graded) if all of its maximal chains have the same length, n. Every ranked poset has a unique rank function $\rho: P \rightarrow \{1,2,3, \ldots \ldots \ldots . . n\}$, Where $\rho(x) = 0$ if x is a minimal element in P that is, there is no z ∈ P such that x >z. If x is not a minimal element in P, and y covers x in P, then the rank function, $\rho(y) = \rho(x) + 1$. An element x∈ P has rank i if ρ(x) = i.

2 Survey of Literature

A monotone triangle of order n is a triangular arrays of integers with i integers in row i for all $1 \le i \le n$, bottom row 1, 2, 3…..n and integer entries for $a_{i,j}$ for $1 \le i \le n$, $n - i \le i \le n - 1$ Such that $a_{i,j-1} \le a_{i-1,j} \le a_{i,j}$ and $a_{i,j} \le a_{i,j+1}$.As in the introduction we already discussed that, alternating sign matrices (ASMs) have been bothering combinatorialists for long time due to lack of an explicit bijection between any two combinatorial objects hence, working upon this author found alternating sign matrices are known to be in bijection with monotone triangles. This monotone triangle of order n is in bijection with $n \times n$ alternating sign matrices [5].

The natural partial order on all ASMs is the distributive lattice of monotone triangles. In fact, as a sub poset the ASM lattice is the smallest lattice to contain the Bruhat order on the permutations. This partial will provide insight on the combinatorics of these objects and the associated outstanding bijection problems [10].

The Birkhoff's fundamental theorem of finite distributive lattices says that given a finite distributive lattice L there exists a unique finite poset P for which L = J(P) [5]. Since alternating sign matrix appear as sub poset of tetrahedron poset with certain edge colors hence, emphasizing the properties of order ideal a nice product formula for finding number of order ideal and bijection between these order ideal an interesting set of combinatorial object has been presented. Later by same author only those subsets of the colors which include all colors whose covering relation are induced by combination of the other colors have been consider which is called admissible subset [12,13].

A Dyck path is a lattice path in the $n \times n$ square consisting of only north and east steps and such that the path doesn't pass below the main diagonal in the grid. It starts at (0,0) and ends at(n, n). A walk of length n along a Dyck path consists of 2n steps, with n in the north direction and n in the east direction. By necessity the first step must be north and the final step must be east. There is a clear bijection between these lattice paths and the balanced strings of parentheses which make up by the Dyck language. The correspondence can easily be seen by considering each north step as a left parenthesis and each east step as a right parenthesis. A partial order can be define on the set of Dyck paths by considering one path to be `less than' another if it lies below the other .This Dyck paths with the each edge colored as weight are found in bijection with ordered ideal posets of colors. Number of order ideals of poset is equal to 2^{n-1} and rank generation function of order ideal of poset is $\prod_{j=1}^{n-1}(1 + q^j)$ [1, 8 and6].

In this field considerable research work has been found [7]. Here structure of pentagonal poset has been presented; idea to construct pentagonal poset has been taken from J. Striker [5]. Cover preserving characterization for lattices of ordered ideal and set of integer array also be present for which sub-posets of this poset having certain ordering relations will be used. Moreover, three-dimensional structure of poset will also be shown. Since this is first result for pentagonal poset hence from this structure lots of new scopes in the field of combinatorial objects are going to be open.

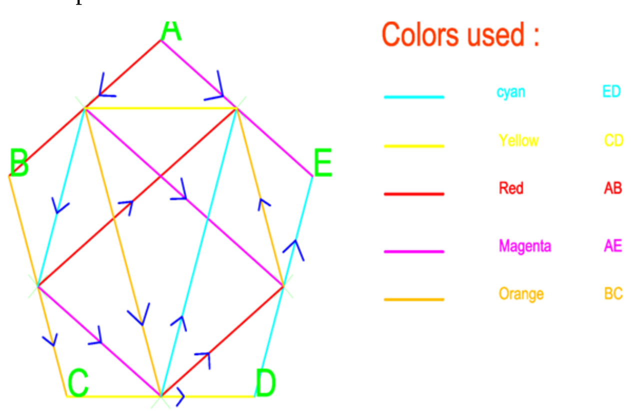

Fig 1: Hass Diagram of P_8

Theorem [11]: Set of order ideals forms distributive lattice under inclusion order.

Theorem [14]: If P is a finite poset, then there are exactly 2|P| sub posets of P.

Theorem [5]: The number of order ideals of P_n is2^{n-1}. The rank generating function of $J(P_n)$ is-

$\prod_{j=1}^{n-1}(1+q^{j})$.

Theorem [5]: If S be an admissible subset of {a,b,c,d,e,f}, $X_n(S)$ is set of integer array x of staircase shape and $J(T_n(S))$ is set of order ideal then there is a cover preserving bijection between them where$I \in J(T_n(S))$ is covers n elements of same rank, and $x \in X_n(S)$ equals the sum of entries of x.

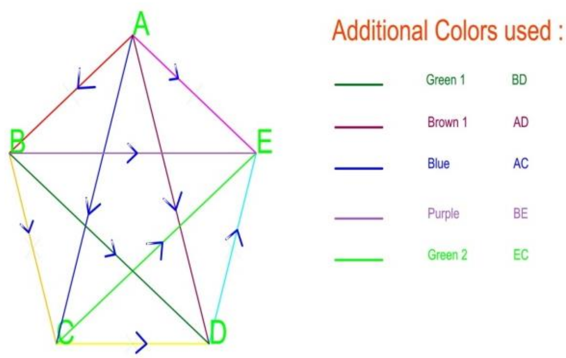

Fig 2: Hass Diagram of P_8(2D view)

3 Construction of Pentagonal Poset

[5] As there are number of bijections between order ideals of sub posets of tetrahedron poset and well–known combinatorial objects such as ASMs, TSSCPPs, Catalan objects, and tournaments. In [5]P_4 is taken to construct T_4 ,So in this direction we have taken pentagonal (T_5) poset.For this we will have to constructP_8.For constructing a pentagonal poset T_5 first we have to construct a poset P_8 using some unit vectors in$\mathbb{R}^3$.In fig.1 we have assigned certain colours to each edge$\overline{AB}$=red= $\bar{r}$,$\overline{BC}$ =green-2= $\overline{g_2}$,$\overline{CD}$ =yellow= $\bar{y}$,$\overline{DE}$ =cyan=$\bar{C}$,$\overline{EA}$=magenta=$\bar{M}$.Unit vector corresponding to each edges will be as:$\bar{r}$=(0.29,0.96,0),$\overline{g_2}$=(0.33,0.96,0),$\bar{y}$=(0.28,0.96,0),$\bar{C}$=(0.31,0.95,0),$\bar{M}$=(0.34,0.94,0).Element of P_8 define as coordinates of all points reached by linear combination of red and magenta colors as-$P_8=\{a\bar{r}+b\bar{M}, a,b \in \mathbb{Z} \geq a+b \leq n-2\}$.To obtain partial order between $\bar{r}$ and $\bar{M}$ in [4] only one additional vector wear taken but in this paper hare three additional vectors $\overline{g_2}, \bar{y}, \bar{C}$ are taken. So all the elements will be in linear combination of following colors:

$$P_8=\{c_1\bar{M}+c_2\bar{r}, c_3\bar{r}+c_4\bar{C}, c_5\bar{M}+c_6\overline{g_2}, c_7\bar{y}+c_8\bar{r}, c_9\bar{C}+c_{10}\overline{g_2}, c_{11}\bar{y}+c_{12}\bar{C},; \text{wherw } c_1, c_2 \dots\dots\dots. c_{12} \in \mathbb{Z} \geq 0, c_1+c_2+\cdots\dots\dots+c_{12} \leq n-2\}$$

By this construction anyone can check Hasse diagram of P_8 has ${}^{n}C_2$ vertices and $3\ {}^{n-1}C_2$ edges.

Now we construct the pentagonal poset T_5 made up of the layers of P_8.Define unit vector

$$\overline{b_1}=(0.473,0.331,-\frac{\sqrt{6}}{3})$$

$$\overline{b_2}=\left(0.178,0.549,\frac{\sqrt{6}}{3}\right)$$

$$\overline{g_3}=(0.195,0.638,-\frac{1}{\sqrt{3}})$$

$$\overline{g_1}=(0.202,0.147,-\frac{\sqrt{3}}{2})$$

Elements of T_5 define as liner combination of $T_5=\{a\bar{r}+b\bar{M}+c\overline{b_1}+d\overline{b_2}, a,b,c,d \in \mathbb{Z} \geq a+b+c+d \leq n-2\}$.To obtain partial order onT_5we consider all edges in Hasse diagram of T_5 directed and three additional colors green-1, green-3 and brown-2 drawn where ever possible. As per partial order defined on Tn corner vertex with edge color red, magenta, blue-1, blue-2 is the smallest element and with edge color megenta,brown-2,green-3 and cyan are largest element and green-1,cyan,blue-1 are above with yello,red,brown-2 and green-3.

Admissible Subset [5]

A subset S of the colors {red, blue-1, blue-2, green-1, green-2, green-3, cyan, yellow, brown-2, magenta} initials $\{r,b_1,b_2,g_1,g_2,g_3,C,Y,B_2,M\}$ be called admissible if all the following inequality conditions hold:

Condition of Admissible Subset

1) If $\{r,g_2\} \subseteq S$ then $B_2 \in S$
2) If $\{b_2 Y\} \subseteq S$ then$b_1 \in S$
3) If $\{b_1,C\} \subseteq S$ then $M \in S$
4) If $\{g_1,C\} \subseteq S$ then $B_2 \in S$
5) If $\{g_2,Y\} \subseteq S$ then $g_1 \in S$
6) If $\{Y,C\} \subseteq S$ then $g_3 \in S$
7) If $\{b_1, C\} \subseteq S$ then $M \in S$
8) If $\{r, b_2\} \subseteq S$ then $M \in S$
9) If $\{g_3,b_2\} \subseteq S$ then $M \in S$

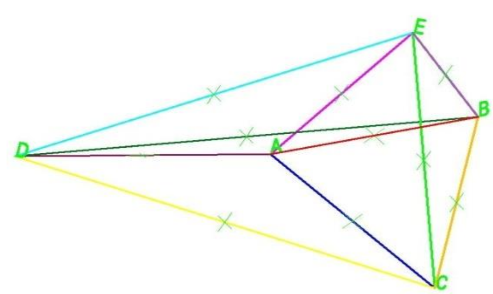

Fig 3: Hass Diagram of P_8(3D view)

Set of all integer arrays [5]

Let S be an admissible subset of $\{r,b_1,b_2,g_1,g_2,g_3,C,Y,B_2,M\}$ and suppose $M \in S$. Then$\boldsymbol{X_n(S)}$ be a set of all integer arrays $\boldsymbol{x}$ of staircase shape $\delta_n = (n-1)(n-2) \ldots \ldots \ldots 3\ 2\ 1$ whose entries $x_{i,j}$ satisfy both $0 \le x_{i,j} \le j$ and the followinginequality conditions corresponding to the additional colors in S:

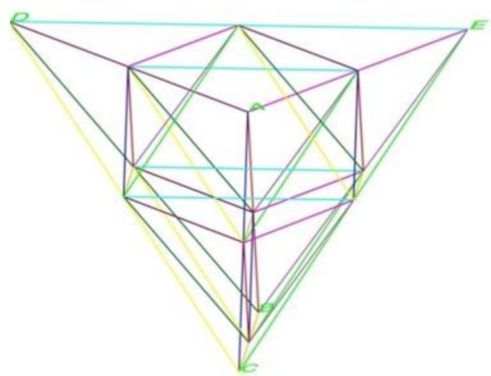

Fig 4: Hass Diagram of T_5

1) Red: $\boldsymbol{x_{i,j} \le x_{i-1,j+1}}$

2) Blue-2: $\boldsymbol{x_{i,j} \le x_{i,j+1}}$

3) Blue-1: $\boldsymbol{x_{i,j} \le x_{i+1,j+1}}$

4) Brown-2: $\boldsymbol{x_{i,j} \le x_{i+1,j-1} + 1}$

5) Green-3: $\boldsymbol{x_{i,j} \le x_{i,j-1} + 1}$

6) Cyan: $\boldsymbol{x_{i,j} \le x_{i-1,j-1} + 1}$

7) Green-2: $\boldsymbol{x_{i,j} \le x_{i+1,j}}$

8) Green-1: $\boldsymbol{x_{i,j} \le x_{i,j-1}}$

9) Yellow: $\boldsymbol{x_{i,j} \le x_{i,-1\ j-1}}$

Theorem: The poset of integer array of staircase paths ordered by inclusion is isomorphic to the distributive lattice generated by poset of order ideals ordered by inclusion of the poset of points (a, b) where $a + b \le n - 2$ and $(a, b) \ge 0$ with iff $a \ge c \ge$ and $b \ge d$.

Proof: Let I_S be the poset of integer array for a $n \times n$ net and P_n be the poset of points (a, b) where $a + b \le n - 2$ with $(a, b) \le (c, d)$ iff $a \ge c \ge$ and $b \ge d$.Let J(P) be distributive lattice of order ideals of P_n.Noted that element of P_n form a staircase path, and There will be nC2 vertices and n-1C2 edges in Hass diagram of P_n .

Define $\emptyset: J(P) \to I_S$ as follows;

$\forall\, I \in J(P)\ \ I_k = \{ (a_1, b_1)(a_2, b_2), (a_3, b_3) \ldots \ldots (a_k, b_k) \}$

Since elements of J (P) are ordered by inclusion then for all (a_i, b_i) there will be (a_j, b_j) such that $(a_j, b_j) \le (a_i, b_i)$.

To show isomorphism of $\emptyset$ it is sufficient to show.

(a) $\emptyset$ is one to one.

(b) $\emptyset$ is onto.

(c) $\emptyset$ preserves $\le$.

Since J (P) is distributive lattice hence each elements in it has a least upper bound and a greatest lower bound satisfying the distributive laws and each order ideal will be associated by a unique integer array through staircase paths [9].

Suppose I_1 and I_2 be two different order ideals that maps to the same staircase paths-

$I_1 = (a_1, b_1)(a_2, b_2), (a_3, b_3) \ldots \ldots (a_k, b_k)$

$I_2 = (c_1, d_1)(c_2, d_2), (c_3, d_3) \ldots \ldots (c_l, d_l)$

Since the order ideals are different, hence either $I_1 - I_2$ is non-empty or $I_2 - I_1$ is non-empty.

Let $I_1 - I_2$ is non-empty this implies that there must be at least one element (a_i, b_i) belongs to I_1 which is not in I_2 . Therefore we have contradicted the assumption that we can have two different order ideals leading to the same staircase path. Which shoes one to one correspondence between J (P) and I_S.

(ii) Let I_S be an arbitrary staircase path then corresponding to each path there will be unique set of lattice point $(a_1, b_1)(a_2, b_2), (a_3, b_3) \ldots \ldots (a_k, b_k)$ such that for each(a, b)

$(a + b) \le (n - 2)$ and a, b≥ 0 with$(a, b) \le (c, d)$ with $a \ge c$ and $b \ge d$.Implies that in J (P) there is unique order ideal for each I_S, this in necessary condition for onto mapping.

(iii) Suppose that we have two order ideals $I_1 = (a_1, b_1)(a_2, b_2), (a_3, b_3) \ldots \ldots (a_k, b_k)$ and $I_2 = (c_1, d_1)(c_2, d_2), (c_3, d_3) \ldots \ldots (c_l, d_l)$ such that $I_1 \le I_2$ This means that the elements of I_1 are a subset of the elements of I_2 , i.e. $I_1 = (a_1, b_1)(a_2, b_2), (a_3, b_3) \ldots \ldots (a_k, b_k) \in I_2$. Then corresponds to I_1 staircase path lies above the cells$(c_1, d_1)(c_2, d_2), (c_3, d_3) \ldots \ldots (c_l, d_l)$.

But since $(a_1, b_1)(a_2, b_2), (a_3, b_3) \ldots \ldots (a_k, b_k) \in I_2$, then the staircase path corresponding to I_1 must lie on or below the staircase path corresponding to I_1 and therefore the $\le$ relation is preserved.

Hence together a,b,c shows isomorphism of $\emptyset$.

Corollary.1 If S be an admissible subset of {a,b,c,d,e,f}, $X_n(S)$ is set of integer array x of staircase shape and $J(T_n(S))$ is set of order ideal then there is a cover preserving bijection between them where$I \in J(T_n(S))$ is covers n elements of same rank, and $x \in X_n(S)$ equals the sum of entries of x.

Proof. Assume number of elements in $J(T_n(S))$ is $s > 0$.$J(T_n(S))$ is a distributive lattice andM is the set of maximal elements of$T_n(S)$, then order ideal$I = T_n(S) - M$. Since $s > 0$,so Iis non empty i.e.$I \ne \emptyset$ let for $x \in I$ there is some $x_i \in M$satisfying $x_1 > x$.Let $x_2, x_3, x_4 \ldots \ldots x_r$ be remaining elements of M. Now define $I_k = M \cup \{x_1, x_2, x_3, x_4 \dots \dots x_k\}$ then each I_k is an order ideal of poset $T_n(S)$and then number of maximal elements of I_k is at most one more then the number of maximal elements of I_{k-1}, since I_i has $< s$ maximal elements and I_r has r maximal elements then some I_k has s maximal elements. This$I_k$is desired I'.By this process we get a sequence of order ideal ,having cover preserving relation ,since each order ideal is an element of $J(T_n(S))$and $J(T_n(S))$ is finite distributive lattice isomorphic with ordinal numbers (chain of integer) That is an integer array x of staircase shape. This is sufficient to shown a covering bijection.

4 Conclusion

In this paper property of (ASM) and their combinatorial problems associated with ***Distributive Lattice*** are investigated. For this a new perspectives of ASM that is poset has been used. Structure of pentagonal poset and the sub-posets consisting elements of this poset having certain ordering relations are presented here. Moreover, by using ordered ideal of these sub-posets, a cover preserving characterization of lattices of ordered ideal and set of integer array and isomorphism between them is also proved. Three dimensional structures of poset and its construction method have also been demonstrated in this paper. Since this is first result for pentagonal poset from this structure a lot of new scopes in the field of combinatorial objects are going to be open.

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