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# Poisson-Odoma Distribution

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**Abstract:** In this paper we propose the probability mass function of the discrete Poisson-Odoma distribution. Then, the hazard function is studied along with some moments based statistical measures are studied such as coefficient of variation, skewness, and kurtosis. Maximum likelihood estimation method is used for estimating the distribution parameter.

**Keywords:** Poisson distribution, Odoma distribution, Moments, Hazard function, Skewness, Kurtosis, Maximum likelihood.

## 1 Introduction

Poisson distribution is the most common discrete probability distribution used for modeling count-data mainly when the mean and variance of the data are equal, but otherwise Poisson distribution is not suitable to use Poisson distribution (see e.g. [4]). Therefore, researchers in the past decades have been introducing new discrete probability distributions using the Poisson mixture of well known lifetime distribution.(see e.g. [1,3,5]).

Recently, [2] introduced a new continuous one parameter lifetime distribution named the Odoma distribution (OD) defined by its probability density function (pdf)

$$g(x) = \frac{\theta^5}{2(\theta^5+\theta^3+24)}(2x^4+\theta x^2+2\theta)e^{-\theta x} \qquad ;x>0,\ \theta>0. \tag{1}$$

with cumulative density function (cdf)

$$G(x) = 1-\left[1+\frac{2\theta^2x^2(\theta^2x^2+4\theta x+12)+\theta x(\theta^4 x+2\theta^3+48)}{2(\theta^5+\theta^3+24)}\right]e^{-\theta x}. \tag{2}$$

In this paper introduce a Poisson mixture of Odoma distribution and study some of its mathematical properties, such as, the probability mass function (pmf), the hazard function, the first four moments about origin and the variance. Also, the maximum likelihood estimation method (MLE) is used to estimate the distribution parameter of the new distribution.

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## 2 Poisson-Odoma Distribution

A random variable $X$ is said to follows Poisson-Odoma distribution (POD) if the parameter $\lambda$ of Poisson distribution follows Odoma distribution. Hence, the density mass function (pmf) of POD can be obtained as follows

$$\begin{aligned} P(X=x) &= \int_0^{\infty} \frac{e^{-\lambda}\lambda^x}{x!}\left[\frac{\theta^5}{2\,(\theta^5+\theta^3+24)}(2\,\lambda^4+\theta\,\lambda^2+2\,\theta)e^{-\theta\lambda}\right].d\lambda, x=0,1,\ldots \\ &= \frac{\theta^5}{2\,x!\,(\theta^5+\theta^3+24)}\int_0^{\infty}(2\,\lambda^{x+4}+\theta\,\lambda^{x+2}+2\,\theta\,\lambda^x)e^{-\lambda(1+\theta)}.\,d\lambda, \\ &= \frac{\theta^5}{2\,x!\,(\theta^5+\theta^3+24)}\left[2\int_0^{\infty}\lambda^{x+4}e^{-\lambda(1+\theta)}.\,d\lambda+\theta\int_0^{\infty}\lambda^{x+2}e^{-\lambda(1+\theta)}.\,d\lambda \right. \\ &\qquad \left. +2\,\theta\int_0^{\infty}\lambda^x e^{-\lambda(1+\theta)}.\,d\lambda\right], \\ &= \frac{\theta^5}{2\,x!\,(\theta^5+\theta^3+24)}\left[\frac{2\,\Gamma(x+5)}{(\theta+1)^{x+5}}+\frac{\theta\,\Gamma(x+3)}{(\theta+1)^{x+3}}+\frac{2\,\theta\,\Gamma(x+1)}{(\theta+1)^{x+1}}\right], \\ &= \frac{\theta^5\left[(x+2)(x+1)\left[2(x+4)(x+3)+\theta(1+\theta)^2\right]+2\theta(1+\theta)^4\right]}{2\,(\theta^5+\theta^3+24)(1+\theta)^{(x+5)}}. \end{aligned} \tag{3}$$

The behavior of the pmf of POD for different values of $\theta$ are shown in Figure 1.

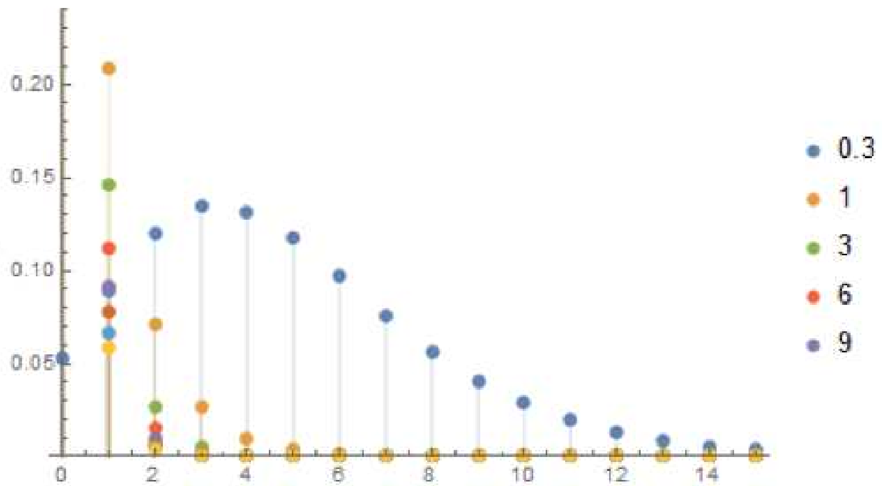


**Fig. 1:** The pmf of POD for different values of $\theta = 0.3, 1, 3, 6, 9$

## 3 Hazard Function

Let $X$ be a POD discrete random variable that takes the value $x_1 < x_2 < x_3 < \ldots$, Then we define the survivor function at $x_j$ as

$$
\begin{aligned}
S(x_j) &= P(X \geq x_j) \\
&= \sum_{k=j}^{\infty} P(X = x_j) \\
&= \frac{\theta^5}{2\left(\theta^5+\theta^3+24\right)} \sum_{k=j}^{\infty} \frac{\left[(x_k+2)(x_k+1)\left[2(x_k+4)(x_k+3)+\theta(1+\theta)^2\right]+2\theta(1+\theta)^4\right]}{(1+\theta)^{(x_k+5)}}.
\end{aligned}
$$

Next, the hazard function for POD is given by

$$
\begin{aligned}
H(x_j) &= P(X = x_j / X \geq x_j) \\
&= \frac{P(X = x_j)}{S(x_j)} \\
&= \frac{(x_j+2)(x_j+1)\left[2(x_j+4)(x_j+3)+\theta(1+\theta)^2\right]+2\theta(1+\theta)^4}{(1+\theta)^{(x_j+5)} \sum_{k=j}^{\infty} \frac{\left[(x_k+2)(x_k+1)\left[2(x_k+4)(x_k+3)+\theta(1+\theta)^2\right]+2\theta(1+\theta)^4\right]}{(1+\theta)^{(x_k+5)}}}.
\end{aligned}
$$

The behavior of the hazard function of POD for different values of $\theta$ are shown in Figure 2.

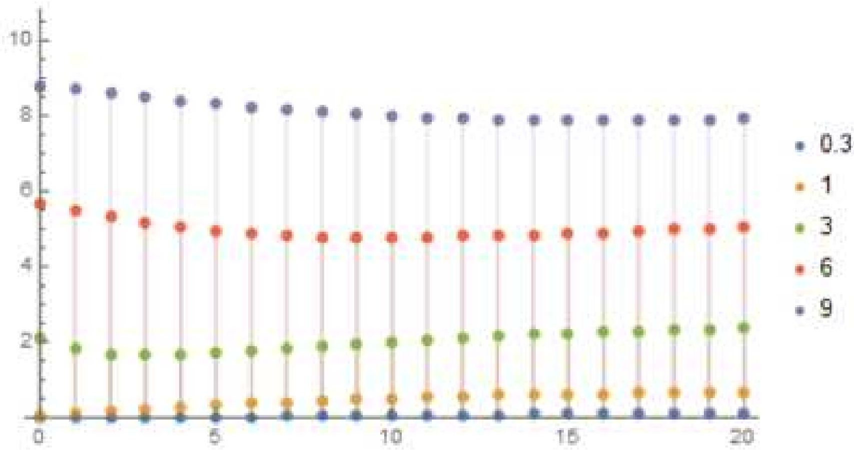


**Fig. 2:** The hazard function of POD for different values of $\theta = 0.3, 1, 3, 6, 9$

## 4 Moments

The $r^{th}$ factorial moment about origin of POD is given by

$$\mu'_{(r)} = E[E(X^{(r)}/\lambda)]$$

where

$$X^{(r)} = X(X-1)(X-2)\ldots(X-r+1)$$

such that

$$
\begin{aligned}
\mu'_{(r)} &= E[E(X^{(r)}/\lambda)] \\
&= \sum_{x=0}^{\infty} x^{(r)} P(X=x) \\
&= \sum_{x=0}^{\infty} x^{(r)} \int_0^{\infty} \left[\frac{e^{-\lambda}\lambda^x}{x!}\right] \left[\frac{\theta^5}{2(\theta^5+\theta^3+24)}(2\lambda^4+\theta\lambda^2+2\theta)e^{-\theta\lambda}\right].\, d\lambda \\
&= \frac{\theta^5}{2(\theta^5+\theta^3+24)} \int_0^{\infty} \sum_{x=0}^{\infty} x^{(r)} \left[\frac{e^{-\lambda}\lambda^x}{x!}\right] (2\lambda^4+\theta\lambda^2+2\theta)e^{-\theta\lambda}.\, d\lambda \\
&= \frac{\theta^5}{2(\theta^5+\theta^3+24)} \int_0^{\infty} \lambda^r \sum_{x=r}^{\infty} \left[\frac{e^{-\lambda}\lambda^{x-r}}{(x-r)!}\right] (2\lambda^4+\theta\lambda^2+2\theta)e^{-\theta\lambda}.\, d\lambda \\
&= \frac{\theta^5}{2(\theta^5+\theta^3+24)} \int_0^{\infty} (2\lambda^{r+4}+\theta\lambda^{r+2}+2\theta\lambda^r)e^{-\theta\lambda}.\, d\lambda \\
&= \frac{r!}{2(\theta^{r+5}+\theta^{r+3}+24\theta^r)} \left[(r+2)(r+1)\left[2(r+4)(r+3)+\theta^3\right]+2\theta^5\right].
\end{aligned}
\tag{4}
$$

Now, for $r = 1,2,3,4$ we have

$$
\begin{aligned}
\mu'_{(1)} &= \frac{120+3\theta^3+\theta^5}{\theta(\theta^5+\theta^4+24)}, \\
\mu'_{(2)} &= \frac{2(360+6\theta^3+\theta^5)}{\theta^2(\theta^5+\theta^4+24)}, \\
\mu'_{(3)} &= \frac{6(840+10\theta^3+\theta^5)}{\theta^3(\theta^5+\theta^4+24)}, \\
\mu'_{(4)} &= \frac{24(1680+105\theta^3+\theta^5)}{\theta^4(\theta^5+\theta^4+24)}.
\end{aligned}
\tag{5}
$$

Using the relation between the $r^{th}$ factorial moment about origin and the raw moments we can calculate the following moments

$$
\begin{aligned}
\mu_0 &= 1, \\
\mu_1 &= 0, \\
\mu_2 &= \mu'_{(2)} - (\mu'_{(1)})^2 \\
&= \frac{2(360+6\theta^3+\theta^5)}{\theta^2(\theta^5+\theta^4+24)} - \frac{(\theta^5+\theta^3+24)^2}{\theta^2(\theta^5+\theta^4+24)^2}, \\
\\
\mu_3 &= \mu'_{(3)} - 3\mu'_{(1)}\mu'_{(2)} + 2(\mu'_{(1)})^2 \\
&= \frac{6(840+10\theta^3+\theta^5)}{\theta^3(\theta^5+\theta^4+24)} - \frac{6(120+3\theta^3+\theta^5)(360+6\theta^3+\theta^5)}{\theta^3(\theta^5+\theta^4+24)^2} \\
&\quad + \frac{2(120+3\theta^3+\theta^5)^3}{\theta^3(\theta^5+\theta^4+24)^3}, \\
\\
\mu_4 &= \mu'_{(4)} - 4\mu'_{(3)}\mu'_{(1)} + 6\mu'_{(2)}(\mu'_{(1)})^2 - 3(\mu'_{(1)})^4 \\
&= \frac{24(1680+105\theta^3+\theta^5)}{\theta^4(\theta^5+\theta^4+24)} - \frac{24(120+3\theta^3+\theta^5)(840+10\theta^3+\theta^5)}{\theta^4(\theta^5+\theta^4+24)^2} \\
&\quad + \frac{12(360+6\theta^3+\theta^5)(120+3\theta^3+\theta^5)^2}{\theta^4(\theta^5+\theta^4+24)^3} - \frac{3(120+3\theta^3+\theta^5)^4}{\theta^4(\theta^5+\theta^4+24)^4}.
\end{aligned}
\tag{6}
$$

The variance of POD can be obtained as

$$\sigma^2 = \mu_2 = \frac{2(360+6\theta^3+\theta^5)}{\theta^2(\theta^5+\theta^4+24)} - \frac{(\theta^5+\theta^3+24)^2}{\theta^2(\theta^5+\theta^4+24)^2}.$$

## 5 Moments Based Measures

The coefficient of variation $(CV)$, index of dispersion $(\gamma)$, coefficient of Skewness $(\sqrt{(\beta_1)})$ and coefficient of Kurtosis $(\beta_2)$ of POD are given by respectively

$$CV = \frac{\sigma}{\mu'_{(1)}} = \frac{\sqrt{\frac{2(360+6\theta^3+\theta^5)}{\theta^2(\theta^5+\theta^4+24)} - \frac{(\theta^5+\theta^3+24)^2}{\theta^2(\theta^5+\theta^4+24)^2}}}{\frac{120+3\theta^3+\theta^5}{\theta(\theta^5+\theta^4+24)}}$$

$$\gamma = \frac{\sigma^2}{\mu'_{(1)}} = \frac{\frac{2(360+6\theta^3+\theta^5)}{\theta^2(\theta^5+\theta^4+24)} - \frac{(\theta^5+\theta^3+24)^2}{\theta^2(\theta^5+\theta^4+24)^2}}{\frac{120+3\theta^3+\theta^5}{\theta(\theta^5+\theta^4+24)}}$$

$$\sqrt{(\beta_1)} = \frac{\mu_3}{[\mu_2]^{3/2}} = \frac{\frac{6(840+10\theta^3+\theta^5)}{\theta^3(\theta^5+\theta^4+24)} - \frac{6(120+3\theta^3+\theta^5)(360+6\theta^3+\theta^5)}{\theta^3(\theta^5+\theta^4+24)^2} + \frac{2(120+3\theta^3+\theta^5)^3}{\theta^3(\theta^5+\theta^4+24)^3}}{\left[\frac{2(360+6\theta^3+\theta^5)}{\theta^2(\theta^5+\theta^4+24)} - \frac{(\theta^5+\theta^3+24)^2}{\theta^2(\theta^5+\theta^4+24)^2}\right]^{3/2}}$$

$$\beta_2 = \frac{\mu_4}{[\mu_2]^2} = \frac{\frac{24(1680+105\theta^3+\theta^5)}{\theta^4(\theta^5+\theta^4+24)} - \frac{24(120+3\theta^3+\theta^5)(840+10\theta^3+\theta^5)}{\theta^4(\theta^5+\theta^4+24)^2}}{\left[\frac{2(360+6\theta^3+\theta^5)}{\theta^2(\theta^5+\theta^4+24)} - \frac{(\theta^5+\theta^3+24)^2}{\theta^2(\theta^5+\theta^4+24)^2}\right]^2} + \frac{\frac{12(360+6\theta^3+\theta^5)(120+3\theta^3+\theta^5)^2}{\theta^4(\theta^5+\theta^4+24)^3} - \frac{3(120+3\theta^3+\theta^5)^4}{\theta^4(\theta^5+\theta^4+24)^4}}{\left[\frac{2(360+6\theta^3+\theta^5)}{\theta^2(\theta^5+\theta^4+24)} - \frac{(\theta^5+\theta^3+24)^2}{\theta^2(\theta^5+\theta^4+24)^2}\right]^2}$$

Table 1 shows the values of the coefficient of variation, index of dispersion, coefficient of Skewness and coefficient of Kurtosis of POD for different values of $\theta$.

**Table 1:** Moments based measures of POD with $\theta = 0.5, 1, 1.5, 2.4.8$.

| $\theta$ | 0.5 | 1 | 1.5 | 2 | 4 | 8 |
|---|---|---|---|---|---|---|
| $CV.$ | 1.0765 | 1.0942 | 1.1978 | 1.3832 | 1.3750 | 1.1479 |
| $\gamma$ | 11.5832 | 5.7097 | 3.5935 | 2.3384 | 0.4842 | 0.1537 |
| $\sqrt{(\beta_1)}$ | 0.0650 | 0.0663 | 0.1296 | 0.4894 | 1.9094 | 1.9766 |
| $\beta_2$ | 0.1375 | 0.2736 | 0.8862 | 3.1068 | 35.5514 | 31.7830 |

## 6 Estimation of Parameter

Suppose $(x_1, x_2, \ldots, x_n)$ is a random sample of size $n$ from POD and suppose $f_x$ be the observed frequency in the sample corresponding to $X = x$ $(x = 1, 2, 3, \ldots, k)$ such that $\sum_{x=1}^{k} f_x = n$,where$k$ is the largest observed value having non-zero frequency. The likelihood function $L$ of POD is given by

$$L = \left[\frac{\theta^5}{2(\theta^5+\theta^3+24)}\right]^n \frac{1}{(1+\theta)^{\sum_{x=1}^{k} f_x(x+5)}}$$
$$\prod_{x=1}^{k}\left[(x+2)(x+1)\left[2(x+4)(x+3)+\theta(1+\theta)^2\right]+2\theta(1+\theta)^4\right]^{f_x}.$$

The log likelihood function is obtained as

$$logL = n\ log\left[\frac{\theta^5}{2(\theta^5+\theta^3+24)}\right] - \sum_{x=1}^{k} f_x(x+5)\ log(1+\theta)$$
$$+\sum_{x=1}^{k} f_x\ log\left[(x+2)(x+1)\left[2(x+4)(x+3)+\theta(1+\theta)^2\right]+2\theta(1+\theta)^4\right].$$

The first derivative of the log likelihood function is given by

$$\frac{logL}{d\theta} = \frac{n(2\theta^3+120)}{\theta(\theta^5+\theta^3+24)} - \frac{n(\bar{x}+5)}{1+\theta}$$
$$+\sum_{x=1}^{k} f_x\left[\frac{(x+2)(x+1)(3\theta^2+4\theta+1)+2(1+5\theta)(1+\theta)^3}{(x+2)(x+1)\left[2(x+4)(x+3)+\theta(1+\theta)^2\right]+2\theta(1+\theta)^4}\right].$$

where $\bar{x}$ is the sample mean. The maximum likelihood estimate (MLE), $\hat{\theta}$ of $\theta$ of POD is the solution of the following non-linear equation

$$\frac{n(2\theta^3+120)}{\theta(\theta^5+\theta^3+24)} - \frac{n(\bar{x}+5)}{1+\theta}+$$
$$\sum_{x=1}^{k} f_x\left[\frac{(x+2)(x+1)(3\theta^2+4\theta+1)+2(1+5\theta)(1+\theta)^3}{(x+2)(x+1)\left[2(x+4)(x+3)+\theta(1+\theta)^2\right]+2\theta(1+\theta)^4}\right] = 0$$

This non-linear equation can be solved by any numerical iteration methods. Table 2 shows simulation results for the MLE of POD for different values of $\theta$ and different sample size $n$.

**Table 2:** Simulation results for the MLE of POD with $\theta = 3,6,8$ and $n = 50,200,500$.

| $\theta$ | | **3** | | | **6** | | | **8** | |
|---|---|---|---|---|---|---|---|---|---|
| $n$ | 50 | 200 | 500 | 50 | 200 | 500 | 50 | 200 | 500 |
| $\hat{\theta}$ | 2.2948 | 4.8126 | 6.3729 | 2.3109 | 4.8178 | 6.3719 | 2.3018 | 4.8043 | 6.3627 |
| *Bais* | 0.7192 | 1.1911 | 1.6283 | 0.6892 | 1.1822 | 1.6281 | 0.6982 | 1.1957 | 1.6372 |
| *MSE* | 0.6935 | 1.6019 | 2.9409 | 0.5211 | 1.4462 | 2.7287 | 0.5059 | 1.4490 | 2.7083 |
| *MRE* | 0.2397 | 0.1985 | 0.2035 | 0.2297 | 0.1970 | 0.2035 | 0.2327 | 0.1993 | 0.2047 |

## 7 Conclusions

Poisson Odoma distribution is a new Poisson mixture of Odoma distribution. We obtained the explicit expressions for the probability mass function Poisson Odoma distribution, the hazard function, the first four moments about origin and the variance. We used the maximum likelihood estimation method (MLE) estimate the distribution parameter.

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## Conflict of interest

The authors declare that they have no conflict of interest.

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