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A Note on Generalized k-Pell Numbers and Their Determinantal Representation

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Abstract: In this paper, we investigate permanents of an $n\times n$ $(0,1,2)$-matrix by contraction method. We show that the permanent of the matrix is equal to the generalized k-Pell numbers.

Keywords: Pell sequence, Permanent, Contraction of a matrix

1 Introduction

The well-known Pell sequence $\{P_n\}$ is defined by the recurrence relation, for $n>2$

$$P_n = 2P_{n-1} + P_{n-2}$$

where $P_1 = 1$ and $P_2 = 2$.

In [1], the authors defined k sequences of the generalized order-k Pell numbers $\left\{P_n^k\right\}$ as shown:

$$P_n^i = 2P_{n-1}^i + P_{n-2}^i + \ldots + P_{n-k}^i \tag{1}$$

for $n>0$ and $1\le i\le k$, with initial conditions

$$P_n^i = \begin{cases} 1 \text{ if } n = 1-i, \\ 0 \text{ otherwise,} \end{cases} \text{ for } 1-k \le n \le 0,$$

where P_n^i is the nth term of the ith sequence. The sequence $\left\{P_n^k\right\}$ is reduced to the usual Pell sequence $\{P_n\}$ for $k=2$.

For example, for $i=4$, they obtained $P_{-3}^4 = 1$, $P_{-2}^4 = P_{-1}^4 = P_0^4 = 0$ and gave first few terms of generalized order 4-Pell sequence as:

$$1, 2, 5, 13, 34, 88, 228, \ldots.$$

The authors called P_n^k the generalized k-Pell number for $i=k$ in (1) [1].

The permanent of an n-square matrix $A = [a_{ij}]$ is defined by

$$perA = \sum_{\sigma \varepsilon S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where the summation extends over all permutations σ of the symmetric group S_n.

Let $A=[a_{ij}]$ be an $m\times n$ real matrix with row vectors $\alpha_1, \alpha_2, \ldots, \alpha_m$. We say A is contractible on column (resp. row) k if column (resp. row) k contains exactly two nonzero entries. Suppose A is contractible on column k with $a_{ik} \ne 0 \ne a_{jk}$ and $i\ne j$. Then the $(m-1)\times(n-1)$ matrix $A_{ij:k}$ obtained from A by replacing row i with $a_{jk}\alpha_i + a_{ik}\alpha_j$ and deleting row j and column k is called the contraction of A on column k relative to rows i and j. If A is contractible on row k with $a_{ki}\ne 0\ne a_{kj}$ and $i\ne j$, then the matrix $A_{k:ij} = \left[A_{ij:k}^T\right]^T$ is called the contraction of A on row k relative to columns i and j. We say that A can be contracted to a matrix B if either $B=A$ or there exist matrices $A_0, A_1, \ldots, A_t$ $(t\ge 1)$ such that $A_0 = A$, $A_t = B$, and A_r is a contraction of A_{r-1} for $r = 1, \ldots, t$.

In [2], Lee defined the following matrix

$$\mathscr{L}^{(n,k)} = \begin{bmatrix} 1 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 0 \\ 1 & 1 & 1 & \ldots & 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 1 & 1 & \ldots & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 1 & 1 \end{bmatrix}$$

and showed that $per\mathscr{L}^{(n,2)} = L_{n-1}$ and also $per\mathscr{L}^{(n,k)} = l_{n-1}^{(k)}$, where L_n and $l_n^{(k)}$ are respectively the nth Lucas number and nth k-Lucas number.

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In [3], Minc defined generalized Fibonacci numbers of order r as

$$f(n,r)=\begin{cases} 0, & \text{if } n<0,\\ 1, & \text{if } n=0,\\ \sum\limits_{k=1}^{r} f(n-k,r), & \text{if } n>0. \end{cases}$$

Minc also defined the $n\times n$ super-diagonal $(0,1)$-matrix $F(n,r)=[f_{ij}]$ as in the following:

$$F(n,r)=\begin{bmatrix} 1 & \cdots & \cdots & 1 & 0 & \cdots & 0\\ 1 & & & & \ddots & \ddots & \vdots\\ 0 & \ddots & & & & \ddots & 0\\ \vdots & \ddots & \ddots & & & & 1\\ \vdots & & \ddots & \ddots & & & \vdots\\ \vdots & & & \ddots & \ddots & & \vdots\\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{bmatrix} \qquad (2)$$

where

$$f_{ij}=\begin{cases} 1, \text{ if } i=j+1 \text{ and } 1\le j\le r-1,\\ 1, \text{ if } 1\le i\le n-r-2 \text{ and } i\le j\le i+r-2,\\ 1, \text{ if } n-r+3\le i\le n \text{ and } i\le j\le n,\\ 0, \text{ otherwise.} \end{cases}$$

Then he proved that

$$perF(n,r)=f(n,r-1).$$

In [4], the authors denoted the matrix $F(n,r)$ in (2) as $\mathscr{F}^{(n,k)}$ and obtained permanent of this matrix, the same result in [3, Theorem 2], by applying contraction to the matrix $\mathscr{F}^{(n,k)}$.

In [5], Kilic defined the $n\times n$ super-diagonal $(0,1,2)$-matrix $S(k,n)$ as:

$$S(k,n)=\begin{bmatrix} 2 & 1 & \cdots & 1 & 0 & \cdots & 0\\ 1 & 2 & 1 & \ldots & 1 & \ddots & \vdots\\ 0 & 1 & 2 & 1 & \ldots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & & 1\\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots\\ \vdots & & & \ddots & \ddots & \ddots & 1\\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 2 \end{bmatrix} \qquad (3)$$

and proved that the permanent of $S(k,n)$ equals to the $(n+1)$th generalized k-Pell number.

In [6], Yilmaz and Bozkurt defined the $n\times n$ upper Hessenberg matrix $H_n=[h_{ij}]$ and $n\times n$ matrix $K_n=[k_{ij}]$ as follows:

$$H_n=\begin{bmatrix} 1 & 1 & -1 & 0 & \cdots & \cdots & \cdots & 0\\ 1 & 1 & 1 & 1 & \ddots & & & \vdots\\ 0 & 1 & 1 & 1 & -1 & \ddots & & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\ \vdots & & \ddots & \ddots & \ddots & \ddots & 1 & 0\\ \vdots & & & \ddots & \ddots & \ddots & \ddots & -1\\ \vdots & & & & \ddots & \ddots & \ddots & 1\\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{bmatrix}$$

and

$$K_n=\begin{bmatrix} 1 & 2 & 3 & 0 & \cdots & \cdots & \cdots & 0\\ 1 & 0 & 0 & 0 & \ddots & & & \vdots\\ 0 & 1 & 0 & 1 & 1 & \ddots & & \vdots\\ \vdots & \ddots & 1 & 0 & 1 & 1 & \ddots & \vdots\\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & 0\\ \vdots & & & \ddots & \ddots & \ddots & \ddots & 1\\ \vdots & & & & \ddots & \ddots & \ddots & 1\\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

They showed that

$$perH_n=perH_n^{(n-2)}=P_n$$

and

$$perK_n=perK_n^{(n-2)}=R_n$$

where P_n is nth Pell number, R_n is nth Perrin number, and $H_n^{(n-2)}$ and $K_n^{(n-2)}$ is $(n-2)$th contraction of the matrix H_n and K_n, respectively.

In this paper, we show that the permanent of the $n\times n$ $(0,1,2)$-matrix $S(k,n)$ equals to the $(n+1)$th generalized k-Pell number by the contraction method.

2 Generalized k-Pell numbers by contraction method

In this section, we use generalized order-k Pell sequence $\{P_n^k\}$ given in (1) for $i=k$. For this sequence, it can be written for $n>k\ge 2$

$$P_{1-k}^k=1,\ P_{2-k}^k=P_{3-k}^k=\ldots=P_{-1}^k=P_0^k=0$$

and

$$P_n^k=2P_{n-1}^k+P_{n-2}^k+\ldots+P_{n-k}^k.$$

First few terms of the generalized k-Pell numbers are follows in [1] as:

$$\begin{aligned}
P_1^k &= 2P_0^k + P_{-1}^k + \dots + P_{1-k}^k = 1,\\
P_2^k &= 2P_1^k + P_0^k + \dots + P_{2-k}^k = 2\times 1 = 2,\\
P_3^k &= 2P_2^k + P_1^k + \dots + P_{3-k}^k = 2\times 2 + 1 = 5,\\
P_4^k &= 2P_3^k + P_2^k + \dots + P_{4-k}^k = 2\times 5 + 2 + 1 = 13,\\
P_5^k &= 2P_4^k + P_3^k + \dots + P_{5-k}^k = 2\times 13 + 5 + 2 + 1 = 34, \dots.
\end{aligned}$$

Then, some terms of this sequence can be given for some k and n values in the following table:

$k \diagdown n$	1	2	3	4	5	6	7	8	9	10
2	1	2	5	12	29	70	169	408	985	2378
3	1	2	5	13	33	84	214	545	1388	3535
4	1	2	5	13	34	88	228	591	1532	3971
5	1	2	5	13	34	89	232	605	1578	4116

It is easily seen that from the above table, $\{P_n^k\}$ is the usual Pell sequence $\{P_n\}$ for $k = 2$.

The following Lemma is well-known from [7].

Lemma 2.1. Let A be a nonnegative integral matrix of order n for $n > 1$ and let B be a contraction of A. Then

$$perA = perB. \tag{4}$$

Let $S(k,n)$ be the $(k+1)$st $(0,1,2)$-matrix of order n given in (3). For convenience, we use the notation $S^{(n,k)}$ instead of $S(k,n)$. If $\mathscr{F}^{(n,k)}$ be the matrix as in (2) and I_n be the identity matrix of order n, then it is immediately seen that $S^{(n,k)} = \mathscr{F}^{(n,k)} + I_n$. Then the matrix $S^{(n,k)}$ is contractible on column 1 relative to rows 1 and 2. In particular, if $k = 2$, the matrix $S^{(n,k)}$ reduced to the tridiagonal Toeplitz matrix $T^{(n)} = tridiag_n(1,2,1)$ of order n as follows:

$$T^{(n)} = \begin{bmatrix} 2 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 2 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 2 \end{bmatrix}. \tag{5}$$

Lemma 2.2. Let the matrix $T^{(n)}$ be as in (5). Then

$$perT^{(n)} = P_{n+1},$$

where P_{n+1} be the $(n+1)$st Pell number.

Proof. Let $T_r^{(n)} = \left[t_{ij}^{(r)}\right]$ be the rth contraction of the matrix $T^{(n)}$ given in (5). The matrix $T^{(n)}$ can be contracted on column 1 such that

$$T_1^{(n)} = \begin{bmatrix} 5 & 2 & 0 & \cdots & \cdots & 0 \\ 1 & 2 & 1 & \ddots & & \vdots \\ 0 & 1 & 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 2 \end{bmatrix},$$

where $t_{11}^{(1)} = 5 = P_3$ and $t_{12}^{(1)} = 2 = P_2$. Also the matrix $T_1^{(n)}$ can be contracted on column 1 such that

$$T_2^{(n)} = \begin{bmatrix} 12 & 5 & 0 & \cdots & \cdots & 0 \\ 1 & 2 & 1 & \ddots & & \vdots \\ 0 & 1 & 2 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 2 \end{bmatrix},$$

where $t_{11}^{(2)} = 12 = P_4$ and $t_{12}^{(2)} = 5 = P_3$. Continuing this process, we obtain the rth contraction of $T^{(n)}$ as follows

$$T_r^{(n)} = \begin{bmatrix} P_{r+2} & P_{r+1} & 0 & \cdots & \cdots & 0 \\ 1 & 2 & 1 & \ddots & & \vdots \\ 0 & 1 & 2 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 2 \end{bmatrix},$$

for $2 \le r \le n-2$. So, we have

$$T_{n-2}^{(n)} = \begin{bmatrix} P_n & P_{n-1} \\ 1 & 2 \end{bmatrix}. \tag{6}$$

By combining (4) and (6), we get

$$\begin{aligned}
perT^{(n)} &= perT_{n-2}^{(n)} \\
&= 2P_n + P_{n-1} \\
&= P_{n+1},
\end{aligned}$$

and the proof is completed.□

Lemma 2.3. Let $S_t^{(n,k)} = \left[s_{ij}^{(t)}\right]$ be the tth contraction of the matrix $S^{(n,k)}$ given in (3) and $1 \le t \le n-3$. Then

$$s_{1,j}^{(t)} = \begin{cases} P_{t+2}^k, & \text{if } j = 1, \\ \sum_{r=1}^{t+1} P_r^k, & \text{if } 2 \le j \le k-t, \\ s_{1,j-1}^{(t)} - P_{t+j-k}^k, & \text{if } k-t+1 \le j \le n-t, \end{cases} \quad \text{for } k > t+2$$

and

$$s_{1,j}^{(t)}=\begin{cases} P_{t+2}^{k}, & \text{if } j=1,\\ \sum\limits_{r=t+3-k}^{t+1} P_r^k, & \text{if } j=2,\\ s_{1,j-1}^{(t)}-P_{t+j-k}^{k}, & \text{if } 3\le j\le n-t, \end{cases} \quad \text{for } k\le t+2.$$

In any case, if $s_{1,j-1}^{(t)}-P_{t+j-k}^{k}<0$, we assume that $s_{ij}=0$.

Proof. We will prove the lemma by induction on t.

If we apply the contraction to the matrix $S^{(n,k)}$ according to column 1, we get

$$S_1^{(n,k)}=\begin{bmatrix} 5 & 3 & 3 & 3 & \dots & 3 & 2 & 0 & \dots & 0\\ 1 & 2 & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 0\\ 0 & \ddots & \ddots & \ddots & & & & & & \\ & & & & & & & \ddots & & \\ & 0 & & & & & & \ddots & \ddots & 1\\ & & & & & & & 0 & 1 & 2 \end{bmatrix}$$

$$=\begin{bmatrix} P_3^k & P_2^k+P_1^k & P_2^k+P_1^k & \dots & P_2^k+P_1^k & P_2^k & 0 & \dots & 0\\ 1 & 2 & 1 & \dots & 1 & 1 & 1 & \dots & 0\\ 0 & \ddots & \ddots & \ddots & & & & & \\ & & & & & & & \ddots & \\ & 0 & & & & & \ddots & \ddots & 1\\ & & & & & & 0 & 1 & 2 \end{bmatrix},$$

where $s_{11}^{(1)}=P_3^k$, $s_{12}^{(1)}=\ldots=s_{1,k-1}^{(1)}=P_2^k+P_1^k$, $s_{1,k}^{(1)}=P_2^k=(P_2^k+P_1^k)-P_1^k=s_{1,k-1}^{(1)}-P_1^k$. So, it is easy to show that the assumption is true for $t=1$.

We now assume the assumption is true for t. Then, there are two cases need to prove.

Case 1. We consider $k>t+2$. Since $s_{11}^{(t)}=P_{t+2}^k$, $s_{12}^{(t)}=\ldots=s_{1,k-t}^{(t)}=\sum_{r=1}^{t+1}P_r^k$, and $s_{1,j}^{(t)}=s_{1,j-1}^{(t)}-P_{t+j-k}^k$ for $k-t+1\le j\le n-t$, we clearly write the matrix $S_t^{(n,k)}=\left[s_{ij}^{(t)}\right]$ as:

$$\begin{bmatrix} P_{t+2}^k & \sum\limits_{r=1}^{t+1}P_r^k & \sum\limits_{r=1}^{t+1}P_r^k & \dots & \sum\limits_{r=1}^{t+1}P_r^k & \sum\limits_{r=2}^{t+1}P_r^k & \dots & P_{t+1}^k+P_t^k & P_{t+1}^k & 0 & \dots & 0\\ 1 & 2 & 1 & 1 & \dots & 1 & 1 & \dots & 1 & 1 & 0 & \\ 0 & \ddots & \ddots & \ddots & \ddots & & & & & & \ddots & \\ & \ddots & & & & & & & & & & 1\\ & & & & & & & & & & & \vdots\\ & & & & & & & & & & \ddots & 1\\ & 0 & & & & & & & \ddots & \ddots & \ddots & 1\\ & & & & & & & & & 0 & 1 & 2 \end{bmatrix}. \tag{7}$$

Now, we must prove our assumption for $t+1$. Since the matrix $S_t^{(n,k)}$ given by (7) is contractible on column 1 relative to rows 1 and 2, we write

$$\begin{aligned} s_{11}^{(t+1)} &= 2s_{11}^{(t)}+s_{12}^{(t)}\\ &= 2P_{t+2}^k+\sum_{r=1}^{t+1}P_r^k\\ &= 2P_{t+2}^k+P_{t+1}^k+\ldots+P_1^k. \end{aligned}$$

Also since $P_0^k=P_{-1}^k=\ldots=P_{t-k+3}^k=0$ for $k>t+2$, we write $P_0^k+P_{-1}^k+\ldots+P_{t-k+3}^k=0$. Consequently, we get

$$\begin{aligned} s_{11}^{(t+1)} &= 2P_{t+2}^k+P_{t+1}^k+\ldots+P_1^k+P_0^k+P_{-1}^k+\ldots+P_{t+3-k}^k\\ &= P_{t+3}^k, \end{aligned}$$

and

$$\begin{aligned} s_{1,q}^{(t+1)} &= s_{11}^{(t)}+s_{1,q+1}^{(t)},\ q=2,3,\ldots,k-t-1\\ &= P_{t+2}^k+\sum_{r=1}^{t+1}P_r^k\\ &= \sum_{r=1}^{t+2}P_r^k. \end{aligned}$$

Thus, $s_{1,q}^{(t+1)}=\sum\limits_{r=1}^{(t+1)+1}P_r^k$ for $q=2,3,\ldots,k-(t+1)$, finally

$$\begin{aligned} s_{1,k-t}^{(t+1)} &= s_{11}^{(t)}+s_{1,k-t+1}^{(t)}\\ &= s_{11}^{(t)}+s_{1,k-t}^{(t)}-P_{t+(k-t+1)-k}^k\\ &= s_{1,k-t-1}^{(t+1)}-P_{(t+1)+(k-t)-k}^k. \end{aligned}$$

So by the recurrence relation, we reach

$$s_{1,j}^{(t+1)}=s_{1,j-1}^{(t+1)}-P_{(t+1)+j-k}^k,$$

for $k-t\le j\le n-(t+1)$ and proof of Case 1 completes.

Case 2. In this case, we will consider $k\le t+2$. For $t=1$ and $k=2$, Lemma 2 is obtained. We assume that $s_{11}^{(t)}=P_{t+2}^k$, $s_{12}^{(t)}=\sum_{r=t+3-k}^{t+1}P_r^k$ and $s_{1,j}^{(t)}=s_{1,j-1}^{(t)}-P_{t+j-k}^k$ for $3\le j\le n-t$. Then, the matrix $S_t^{(n,k)}=\left[s_{ij}^{(t)}\right]$ follows as:

$$\begin{bmatrix} P_{t+2}^k & \sum\limits_{r=t+3-k}^{t+1}P_r^k & \sum\limits_{r=t+4-k}^{t+1}P_r^k & \sum\limits_{r=t+5-k}^{t+1}P_r^k & & \dots & P_{t+1}^k+P_t^k & P_{t+1}^k & 0 & \dots & 0\\ 1 & 2 & 1 & 1 & 1 & \dots & 1 & 1 & 1 & 0 & \\ 0 & \ddots & \ddots & \ddots & \ddots & & & & & \ddots & \\ & \ddots & & & & & & & & & 1\\ & & & & & & & & & & \vdots\\ & & & & & & & & & \ddots & 1\\ & 0 & & & & & & \ddots & \ddots & \ddots & 1\\ & & & & & & & & 0 & 1 & 2 \end{bmatrix}. \tag{8}$$

If we apply the contraction to the matrix $S_t^{(n,k)}$ given in (8) according to its first column, we get

$$\begin{aligned} s_{11}^{(t+1)} &= 2s_{11}^{(t)} + s_{12}^{(t)} \\ &= 2P_{t+2}^{(k)} + \sum_{r=t+3-k}^{t+1} P_r^{(k)} \\ &= 2P_{t+2}^{(k)} + P_{t+1}^{(k)} + \ldots + P_{t-k+3}^{(k)} \\ &= P_{t+3}^{(k)}, \end{aligned}$$

$$\begin{aligned} s_{12}^{(t+1)} &= s_{11}^{(t)} + s_{13}^{(t)} \\ &= P_{t+2}^{k} + s_{12}^{(t)} - P_{t+3-k}^{k} \\ &= P_{t+2}^{k} + \sum_{r=t+3-k}^{t+1} P_r^{k} - P_{t+3-k}^{k} \\ &= P_{t+2}^{k} + P_{t+1}^{k} + \ldots + P_{t+4-k}^{k} + P_{t+3-k}^{k} - P_{t+3-k}^{k} \\ &= \sum_{r=t+4-k}^{t+2} P_r^{k} \\ &= \sum_{r=(t+1)+3-k}^{(t+1)+1} P_r^{k} \end{aligned}$$

and

$$\begin{aligned} s_{13}^{(t+1)} &= s_{11}^{(t)} + s_{14}^{(t)} \\ &= s_{11}^{(t)} + s_{13}^{(t)} - P_{t+4-k}^{k} \\ &= s_{12}^{(t+1)} - P_{(t+1)+3-k}^{k}. \end{aligned}$$

Thus, we get the recurrence relation

$$s_{1,j}^{(t+1)} = s_{1,j-1}^{(t+1)} - P_{(t+1)+j-k}^{k},$$

and the proof of Case 2 completes.□

Theorem 2.4. Let P_{n+1}^{k} be the $(n+1)$st generalized k-Pell number for $n \geq k$. Then

$$perS^{(n,k)} = P_{n+1}^{k}.$$

Proof. Since the matrix $S^{(n,k)}$ is contractible, by Lemma 3 we obtain

$$S_{n-3}^{(n,k)} = \begin{bmatrix} P_{n-1}^{k} & \sum_{r=n-k}^{n-2} P_r^{k} & \sum_{r=n-k}^{n-2} P_r^{k} - P_{n-k}^{k} \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix},$$

where $\sum_{r=n-k}^{n-2} P_r^{k} - P_{n-k}^{k} = \sum_{r=n-k+1}^{n-2} P_r^{k}$. So we get

$$S_{n-2}^{(n,k)} = \begin{bmatrix} P_n^{k} & P_{n-1}^{k} + P_{n-2}^{k} + \ldots + P_{n-k+1}^{k} \\ 1 & 2 \end{bmatrix}.$$

By combining the last equality and (4), we reach

$$\begin{aligned} perS^{(n,k)} &= perS_{n-2}^{(n,k)} \\ &= 2P_n^{k} + P_{n-1}^{k} + \ldots + P_{n-k+1}^{k} \\ &= P_{n+1}^{k}, \end{aligned}$$

and the proof is completed.□

3 Conclusion

The famous integer sequences (e.g. Fibonacci, Pell) provide invaluable opportunities for exploration, and contribute handsomely to the beauty of mathematics, especially number theory. Among these sequences, Pell numbers have achieved a kind of celebrity status. There is a vast literature concerned with this sequence. This study is a different application which consider the connection between the permanents of a matrix and Pell numbers.

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