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Some Generalized Classes of Efficient Circular Repeated Measurements Designs Which are Strongly Balanced

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Abstract: Repeated measurement designs (RMDs) are frequently used in the fields of psychology, animal sciences, medicine and pharmacology, etc. Minimal strongly balanced RMDs are important to estimate the residual and direct effects independently and economically. Minimal strongly balanced generalized RMDs are preferred where minimal strongly balanced RMDs are impossible. Here, some new generators are developed for minimal strongly balanced generalized RMDs in periods of (i) two, (ii) three, and (iii) four different sizes. Proposed designs possess the high efficiency of Separability.

Keywords: Balanced RMDs; Efficiency of Separability; CBRMDs; CSBRMDs.
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1 Introduction

Repeated Measurements Design (RMD) is an experimental design where experimental units are measured repeatedly by giving a sequence of treatments. RMD is minimal (i) balanced if each treatment follows exactly once by every other treatment (excluding itself), (ii) strongly balanced if each treatment follows exactly once by every other treatment (including itself), and (iii) generalized strongly balanced if each treatment follows once and no time, or once and twice by every other treatment (including itself). [1] introduced RMDs. [2] constructed some RMDs for $p < v$, where p is period size and v is number of treatments. Circular balanced RMDs (BRMDs) were introduced by [3]. Some classes of BRMDs and strongly BRMDs (SBRMDs) were presented by [4]. In unequal period sizes, minimal BRMDs and SBRMDs were constructed by [5]. In periods of two different sizes, [6] presented some efficient RMDs and SBRMDs. For some cases, [7], [8], [9], [10], [11] and [12] presented circular BRMDs (CBRMDs) and circular SBRMDs (CSBRMDs). [13], [14] and [15] presented minimal circular partially and strongly partially BRMDs in periods of equal and two different sizes. Some more references can be found in [16]. Here, some generators are presented for some classes of efficient minimal circular generalized SBRMDs (CGSBRMDs) in periods of (i) two, (ii) three, and (iii) four, different sizes. Design construction method is explained in Section 2 for the construction of minimal CGSBRMDs. Procedure for finding efficiency of Separability is also described. Generators for minimal CGSBRMDs in periods of (i) two, (ii) three, and (iii) four different sizes are developed in Section 3, 4 and 5 respectively.

2 Method of cyclic shifts

[17] developed method of cyclic shifts for the construction of designs. Rule II of this method is explained here for the construction of minimal CGSBRMDs.

Let $S_1 = [q_{j1}, q_{j2}, ..., q_{j(r-1)}]$ and $S_2 = [q_{i1}, q_{i2}, ..., q_{i(s-2)}]$t, where $0 \leq q_{ij} \leq v-2$, if each of $0, 1, 2, ..., v-2$ appears once in $S^* = [q_{j1}, q_{j2}, ..., q_{j(r-1)}, q_{i1}, q_{i2}, ..., q_{i(s-2)}, v-1-(q_{j1}+q_{j2}+, ..., +q_{j(r-1)}) mod (v-1)]$, it is minimal CSBRMD

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in $p_1 = r$ and $p_2 = s$. It will be minimal CGSBRMDs if each of $0,1,2,...,v-2$ appears either (i) 0 and 1, or (ii) 1 and 2 times.

Example 2.1: $S_1 = [1,5,4,11,10,9,8], S_2 = [2,3,7]$t generate minimal CGSBRMD in $p_1 = 8, p_2 = 5$ for $v = 13$. Here, $S^* = [1,5,4,11,10,9,8,0,2,3,7]$ it means each of $0,1,...,5,7,8,...,11$ appears once while 6 does not appear. So it is minimal CGSBRMD.

Take v-1 subjects for $S_1 = [1,5,4,11,10,9,8]$. First period is assigned as $0,1,...,v-2$ respectively. To get the elements of period 2, add 1 (mod 12) to the elements of period 1. Similarly add 5 (mod 12) to the elements of second period to get the elements of third period. Similarly, the process of adding $4,11,10,9$ and 8 gives the following array, see Table 1.

Table 1: Arrays obtained from $S_1 = [1,5,4,11,10,9,8]$

Periods	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1	0	1	2	3	4	5	6	7	8	9	10	11
2	1	2	3	4	5	6	7	8	9	10	11	0
3	6	7	8	9	10	11	0	1	2	3	4	5
4	10	11	0	1	2	3	4	5	6	7	8	9
5	9	10	11	0	1	2	3	4	5	6	7	8
6	7	8	9	10	11	0	1	2	3	4	5	6
7	4	5	6	7	8	9	10	11	0	1	2	3
8	0	1	2	3	4	5	6	7	8	9	10	11

For $S_2 = [2,3,7]$t, 12 more subjects are taken. Arrays will be obtained in the similar way as of S_1 except the last period where v-1 will be inserted which is 12 here, table 2

Table 2: Arrays obtained from $S_2 = [2,3,7]$t

13	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**
0	1	2	3	4	5	6	7	8	9	10	11
2	3	4	5	6	7	8	9	10	11	0	1
5	6	7	8	9	10	11	0	1	2	3	4
0	1	2	3	4	5	6	7	8	9	10	11
12	12	12	12	12	12	12	12	12	12	12	12

Table 1 and Table 2 together present the required minimal CGSBRMD using 24 subjects.

2.1 Efficiency of Separability

[18] formula to find Es is modified here for CGSBRMDs and is given by.

$$Es = [1 - 1/(v\sqrt{v-1})] * 100\%$$

3 Minimal CGSBRMDs in period of two different sizes

[19] presented minimal CGSBRMDs in p_1 and p_2 for some specific cases. Here, some more generators are presented to generate these designs for some of the remaining cases.

3.1 Minimal CGSBRMDs in which some ordered pairs do not appear

Generator 3.1.1: Minimal CGSBRMDs can be obtained for $v = 2c_1i + 2c_2 + 1,\ p_1 = 2c_1,\ p_2 = 2c_2 + 1,\ c_2 > 1$, from following sets.

$S_{l+1} = [c_1l+1, c_1l+2, ..., c_1l+c_1, (v-1)-(c_1l+1), (v-1)-(c_1l+2, ..., (v-1)-(c_1l+c_1-1))]$.

$l = 0,1,2,...,i-1$

$S_{i+1} = [0, c_1i+1, c_1i+2, ..., c_1i+c_2-1, (v-1)-(c_1i+1), (v-1)-(c_1i+2), ..., (v-1)-(c_1i+c_2-1)]$t

Table 3: Examples of Generator 3.1.1

v	p_1	p_2	Sets of Shifts	Es
13	8	5	[2,1,4,3,11,10,9]+[0,5,7]t	0.92
15	10	5	[2,1,3,4,5,12,13,11,10]+[0,6,8]t	0.93
17	10	7	[1,2,3,4,5,15,14,13,12]+[0,6,7,10,9]t	0.94
19	10	9	[1,2,3,4,5,17,16,15,14]+[0,6,7,8,12,11,10]t	0.94

Generator 3.1.2: Minimal CGSBRMDs can be obtained for $v = 2c_1i + 4c_2 + 2$, $p_1 = 2c_1$, $p_2 = 2c_2 + 1$, from following sets.

$S_{l+1} = [c_1l+1, c_1l+2, ..., c_1l+c_1, (v-1)-(c_1l+1), (v-1)-(c_1l+2), ..., (v-1)-(c_1l+c_1-1)]$;
$l = 0, 1, 2, ..., i-1$
$S_{i+1} = [c_1i+1, c_1i+2, ..., c_1i+c_2, (v-1)-(c_1i+1), (v-1)-(c_1i+2), ..., (v-1)-(c_1i+c_2)]$
$S_{i+2} = [1+c_1i+c_2, 2+c_1i+c_2, ..., c_1i+2c_2, (v-1)-(c_1i+c_2+1), (v-1)-(c_1i+c_2+2), ..., (v-1)-(c_1i+2c_2-1)]$t

Table 4: Examples of Generator 3.1.2

v	p_1	p_2	Sets of Shifts	Es
18	8	5	[2,1,4,3,16,15,14]+[5,6,12,11]+[7,8,10]t	0.94
22	8	7	[1,2,3,4,20,19,18]+[5,6,7,16,15,14]+[8,9,10,13,12]t	0.95
24	10	7	[1,2,3,4,5,22,21,20,19]+[6,7,8,17,16,15]+[9,10,11,14,13]t	0.96
28	10	9	[1,2,3,4,5,26,25,24,23]+[6,7,8,9,21,20,19,18]+[10,11,12,13,17,16,15]t	0.96

Generator 3.1.3: Minimal CGSBRMDs can be obtained for $v = 2c_1i + 2c_2$, $p_1 = 2c_1$, $p_2 = 2c_2$, from following sets.

$S_{l+1} = [c_1l+1, c_1l+2, ..., c_1l+c_1, (v-1)-(c_1l+1), (v-1)-c_1l+2, ..., (v-1)-(c_1l+c_1-1)]$
$l = 0, 1, 2, ..., i-1$.
$S_{i+1} = [0, c_1i+1, c_1i+2, ..., c_1i+c_2-1, (v-1)-(c_1i+1), (v-1)-(c_1i+2), ..., (v-1)-(c_1i+c_2-1)]$t

Table 5: Examples of Generator 3.1.3

v	p_1	p_2	Sets of Shifts	Es
14	8	6	[1,2,3,4,12,11,10]+[0,5,6,8]t	0.92
18	10	8	[1,2,3,4,5,16,15,14,13]+[0,6,7,8,11,10]t	0.94
22	12	10	[1,2,4, 3,20,6,5,19,17 ,18,16]+[0,7,8,9,10,14,13,12]t	0.95
19	10	9	[1,3,2,4,5,6,7,24,22,23,21,20,19]+[0,8,9,10,11,12,17,16,15,14]t	0.96

Generator 3.1.4: Minimal CGSBRMDs can be obtained for $v = 2c_1i + 4c_2$, $p_1 = 2c_1$, $p_2 = 2c_2$, from following sets.

$S_{l+1} = [c_1l+1, c_1l+2, ..., c_1l+c_1, (v-1)-(c_1l+1), (v-1)-c_1l+2, ..., (v-1)-(c_1l+c_1-1)]$
$l = 0, 1, 2, ..., i-1$.
$S_{i+1} = [1+c_1i, 2+c_1i, ..., c_1i+c_2, (v-1)-(c_1i+1), (v-1)-(c_1i+2), ..., (v-1)-(c_1i+c_2-1)]$
$S_{i+2} = [0, 1+c_1i+c_2, 2+c_1i+c_2, ..., c_1i+2c_2-1, (v-1)-(c_1i+c_2+1), (v-1)-(c_1i+c_2+2), ..., (v-1)-(c_1i+2c_2-2)]$t

Table 6: Examples of Generator 3.1.4

v	p_1	p_2	Sets of Shifts	Es
20	8	6	[2,1,4,3,16,17,18]+[6,5,7,13,14]+[0,8,9,11]t	0.95
26	10	8	[1,2,3,4,5,24,23,22,21]+[6,7,8,9,19,18,17]+[0,10,11,12,15,14]t	0.96
32	12	10	[1,2,4,3,5,6,30,29,27,28,26]+[8,7,9,10,11,24,23,22,21]+ [0,12,13,14,15,19,18,17]t	0.97
38	14	12	[1,2,3,4,5,6,7,36,35,34,33,32,31]+[8,9,10,11,13,12,29,28,27,26,25]+ [0,14,15,16,17,18,23,22,21,20]t	0.97

3.2 Minimal CGSBRMDs in which some ordered pairs appear twice together

Generator 3.2.1: Minimal CGSBRMDs can be obtained for $v = 2c_1i + 2c_2 - 1$, $p_1 = 2c_1$, $p_2 = 2c_2 + 1$, $c_2 > 1$, from following sets.

$S_{l+1} = [c_1l+1, c_1l+2, ..., c_1l+c_1, (v-1)-(c_1l+1), (v-1)-(c_1l+2), ..., (v-1)-(c_1l+c_1-1)]$
$l = 0, 1, 2, ..., i-1$.
$S_{i+1} = [0, 1+c_1i, 2+c_1i, ..., c_1i+c_2, (v-1)-(c_1i+1), (v-1)-(c_1i+2), ..., (v-1)-(c_1i+c_2-1)]$ t

Table 7: Examples of Generator 3.2.1

v	p_1	p_2	**Sets of Shifts**	**Es**
13	8	7	[2,1,4,3,11,10,9]+[0,5,6,7,7]t	0.87
17	10	9	[1,2,3,4,5,15,14,13,12]+[0,6,7,8,9,10,9]t	0.90
21	12	11	[2, 1,4,3,5,6,18,19,16,17,15]+[0,7,8,9,10,11,13,12,11]t	0.92
25	14	13	[1,4,3,2,5,7,6,23,20,22,21,19,18]+[0,8,9,10,11,12,13,16,15,14,13]t	0.93

Generator 3.2.2: Minimal CGSBRMDs can be obtained for $v = 2c_1i + 4c_2$, $p_1 = 2c_1$, $p_2 = 2c_2 + 1$, from following sets.

$S_{l+1} = [c_1l+1, c_1l+2, ..., c_1l+c_1, (v-1)-(c_1l+1), (v-1)-(c_1l+2), ..., (v-1)-(c_1l+c_1-1)]$

$l = 0, 1, 2, ..., i-1.$

$S_{i+1} = [1+c_{1i}, 2+c_{1i}, ..., c_{1i}+c_2, (v-1)-(c_{1i}+1), (v-1)-(c_{1i}+2), ..., (v-1)-(c_{1i}+c_2)]$

$S_{i+2} = [1+c_1i+c_2, 2+c_1i+c_2, ..., c_1i+2c_2, (v-1)-(c_1i+c_2+1), (v-1)-(c_1i+c_2+2), ..., (v-1)-(c_1i+2c_2-1)]t$

Table 8: Examples of Generator 3.2.2

v	p_1	p_2	**Sets of Shifts**	**Es**
20	8	7	[2,1,4,3,18,17,16]+[6,5,7,14,13,12]+[8,9,10,11,10]t	0.92
26	10	9	[1,2,3,4,5,24,23,22,21]+[6,7,8,9,19,18,17,16]+[10,11,12,13,15,14,13]t	0.94
32	12	11	[1,2,4,3,5,6,30,29,27,28,26]+[7,8,9,10,11,24,23,22,21,20]+[12,13,14,15,16,19,18,17,16]t	0.95
38	14	13	[1,2,3,4,5,6,7,36,35,34,33,32,31]+[8,9,10,11,12,13,29,28,27,26,25,24]+ [14,15,16,17,18,19,23,22,21,20,19]t	0.96

Generator 3.2.3: Minimal CGSBRMDs can be obtained for $v = 2c_1i + 2c_2$, $p_1 = 2c_1$, $p_2 = 2c_2$, from following sets.

$S_{l+1} = [c_1l+1, c_1l+2, ..., c_1l+c_1, (v-1)-(c_1l+1), (v-1)-(c_1l+2), ..., (v-1)-(c_1l+c_1-1)]$

$l = 0, 1, 2, ..., i-1.$

$S_{i+1} = [0, 1+c_1i, 2+c_1i, ..., c_1i+c_2-1, (v-1)-(c_1i+1), (v-1)-(c_1i+2), ..., (v-1)-(c_1i+c_2)-2]t$

Table 9: Examples of Generator 3.2.3

v	p_1	p_2	**Sets of Shifts**	**Es**
12	8	6	[1,2,3,4,10,9,8]+[0,5,6,6]t	0.86
16	10	8	[1,2,3,4,14,5,13,11,12]+[0,6,7,8,9,8]t	0.90
20	12	10	[2,1,4,3,6,18,5,17,15,16,14]+[0,7,8,9,10,12,11,10]t	0.92
24	14	12	[1,2,3,4,5,6,7,22,21,20,19,18,17]+[0,8,9,10,11,12,15,14,13,12]t	0.93

Generator 3.2.4: Minimal CGSBRMDs can be obtained for $v = 2c_{1i} + 4c_2 - 2$, $p_1 = 2c_1$, $p_2 = 2c_2$, from following sets.

$S_{l+1} = [c_1l+1, c_1l+2, ..., c_1l+c_1, (v-1)-(c_1l+1), (v-1)-(c_1l+2), ..., (v-1)-(c_1l+c_1-1)]$

$l = 0, 1, 2, ..., i-1.$

$S_{i+1} = [1+c_1i, 2+c_1i, ..., c_1i+c_2, (v-1)-(c_1i+1), (v-1)-(c_1i+2), ..., (v-1)-(c_1i+c_2)-1]$

$S_{i+2} = [0, 1+c_1i+c_2, 2+c_1i+c_2, ..., c_1i+2c_2-1, (v-1)-(c_1i+c_2+1), (v-1)-(c_1i+c_2+1), ..., (v-1)-(c_1i+2c_2-2)]t$

Table 10: Examples of Generator 3.2.4

v	p_1	p_2	**Sets of Shifts**	**Es**
18	8	6	[2,1,4,3,16,15,14]+[6,5,7,12,11]+[0,8,9,9]t	0.91
24	10	8	[1,2,3,4,5,22,21,20,19]+[6,7,8,9,17,16,15]+[0,10,11,12,13,12]t	0.93
30	12	10	[1,2,4,3,5,6,28,27,25,26,24]+[7,8,9,10,11,22,21,20,19]+[0,12,13,14,15,17,16,15]t	0.94
36	14	12	[1,2,3,4,5,6,7,34,33,32,31,30,29]+[8,9,10,11,12,13,27,26,25,24,23]+[0,14,15,16,17,18,21,20,19,18]t	0.95

4 Minimal CGSBRMDs in three different periods sizes

4.1 Minimal CGSBRMDs for v odd

Generators developed in this section provide minimal CGSBRMDs for v odd in p_1, p_2 and p_3. Here, some ordered pairs do not appear.

Generator 4.1.1: Minimal CGSBRMDs can be obtained for $v = 2c_1 i + 2c_2 + 2c_3 + 1$, i integer, $p_1 = 2c_1$, $p_2 = 2c_2 + 1$, and $p_3 = 2c_3$, from following sets.

$S_{l+1} = [c_1 l + 1, c_1 l + 2, ..., c_1 l + c_1, (v-1) - (c_1 l + 1), (v-1) - (c_1 l + 2), ..., (v-1) - (c_1 l + c_1 - 1)];$
$l = 0, 1, 2, ..., i-1.$
$S_{i+1} = [1 + c_1 i, 2 + c_1 i, ..., c_1 i + c_2, (v-1) - (c_1 i + 1), (v-1) - (c_1 i + 2), ..., (v-1) - (c_1 i + c_2)]$
$S_{i+2} = [1 + c_1 i + c_2, 2 + c_1 i + c_2, ..., c_1 i + c_2 + c_3 - 1, (v-1) - (c_1 i + c_2 + 1), (v-1) - (c_1 i + c_2 + 2), ..., (v-1) - (c_1 i + c_2 + c_3 - 1)]$

Table 11: Examples of Generator 4.1.1

v	p_1	p_2	p_3	**Sets of Shifts**	**Es**
19	8	7	4	[2,1,4,3,15,16,17]+[6,5,7,13,12,11]+[8,10]t	0.94
21	10	7	4	[1,2,3,4,5,19,18,17,16]+[6,7,8,14,13,12]+[9,11]t	0.95
25	10	9	6	[1,2,3,4,5,23,22,21,20]+[6,7,8,9,18,17,16,15]+[10,11,14,13]t	0.96
27	12	9	6	[1,2,3,4,5,6,25,24,23,22,21]+[7,8,9,10,19,18,17,16]+[11,12,15,14]t	0.96

Generator 4.1.2: Minimal CGSBRMDs can be obtained for $v = 2c_1 i + 2c_2 + 4c_3 + 1$, i integer, $p_1 = 2c_1, p_2 = 2c_2 + 1$, and $p_3 = 2c_3$, from the following sets.

$S_{l+1} = [c_1 l + 1, c_1 l + 2, ..., c_1 l + c_1, (v-1) - (c_1 l + 1), (v-1) - (c_1 l + 2), ..., (v-1) - (c_1 l + c_1 - 1)];$
$l = 0, 1, 2, ..., i-1.$
$S_{i+1} = [1 + c_1 i, 2 + c_1 i, ..., c_1 i + c_2, (v-1) - (c_1 i + 1), (v-1) - (c_1 i + 2), ..., (v-1) - (c_1 i + c_2)]$
$S_{i+2} = [1 + c_1 i + c_2, 2 + c_1 i + c_2, ..., c_1 i + c_2 + c_3, (v-1) - (c_1 i + c_2 + 1), (v-1) - (c_1 i + c_2 + 2), ..., (v-1) - (c_1 i + c_2 + c_3 - 1)].$
$S_{i+3} = [1 + c_1 i + c_2 + c_3, 2 + c_1 i + c_2 + c_3, ..., c_1 i + c_2 + 2c_3, (v-1) - (c_1 i + c_2 + c_3 + 1), (v-1) - (c_1 i + c_2 + c_3 + 2), ..., (v-1) - (c_1 i + c_2 + 2c_3 - 2)]t.$

Table 12: Examples of Generator 4.1.2

v	p_1	p_2	p_3	**Sets of Shifts**	**Es**
23	8	7	4	[1,2,3,4,21,20,19]+[5,6,7,17,16,15]+[8,9,14]+[10,11]t	0.95
27	10	9	4	[1,2,3,4,5,25,24,23,22]+[6,7,8,9,20,19,18,17]+[10,11,16]+[12,13]t	0.96
35	12	11	6	[1,2,3,4,5,6,33,32,31,30,29]+[7,8,9,10,11,27,26,25,24,23]+ [12,13,14,22,21]+[15,16,17,19]t	0.97
39	12	11	8	[1,2,3,4,5,6,37,36,35,34,33]+[7,8,9,10,11,31,30,29,28,27]+ [12,13,14,15,26,25,24]+[16,17,18,19,22,21]t	0.97

4.2 Minimal CGSBRMDs for v even

Generators developed in this section provide minimal CGSBRMDs for v even in p_1, p_2 and p_3. Here, some ordered pairs do not appear.

Generator 4.2.1: Minimal CGSBRMDs can be obtained for $v = 2c_1 i + 2c_2 + 4c_3 + 2$, i integer, $p_1 = 2c_1, p_2 = 2c_2$ and $p_3 = 2c_3 + 1$, from following sets.

$S_{l+1} = [c_1 l + 1, c_1 l + 2, ..., c_1 l + c_1, (v-1) - (c_1 l + 1), (v-1) - (c_1 l + 2), ..., (v-1) - (c_1 l + c_1 - 1)];$
$l = 0, 1, 2, ..., i-1.$
$S_{i+1} = [1 + c_1 i, 2 + c_1 i, ..., c_1 i + c_2, (v-1) - (c_1 i + 1), (v-1) - (c_1 i + 2), ..., (v-1) - (c_1 i + c_2 - 1)]$
$S_{i+2} =$
$[1 + c_1 i + c_2, 2 + c_1 i + c_2, ..., c_1 i + c_2 + c_3, (v-1) - (c_1 i + c_2 + 1), (v-1) - (c_1 i + c_2 + 2), ..., (v-1) - (c_1 i + c_2 + c_3)]$
$S_{i+3} = [1 + c_1 i + c_2 + c_3, 2 + c_1 i + c_2 + c_3, ..., c_1 i + c_2 + 2c_3, (v-1) - (c_1 i + c_2 + c_3 + 1), (v-1) - (c_1 i + c_2 + c_3 + 2), ..., (v-1) - (c_1 i + c_2 + 2c_3 - 1)]t$

Table 13: Examples of Generator 4.2.1

v	p_1	p_2	p_3	**Sets of Shifts**	**Es**
24	8	6	5	[2,1,4,3,21,22,20]+[6,5,7,18,17]+[9,8,14,15]+[10,11,13]t	0.96
28	10	8	5	[1,2,3,4,5,26,25,24,23]+[6,7,8,9,21,20,19]+[10,11,17,16]+[12,13,15]t	0.96
32	10	8	7	[1,2,3,4,5,30,29,28,27]+[6,7,8,9,25,24,23]+[10,11,12,21,20,19]+[13,14,15,18,17]t	0.97
40	12	10	9	[1,2,4,3,5,6,38,37,35,36,34]+[7,8,9,10,11,32,31,30,29]+[12,13,14,15,27,26,25,24]+[16,17,18,19,23,22,21]t	0.97

Generator 4.2.2: Minimal CGSBRMDs can be obtained for $v = 2c_1i + 4c_2 + 4c_3 + 2$, i integer, $p_1 = 2c_1, p_2 = 2c_2$, and $p_3 = 2c_3 + 1$, from the following sets.
$S_{l+1} = [c_1l+1, c_1l+2, ..., c_1l+c_1, (v-1)-(c_1l+1), (v-1)-(c_1l+2), ..., (v-1)-(c_1l+c_1-1)]$;
$l = 0, 1, 2, ..., i-1$.
$S_{i+1} = [1+c_1i, 2+c_1i, ..., c_1i+c_2, (v-1)-(c_1i+1), (v-1)-(c_1i+2), ..., (v-1)-(c_1i+c_2-1)]$
$S_{i+2} = [1+c_1i+c_2, 2+c_1i+c_2, ..., c_1i+2c_2, (v-1)-(c_1i+c_2+1), (v-1)-(c_1i+c_2+2), ..., (v-1)-(c_1i+2c_2-1),]$
$S_{i+3} = [1+c_1i+2c_2, 2+c_1i+2c_2, ..., c_1i+2c_2+c_3, (v-1)-(c_1i+2c_2+1), (v-1)-(c_1i+2c_2+1), ..., (v-1)-(c_1i+2c_2+c_3)]$
$S_{i+4} = [1+c_1i+2c_2+c_3, 2+c_1i+2c_2+c_3, ..., c_1i+2c_2+2c_3, (v-1)-(c_1i+2c_2+c_3+1), (v-1)-(c_1i+2c_2+c_3+2), ..., (v-1)-(c_1i+2c_2+2c_3-1)]$t

Table 14: Examples of Generator 4.2.2

v	p_1	p_2	p_3	**Sets of Shifts**	**Es**
30	8	6	5	[2,1,4,3,28,27,26]+[6,5,7,24,23]+[9,8,10,21,20]+11,12,18,17]+[13,14,16]t	0.97
40	10	8	7	[1,2,3,4,5,26,25,24,23]+[6,7,8,9,21,20,19]+[10,11,17,16]+[12,13,15]t	0.96
36	10	8	5	[1,2,3,4,5,34,33,32,31]+[6,7,8,9,29,28,27]+[10,11,12,13,25,24,23]+[14,15,21,20]+[16,17,19]t	0.97
50	12	10	9	[1,2,3,4,5,6,48,47,46,45,44]+[7,8,9,10,11,42,41,40,39]+ [12,13,14,15,16,37,36,35,34]+[17,18,19,20,32,31,30,29]+[21,22,23,24,28,27,26]t	0.98

5 Minimal CGSBRMDs in four different periods sizes

Generators developed in this section provide minimal CGSBRMDs in p_1, p_2, p_3 and p_4. Here, some ordered pairs do not appear.

5.1 Minimal CGSBRMDs for v odd

Generators developed in this section provide minimal CGSBRMDs for v odd in p_1, p_2, p_3 and p_4.
Generator 5.1.1: Minimal CGSBRMDs can be obtained for $v = 2c_1i + 2c_2 + 2c_3 + 2c_4 + 1, i$ integer, $p_1 = 2c_1$, $p_2 = 2c_2$, $p_3 = 2c_3 + 1$ and $p_4 = 2c_4$, from the following sets.
$S_{l+1} = [c_1l+1, c_1l+2, ..., c_1l+c_1, (v-1)-(c_1l+1), (v-1)-(c_1l+2), ..., (v-1)-(c_1l+c_1-1)]$;
$l = 0, 1, 2, ..., i-1$.
$S_{i+1} = [1+c_1i, 2+c_1i, ..., c_1i+c_2, (v-1)-(c_1i+1), (v-1)-(c_1i+2), ..., (v-1)-(c_1i+c_2-1)]$
$S_{i+2} = [1+c_1i+c_2, 2+c_1i+c_2, ..., c_1i+c_2+c_3, (v-1)-(c_1i+c_2+1), (v-1)-(c_1i+c_2+2), ..., (v-1)-(c_1i+c_2+c_3)]$
$S_{i+3} = [1+c_1i+c_2+c_3, 2+c_1i+c_2+c_3, ..., c_1i+c_2+c_3+c_4, (v-1)-(c_1i+c_2+c_3+1), (v-1)-(c_1i+c_2+c_3+2), ..., (v-1)-(c_1i+c_2+c_3+c_4-2)]$t.

Table 15: Examples of Generator 5.1.1

v	p_1	p_2	p_3	p_4	**Sets of Shifts**	**Es**
23	8	6	5	4	[2,1,4,3,21,20,19]]+[7,8,9,15,14]+[5,6,17,16]+[10,11]t	0.95
29	10	8	7	4	[1,2,3,4,5,27,26,25,24]+[6,7,9,8,22,21,20]+[10,11,12,18,17,16]+[13,14]t	0.96
31	10	8	7	6	[1,2,3,4,5,29,28,27,26]+[6,7,8,24,9,23,22]+[10,11,12,20,19,18]+[13,14,15,17]t	0.97
37	12	10	9	6	[1,2,4,3,6,5,35,34,33,32,31]+[7,8,9,10,11,29,28,27,26]+[12,13,14,15,24,23,22,21]+[16,17,18,20]t	0.97

Generator 5.1.2: Minimal CGSBRMDs can be obtained for $v = 2c_1i + 4c_2 + 2c_3 + 2c_4 + 1$, i integer, $p_1 = 2c_1, p_2 = 2c_2, p_3 = 2c_3 + 1$ and $p_4 = 2c_4$, from the following sets.
$S_{l+1} = [c_1l+1, c_1l+2, ..., c_1l+c_1, (v-1)-(c_1l+1), (v-1)-(c_1l+2), ..., (v-1)-(c_1l+c_1-1)]$;
$l = 0, 1, 2, ..., i-1$.
$S_{i+1} = [1+c_1i, 2+c_1i, ..., c_1i+c_2, (v-1)-(c_1i+1), (v-1)-(c_1i+2), ..., (v-1)-(c_1i+c_2-1)]$

$S_{i+2} = [1+c_1i+c_2, 2+c_1i+c_2, ..., c_1i+2c_2, (v-1)-(c_1i+c_2+1), (v-1)-(c_1i+c_2+1), ..., (v-1)-(c_1i+2c_2-1)]$
$S_{i+3} = [1+c_1i+2c_2, 2+c_1i+2c_2, ..., c_1i+2c_2+c_3, (v-1)-(c_1i+2c_2+1), (v-1)-(c_1i+2c_2+1), ..., (v-1)-(c_1i+2c_2+c_3)]$
$S_{i+4} = [1+c_1i+2c_2+c_3, 2+c_1i+2c_2+c_3, ..., c_1i+2c_2+c_3+c_4, (v-1)-(c_{1_i}+2c_2+c_3+1), (v-1)-(c_1i+2c_2+c_3+1), ..., (v-1)-(c_1i+2c_2+c_3+c_4-2)]$t.

Table 16: Examples of Generator 5.1.2

v	p_1	p_2	p_3	p_4	**Sets of Shifts**	**Es**
29	8	6	5	4	[2,1,4,3,27,26,25]+[6,5,7,23,22]+[9,8,10,20,19]+[11,12,17,16]+[13,14]t	0.95
31	10	6	5	4	[1,2,3,4,5,29,28,27,26]+[6,7,8,24,23]+[9,10,21,11,20]+[12,13,18,17]+[14,15]t	0.97
37	10	8	7	4	[1,2,3,4,5,35,34,33,32]+[6,7,8,9,30,29,28]+[11,10,12,13,25,26,24]+[14,15,16,22,21,20]+[17,18]t	0.97
41	12	8	7	6	[1,2,4,3,6,5,39,38,37,36,35]+[7,8,9,10,33,32,31]+[11,12,13,14,29,28,27]+ [15,16,17,25,24,23]+[18,19,20,22]t	0.98

Generator 5.1.3: Minimal CGSBRMDs can be obtained for $v = 2c_1i+4c_2+2c_3+4c_4+1$, i integer, $p_1 = 2c_1, p_2 = 2c_2, p_3 = 2c_3+1$ and $p_4 = 2c_4$, from the following sets.
$S_{l+1} = [c_1l+1, c_1l+2, ..., c_1l+c_1, (v-1)-(c_1l+1), (v-1)-(c_1l+2), ..., (v-1)-(c_1l+c_1-1)]$;
$l = 0,1,2,...,i-1$.
$S_{i+1} = [1+c_1i, 2+c_1i, ..., c_1i+c_2, (v-1)-(c_1i+1), (v-1)-(c_1i+2), ..., (v-1)-(c_1i+c_2)]$
$S_{i+2} = [1+c_1i+c_2, 2+c_1i+c_2, ..., c_1i+2c_2, (v-1)-(c_1i+c_2+1), (v-1)-(c_1i+c_2+2), ..., (v-1)-(c_1i+2c_2-1)]$
$S_{i+3} = [1+c_1i+2c_2, 2+c_1i+2c_2, ..., c_1i+2c_2+c_3, (v-1)-(c_1i+2c_2+1), (v-1)-(c_1i+2c_2+2), ..., (v-1)-(c_1i+2c_2+c_3-1)]$
$S_{i+4} = [1+c_1i+2c_2+c_3, 2+c_1i+2c_2+c_3, ..., c_1i+2c_2+c_3+c_4, (v-1)-(c_1i+2c_2+c_3+1), (v-1)-(c_1i+2c_2+c_3+2), ..., (v-1)-(c_1i+2c_2+c_3+c_4-1)]$
$S_{i+5} = [1+c_1i+2c_2+c_3+c_4, 2+c_1i+2c_2+c_3+c_4, ..., c_1i+2c_2+c_3+2c_4, (v-1)-(c_1i+2c_2+c_3+c_4+1), (v-1)-(c_1i+2c_2+c_3+c_4+2), ..., (v-1)-(c_1i+2c_2+c_3+2c_4-1)]$t

Table 17: Examples of Generator 5.1.3

v	p_1	p_2	p_3	p_4	**Sets of Shifts**	**Es**
24	8	6	4	3	[2,1,4,3,22,21,20]+[6,5,7,18,17]+[8,9,15]+[10,13]+[11]t	0.96
30	10	8	6	3	[1,2,3,4,5,28,27,26,25]+[6,7,8,9,23,22,21]+[10,11,12,19,18]+[13,16]+[14]t	0.97
34	10	8	6	5	[1,2,3,4,5,32,31,30,29]+[6,7,8,9,27,26,25]+[10,11,12,24,23]+[13,14,21,20]+[15,16,18]t	0.97
44	12	10	8	7	[1,2,4,3,6,5,42,41,40,39,38]+[7,8,9,10,11,36,35,34,33]+[12,13,14,15,31,30,29]+ [16,17,18,27,26,25]+[19,20,21,24,23]t	0.98

5.2 Minimal CGSBRMDs for v even

Generators developed in this section provide minimal CGSBRMDs for v odd in p_1, p_2, p_3 and p_4.
Generator 5.2.1: Minimal CGSBRMDs can be obtained for $v = 2c_1i+2c_2+2c_3+4c_4+1$, i integer, $p_1 = 2c_1, p_2 = 2c_2, p_3 = 2c_3$ and $p_4 = 2c_4+1$, from the following sets.
$S_{l+1} = [c_1l+1, c_1l+2, ..., c_1l+c_1, (v-1)-(c_1l+1), (v-1)-(c_1l+2), ..., (v-1)-(c_1l+c_1-1)]$;
$l = 0,1,2,...,i-1$.
$S_{i+1} = [1+c_1i, 2+c_1i, ..., c_1i+c_2, (v-1)-(c_1i+1), (v-1)-(c_1i+2), ..., (v-1)-(c_1i+c_2-1)]$
$S_{i+2} = [1+c_1i+c_2, 2+c_1i+c_2, ..., c_1i+c_2+c_3, (v-1)-(c_1i+c_2+1), (v-1)-(c_1i+c_2+2), ..., (v-1)-(c_1i+c_2+c_3-1)]$
$S_{i+3} = [1+c_1i+c_2+c_3, 2+c_1i+c_2+c_3, ..., c_1i+c_2+c_3+c_4, (v-1)-(c_1i+c_2+c_3+1), (v-1)-(c_1i+c_2+c_3+2), ..., (v-1)-(c_1i+c_2+c_3+c_4)]$
$S_{i+4} = [1+c_1i+c_2+c_3+c_4, 2+c_1i+c_2+c_3+c_4, ..., c_1i+c_2+c_3+2c_4, (v-1)-(c_1i+c_2+c_3+c_4+1), (v-1)-(c_1i+c_2+c_3+c_4+2), ..., (v-1)-(c_1i+c_2+c_3+2c_4-1)]$t

Table 18: Examples of Generator 5.2.1

v	p_1	p_2	p_3	p_4	Sets of Shifts	Es
24	8	6	4	3	[2,1,4,3,22,21,20]+[6,5,7,18,17]+[8,9,15]+[10,13]+[11]t	0.96
30	10	8	6	3	[1,2,3,4,5,28,27,26,25]+[6,7,8,9,23,22,21]+[10,11,12,19,18]+[13,16]+[14]t	0.97
34	10	8	6	5	[1,2,3,4,5,32,31,30,29]+[6,7,8,9,27,26,25]+[10,11,12,24,23]+[13,14,21,20]+[15,16,18]t	0.97
44	12	10	8	7	[1,2,4,3,6,5,42,41,40,39,38]+[7,8,9,10,11,36,35,34,33]+[12,13,14,15,31,30,29]+ [16,17,18,27,26,25]+[19,20,21,24,23]t	0.98

Generator 5.2.2: Minimal CGSBRMDs can be obtained for $v = 2c_1i + 2c_2 + 4c_3 + 4c_4 + 1$, i integer, $p_1 = 2c_1, p_2 = 2c_2, p_3 = 2c_3$ and $p_4 = 2c_4 + 1$, from the following sets.
$S_{l+1} = [c_1l+1, c_1l+2, ..., c_1l+c_1, (v-1)-(c_1l+1), (v-1)-(c_1l+2), ..., (v-1)-(c_1l+c_1-1)]$;
$l = 0,1,2,...,i-1$.
$S_{i+1} = [1+c_1i, 2+c_1i, ..., c_1i+c_2, (v-1)-(c_1i+1), (v-1)-(c_1i+2), ..., (v-1)-(c_1i+c_2-1)]$
$S_{i+2} = [1+c_1i+c_2, 2+c_1i+c_2, ..., c_1i+c_2+c_3, (v-1)-(c_1i+c_2+1), (v-1)-(c_1i+c_2+2), ..., (v-1)-(c_1i+c_2+c_3-1)]$
$S_{i+3} = [1+c_1i+c_2+c_3, 2+c_1i+c_2+c_3, ..., c_1i+c_2+2c_3, (v-1)-(c_1i+c_2+c_3+1), (v-1)-(c_1i+c_2+c_3+2), ..., (v-1)-(c_1i+c_2+2c_3-1)]$
$S_{i+4} = [1+c_1i+c_2+2c_3, 2+c_1i+c_2+2c_3, ..., c_1i+c_2+2c_3+c_4, (v-1)-(c_1i+c_2+2c_3+1), (v-1)-(c_1i+c_2+2c_3+2), ..., (v-1)-(c_1i+c_2+2c_3+c_4)]$
$S_{i+5} = [1+c_1i+c_2+2c_3+c_4, 2+c_1i+c_2+2c_3+c_4, ..., c_1i+c_2+2c_3+2c_4, (v-1)-(c_1i+c_2+2c_3+c_4+1), (v-1)-(c_1i+c_2+2c_3+c_4+2), ..., (v-1)-(c_1i+c_2+2c_3+2c_4-1)]$t.

Table 19: Examples of Generator 5.2.2

v	p_1	p_2	p_3	p_4	Sets of Shifts	Es
28	8	6	4	3	[2,1,4,3,26,25,24]+[6,5,7,22,21]+[9,8,19]+[11,10,17]+[12,15]+[13]t	0.96
36	10	8	6	3	[1,2,3,4,5,34,33,32,31]+[6,7,8,9,29,28,27]+[10,11,12,25,24]+[13,14,15,22,21]+[16,19]+[17]t	0.97
40	10	8	6	5	[1,2,3,4,5,38,37,39,35]+[6,7,8,9,33,32,31]+[10,11,12,29,28]+[13,14,15,26,25]+[16,17,23,22]+[18,19,21]t	0.97
52	12	10	8	7	[1,2,3,4,5,6,50,49,48,47,46]+[7,8,9,10,11,44,43,42,41]+[12,13,14,15,39,38,37]+ [16,17,18,19,35,34,33]+[20,21,22,31,30,29]+[23,24,25,28,27]t	0.98

Generator 5.2.3: Minimal CGSBRMDs can be obtained for $v = 2c_{1i} + 4c_2 + 4c_3 + 4c_4 + 1$, i integer, $p_1 = 2c_1, p_2 = 2c_2, p_3 = 2c_3$ and $p_4 = 2c_4 + 1$, from the following sets.
$S_{l+1} = [c_1l+1, c_1l+2, ..., c_1l+c_1, (v-1)-(c_1l+1), (v-1)-(c_1l+2), ..., (v-1)-(c_1l+c_1-1)]$;
$l = 0,1,2,...,i-1$.
$S_{i+1} = [1+c_1i, 2+c_{1i}, ..., c_1i+c_2, (v-1)-(c_1i+1), (v-1)-(c_1i+2), ..., (v-1)-(c_{1i}+c_2-1)]$
$S_{i+2} = [1+c_1i+c_2, 2+c_1i+c_2, ..., c_1i+2c_2, (v-1)-(c_1i+c_2+1), (v-1)-(c_1i+c_2+2), ..., (v-1)-(c_1i+2c_2-1)]$
$S_{i+3} = [1+c_1i+2c_2, 2+c_1i+2c_2, ..., c_1i+2c_2+c_3, (v-1)-(c_1i+2c_2+1), (v-1)-(c_1i+2c_2+2), ..., (v-1)-(c_1i+2c_2+c_3-1)]$
$S_{i+3} = [1+c_1i+2c_2+c_3, 2+c_1i+2c_2+c_3, ..., c_{1i}+2c_2+2c_3, (v-1)-(c_1i+2c_2+c_3+1), (v-1)-(c_1i+2c_2+c_3+2), ..., (v-1)-(c_1i+2c_2+2c_3-1)]$
$S_{i+4} = [1+c_1i+2c_2+2c_3, 2+c_1i+2c_2+2c_3, ..., c_1i+2c_2+2c_3+c_4, (v-1)-(c_1i+2c_2+2c_3+1), (v-1)-(c_1i+2c_2+2c_3+2), ..., (v-1)-(c_1i+2c_2+2c_3+c_4)]$
$S_{i+5} = [1+c_1i+2c_2+2c_3+c_4, 2+c_1i+2c_2+2c_3+c_4, ..., c_1i+2c_2+2c_3+2c_4, (v-1)-(c_1i+2c_2+2c_3+c_4+1), (v-1)-(c_1i+2c_2+2c_3+c_4+2), ..., (v-1)-(c_1i+2c_2+2c_3+2c_4-1)]$t

Table 20: Examples of Generator 5.2.3

v	p_1	p_2	p_3	p_4	Sets of Shifts	Es
34	8	6	4	3	[2,1,4,3,32,31,30]+[6,5,7,28,27]+[9,8,10,25,24]+[11,12,22]+[13,14,20]+[15,18]+[16]t	0.97
44	10	8	6	3	[1,2,3,4,5,42,41,40,39]+[6,7,8,9,37,36,35]+[10,11,12,13,33,32,31]+[14,15,16,29,28]+ [17,18,19,26,25]+[20,23]+[21]t	0.98
48	10	8	6	5	[1,2,3,4,5,46,45,44,43]+[6,7,8,9,41,40,39]+[10,11,12,13,37,36,35]+ [14,15,16,33,32]+[17,18,19,30,29]+[20,21,27,26]+[22,23,25]t	0.98
62	12	10	8	7	[1,2,3,4,5,6,60,59,58,57,56]+[7,8,9,10,11,54,53,52,51]+[12,13,14,15,16,49,48,47,46]+ [17,18,19,20,44,43,42]+[21,22,23,24,40,39,38]+[25,26,27,36,35,34]+[28,29,30,33,32]t	0.98

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Conflict of interest

The authors declare that they have no conflict of interest.

References

[1] E. R. Williams, Experimental designs balanced for the estimation of residual effects of treatments. Australian Journal of Science Research, **2**(2), 149-168 (1949).
[2] E. R. Williams, Experimental designs balanced for pairs of residual effects. Australian Journal of Science Research, **3**(3), 351-363 (1950).
[3] C. G. Magda, Circular balanced repeated measurements designs. Communications in Statistics-Theory and Methods, **9**, 1901-1918 (1980).
[4] C. S. Cheng, and C. F. Wu, Balanced repeated measurements designs. The Annals of Statistics, **8**(6), 1272-1283 (1980).
[5] K. Afsarinejad, Repeated measurements designs with unequal period sizes. Journal of the Italian Statistical Society, **3**(2), 161-168 (1994).
[6] I. Iqbal, and B. Jones, Efficient repeated measurements designs with equal and unequal period sizes. Journal of Statistical Planning and Inference, 42 (1-2), 79-88. and Applied Statistics, **3**(5), 125-129 (1994).
[7] I. Iqbal, and M. H. Tahir, Circular strongly balanced repeated measurements designs. Communications in Statistics-Theory and Methods, **38**, 3686-3696 (2009).
[8] I. Iqbal, M. H. Tahir, and S. S. A. Ghazali, Circular first and second order balanced repeated measurements designs. Communications in Statistics-Theory and Methods, **39**, 228-240 (2010).
[9] M. Rajab, R. Ahmed, F. Shehzad, and M. H. Tahir, Some new constructions of circular balanced repeated measurements designs. Communications in Statistics-Theory and Methods, **47**, 4142-4151 (2018).
[10] U. Rasheed, H. M. K. Rasheed, M. Rasheed, and R. Ahmed, Minimal circular strongly balanced repeated measurements designs in periods of three different sizes. Communications in Statistics-Theory and Methods, **47**, 4088-4094 (2018).
[11] R. Ahmed, F. Shehzad, M. Rajab, M. Daniyal, and M. H. Tahir, Minimal circular balanced repeated measurements designs in periods of unequal sizes. Communications in Statistics-Theory and Methods, **48**(21), 5223-5232 (2019).
[12] M. Daniyal, R. Ahmed, F. Shehzad, M. H. Tahir, and Z. Iqbal, Construction of repeated measurements designs strongly balanced for residual effects. Communications in Statistics-Theory and Methods, **49**(17), 4288-4297 (2019).
[13] A. Khan, R. Ahmed, F. Shehzad, M. H. Tahir, and S. S. A. Ghazali, Construction of circular partially balanced repeated measurement designs. Communications in Statistics - Simulation and Computation, **48**(2), 506-515 (2019).
[14] R. Jabeen, R. Ahmed, M. Sajjad, H. M. K. Rasheed, and A. Khan, Circular strongly partially-balanced repeated measurement designs in periods of two different sizes using method of cyclic shifts (Rule II). Journal of King Saud University-Science, **31**, 519-524 (2019b).
[15] Y. Nazeer, R. Ahmed, R. Jabeen, M. Daniyal, and M. H. Tahir, Circular strongly partially-balanced repeated measurements designs in periods of two different sizes. Journal of Probability and Statistical Science, **17**, 85-95 (2019).
[16] R. Ahmed, M. H. Tahir, M. Rajab, and M. Daniyal, An overview on method of cyclic shifts for the construction of experimental designs useful in business and commerce. Pakistan Journal of Commerce and Social Sciences, **14**(1), 50-63 (2020).
[17] I. Iqbal, Construction of experimental design using cyclic shifts, Ph.D. Thesis, University of Kent at Canterbury, U.K (1991).
[18] J. Divecha, and J. Gondaliya, Construction of minimal balanced crossover designs having good efficiency of separability. Electronic Journal of Statistics, **8**, 2923-36 (2014).
[19] R. Jabeen, R. Ahmed, H. M. K. Rasheed, and F. Shehzad, Construction of circular strongly partially balanced repeated measurements designs. Journal of King Saud University-Science, **31**, 345-351 (2019a).