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Chaotic Dynamics and Synchronization of Cournot Duopoly Game with a Logarithmic Demand Function

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Abstract: This paper analyzed the dynamics of Cournot duopoly game with a logarithmic demand function. We assumed that the owners of both firms played with bounded rationality expectation. The existence of equilibrium points and its local stability of the output game are investigated. The complex dynamics, bifurcations and chaos are displayed by numerical experiments. Numerical methods showed that the higher values of speeds of adjustment act on the Nash equilibrium that becomes unstable through period doubling bifurcations, more complex attractors are created around it. The chaotic behavior of the game has been controlled by using feedback control method. we investigated the mechanisms that lead the firms to behave in the same way in the long run (synchronization phenomena).

Keywords: Cournot duopoly game; Homogeneous players; Logarithmic inverse demand functions; Nash Equilibrium point; Bifurcation; Chaos; Feedback control method; synchronization; Natural Lyapunov exponent.

1 Introduction

Expectations play an important role in modeling economic phenomena. A producer can choose his expectations rules of many available techniques to adjust his production outputs. There exist three different firms'expectations: naive, bounded rational and adaptive. Recently, several works have considered more realistic mechanisms through which players form their expectations about decisions of competitors, the game with bounded rationality has been the hot spot of research. Generalization of the duopoly model of Bowley to the case of cost function with linear terms has been discussed in [1]. The dynamics of a nonlinear Cournot duopoly with managerial delegation and homogeneous players based on the bounded rational expectation has been studied by Luciano Fanti and Luca Gori [2]. The complex dynamic features of a nonlinear mixed Cournot model with bounded rationality where one semipublic firm endeavors to maximize the weighted average on social welfare and its own profit while the other private firm only intends to maximize its own profit has been considered in [3]. Wang and Ma 2013 [4] analyzed the Cournot-Bertrand mixed duopoly game model with limited information about the market, where the market has linear demand and two firms have the same fixed marginal cost. Duopolists with dynamic adjustment behavior of bounded rationality based on constant conjectural variation have been considered in [5]. Dynamics of a four dimensional system which generalizes the classical Cournot competition in a local way has been analyzed in [6].

Currently, under the assumption of bounded rationality, the research result of the duopoly game with heterogeneous and homogeneous players (as regards the type of expectations' formation) has been widely used in realistic problems of quantity competition see in ([7], [8], [9], [10], [11] and [12]).

The Bertrand's economic competition model which describes a duopoly market in which two firms are competing with each other through a price war for maximizing there profits. A duopoly game was modelled by two nonlinear difference equations with bounded rational has been done in ([13], [14], [15]). A Bertrand duopoly model with heterogeneous players in which each of duopoly firm sets its optimal product's price by competitors' price has been formulated in ([16], [17] and [18]).

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There exists large number of applications that deal with quantity and price competitions of homogeneous or heterogeneous products in static oligopoly, for examples: the mechanisms of carbon emission trading is formulated in [17], works in topics within electricity market system literature ([19], [20]) and airlines' price competition modelled in [21]. Mathematical modelling for an economy viewed as a transport network for energy in which the law of motion of capital occurs with a time delay and by choosing time delay as a bifurcation parameter, see [22].

In the present paper, we endeavor to bring the duopoly game model of bounded rationality with homogeneous players and set up a logarithmic inverse demand function. And the other objective of this paper is to study the complex dynamic features as (local stability of equilibrium points, behaviors under some change of control parameters of the game, control chaos of the system and the phenomenon of synchronization).

The general scheme of this paper is as follows. In Section (2) we determine the dynamical system of a Cournot duopoly game with bounded rationality by a two-dimensional map. In Section (3) we study the existence and local stability of equilibrium points of duopoly game. In Section (4) we simulate complex dynamic of this system via changing control parameters of the model. Delay feedback control method is proposed to control chaos of the system in Section (5). The aim of Section (6) is to analyze a synchronization phenomena of the model. Finally, conclusions are drawn in Section (7).

2 Model

We assume the existence of an economy model with two types of agents: firms and consumers. There exists a duopolistic sector with two firms, firm 1 and firm 2, every firm i produces goods which are perfect substitutes in a oligopoly market. Let the price and the quantity are given by p_i and q_i, respectively with $i=\{1,2\}$.

The nonlinear inverse demand functions of products of variety 1 and 2 (as a function of quantities) are determined from the logarithmic total supply function

$$Q(t)=\ln(q_i(t)+q_j(t)),\ i\neq j \text{ and } i,j=1,2$$

in period t are given by the following equations:

$$p_1(q_1,q_2)=a-b\ \ln(q_1+q_2), \tag{1}$$

$$p_2(q_1,q_2)=a-b\ \ln(q_1+q_2),\ a>0,\ b>0. \tag{2}$$

The firm $i's$ cost function is linear and described by:

$$C_i(q_i)=c_iq_i, \qquad i=1,2 \tag{3}$$

where the positive parameters c_i are the marginal costs. The Profits of firm i in every period can be written as follows:

$$\pi_i\left(q_i,q_j\right)=p_i\left(q_i,q_j\right)q_i-C_i(q_i). \tag{4}$$

From the profit maximization by firm $i=\{1,2\}$, marginal profits are obtained as:

$$\frac{\partial\pi_1\left(q_1,q_2\right)}{q_1}=a-b\left(\frac{q_1}{q_1+q_2}+\ln(q_1+q_2)\right)-c_1, \tag{5}$$

$$\frac{\partial\pi_2\left(q_1,q_2\right)}{q_2}=a-b\left(\frac{q_2}{q_1+q_2}+\ln(q_1+q_2)\right)-c_2. \tag{6}$$

In order to get the maximum profit, every firm carries out the output decision-making. In this work, we consider two firm with same expectation. Suppose that the two firms with bounded rationality adjusts which production based on a local estimate of the marginal profit $\frac{\partial\pi_i\left(q_i,q_j\right)}{q_i}$.

Therefore, given this type of expectations formation mechanisms, the two-dimensional system that characterizes the dynamics of the economic model is the following:

$$\begin{cases} q_1(t+1)=q_1\left(t\right)+\alpha_1q_1\left(t\right)\left[a-b\left(\frac{q_1}{q_1+q_2}+\ln(q_1+q_2)\right)-c_1\right] \\ q_2(t+1)=q_2\left(t\right)+\alpha_2q_2\left(t\right)\left[a-b\left(\frac{q_2}{q_1+q_2}+\ln(q_1+q_2)\right)-c_2\right] \end{cases} \tag{7}$$

where the parameters $\alpha_1,\alpha_2>0$, stand for the rates of adjustment.

3 Dynamics analysis of the model

3.1 Existence of the equilibrium points

From an economic point of view we are only interested to study the local stability properties of the unique positive output equilibrium, which is determined by setting $q_1\left(t+1\right)=q_1\left(t\right)=q_1$ and $q_2\left(t+1\right)=q_2\left(t\right)=q_2$ in Eq.(7) and solving the following algebraic nonnegative solution:

$$\begin{cases} q_1\left(t\right)\left[a-b\left(\frac{q_1}{q_1+q_2}+\ln(q_1+q_2)\right)-c_1\right]=0 \\ q_2\left(t\right)\left[a-b\left(\frac{q_2}{q_1+q_2}+\ln(q_1+q_2)\right)-c_2\right]=0 \end{cases} \tag{8}$$

We can have four fixed points: the boundary equilibria

$$E_1(0,0),\ E_2\left(0,e^{\frac{a-b-c_2}{b}}\right),\ E_3\left(e^{\frac{a-b-c_1}{b}},0\right) \tag{9}$$

and the unique Nash equilibrium $E_4(q_1^*,q_2^*)$ where,

$$q_1^*=\frac{(b-c_1+c_2)\,e^{\frac{2a-b-c_1-c_2}{2b}}}{2b},\quad q_2^*=\frac{(b+c_1-c_2)\,e^{\frac{2a-b-c_1-c_2}{2b}}}{2b} \tag{10}$$

provided that

$$\begin{cases} b>c_1-c_2 \\ b>c_2-c_1. \end{cases} \tag{11}$$

The study of the local stability of fixed points is based on the localization, on the complex plane of the eigenvalues of the Jacobian matrix of the two-dimensional map Eq.(8). The Jacobian matrix has the form:

$$J_{(q_1,q_2)} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}, \text{where} \tag{12}$$

$$\begin{aligned} J_{11} &= 1+\alpha_1(a-c_1) \\ &\quad -\alpha_1 b\left[\frac{3q_1}{q_1+q_2}+\ln(q_1+q_2)-\frac{q_1^2}{q_1+q_2}\right] \\ J_{12} &= \frac{-\alpha_1 b q_1 q_2}{(q_1+q_2)^2} \\ J_{21} &= \frac{-\alpha_2 b q_1 q_2}{(q_1+q_2)^2} \\ J_{22} &= 1+\alpha_2(a-c_2) \\ &\quad -\alpha_2 b\left[\frac{3q_2}{q_1+q_2}+\ln(q_1+q_2)-\frac{q_2^2}{q_1+q_2}\right] \end{aligned}$$

3.2 Stability analysis of the boundary equilibria

The equilibrium E_1 has no economic implications, so we exclude E_1 from the analysis.

Theorem 1. *Both boundary equilibrium E_2 and E_3 are unstable.*

Proof. An equilibrium is stable if and only if all eigenvalues of the related Jacobian matrix are less than one in the absolute value. The Jacobian matrix at equilibrium E_2 takes the form of

$$J(E_2) = \begin{bmatrix} 1+\alpha_1(b-c_1+c_2) & 0 \\ 0 & 1-\alpha_2 b \end{bmatrix} \tag{13}$$

which has two eigenvalues

$$\lambda_1 = 1-\alpha_2 b$$
$$\lambda_2 = 1+\alpha_1(b-c_1+c_2)$$

since the parameters a, b, c_i, α_i , $i=\{1,2\}$ are positive parameters and according to Eq.(11) which obviously imply, $\lambda_1 < 1$ and $\lambda_2 > 1$.Hence the unstable equilibrium E_2 is a saddle point. At the same time, equilibrium E_3 one can get the Jacobian matrix

$$J(E_3) = \begin{bmatrix} 1-\alpha_1 b & 0 \\ 0 & 1+\alpha_2(b+c_1-c_2) \end{bmatrix} \tag{14}$$

We are able to derive two related eigenvalues of such Jacobian matrix

$$\lambda_1 = 1-\alpha_1 b$$
$$\lambda_2 = 1+\alpha_2(b+c_1-c_2)$$

obviously $\lambda_1 < 1$ and $\lambda_2 > 1$.Then the unstable equilibrium E_3 is a saddle.

Theorem 2. *The Nash equilibrium point $E_4(q_1^*,q_2^*)$ of system (7) is stable if condition (20) defined by*

$4-\left(\frac{\alpha_1+\alpha_2}{4b}\right)\left(3b^2-(c_1-c_2)^2\right)+\left(\frac{\alpha_1-\alpha_2}{2}\right)(c_1-c_2) > \left(\frac{\alpha_1+\alpha_2}{4b}\right)\left(3b^2-(c_1-c_2)^2\right) - \left(\frac{\alpha_1-\alpha_2}{2}\right)(c_1-c_2) - \frac{\alpha_1\alpha_2}{2}\left(b^2-(c_1-c_2)^2\right) > 0$ holds.

Proof. One can simplify the Jacobian matrix at the Nash equilibrium as follows:

$$J(E_4) = \begin{bmatrix} J_{11}(E_4) & J_{12}(E_4) \\ J_{21}(E_4) & J_{22}(E_4) \end{bmatrix}, \textit{where} \tag{15}$$

$$\begin{aligned} J_{11}(E_4) &= 1-\alpha_1(b-c_1+c_2)\left[1-\frac{(b-c_1+c_2)}{4b}\right], \\ J_{12}(E_4) &= -\alpha_1\frac{(b+c_2-c_1)}{2}\left[1-\frac{(b-c_1+c_2)}{2b}\right], \\ J_{21}(E_4) &= -\alpha_2\frac{(b-c_2+c_1)}{2}\left[1-\frac{(b+c_1-c_2)}{2b}\right], \\ J_{22}(E_4) &= 1-\alpha_2(b+c_1-c_2)\left[1-\frac{(b+c_1-c_2)}{4b}\right] \end{aligned} \tag{16}$$

The characteristic equations of the $J(E_4)$ has the form

$$P(\lambda) = \lambda^2 - Tr(J_{E_4})\lambda + Det(J_{E_4}) = 0$$

where $Tr(J_{E_4})$ is the trace and $Det(J_{E_4})$ is the determinant of the Jacobian matrix which are given by:

$$\begin{aligned} Tr(J_{E_4}) &= 2-\left(\frac{\alpha_1+\alpha_2}{4b}\right)\left(3b^2-(c_1-c_2)^2\right) \\ &\quad +\left(\frac{\alpha_1-\alpha_2}{2}\right)(c_1-c_2) \end{aligned} \tag{17}$$

$$\begin{aligned} Det(J_{E_4}) &= 1-\left(\frac{\alpha_1+\alpha_2}{4b}\right)\left(3b^2-(c_1-c_2)^2\right) \\ &\quad +\left(\frac{\alpha_1-\alpha_2}{2}\right)(c_1-c_2) \\ &\quad +\frac{\alpha_1\alpha_2}{2}\left(b^2-(c_1-c_2)^2\right) \end{aligned}$$

Note that E_4 is stable if and only if the following Jury's conditions are satisfied [15] which are,

$$\begin{cases} i)\ 1-Tr(J_{E_4})+Det(J_{E_4}) > 0 \\ ii)\ 1+Tr(J_{E_4})+Det(J_{E_4}) > 0 \\ iii)\ 1-Det(J_{E_4}) > 0. \end{cases} \tag{18}$$

Then Jury's conditions become

$$\begin{cases} i)\ 2\alpha_1\alpha_2(b^2-(c_1-c_2)^2) > 0, \\ ii)\ 4-\left(\frac{\alpha_1+\alpha_2}{2b}\right)\left(3b^2-(c_1-c_2)^2\right)+(\alpha_1-\alpha_2)(c_1-c_2) \\ \quad +\frac{\alpha_1\alpha_2}{2}\left(b^2-(c_1-c_2)^2\right) > 0, \\ iii)\ \left(\frac{\alpha_1+\alpha_2}{4b}\right)\left(3b^2-(c_1-c_2)^2\right)-\frac{1}{2}(\alpha_1-\alpha_2)(c_1-c_2) \\ \quad -\frac{\alpha_1\alpha_2}{2}\left(b^2-(c_1-c_2)^2\right) > 0. \end{cases} \tag{19}$$

The first condition is satisfied. Then the second and third conditions become,

$$4-\frac{(\alpha_1+\alpha_2)}{4b}\left(3b^2-(c_1-c_2)^2\right)+\frac{(\alpha_1-\alpha_2)}{2}(c_1-c_2)>$$
$$\frac{(\alpha_1+\alpha_2)}{4b}\left(3b^2-(c_1-c_2)^2\right)-$$
$$\frac{(\alpha_1-\alpha_2)}{2}(c_1-c_2)-\frac{\alpha_1\alpha_2}{2}\left(b^2-(c_1-c_2)^2\right)>0 \quad (20)$$

This equation defines a region of stability in the plane of the rates of adjustment (α_1,α_2).

4 Numerical illustration

The main purpose of this section is to show the qualitative behavior of the solutions of the duopoly game with homogeneous players described by the dynamic system Eq. (7). We just focus on the Nash equilibrium $E_4(q_1^*,q_2^*)$, inequality (20) define the region of stability in the plane of the speed of adjustments (α_1,α_2). Assume that $a=4$, $b=3$, $c_1=0.01$, $c_2=0.2$, then we can get the region of stability of Nash equilibrium point, which is shown in Fig. (1). If we set all parameters as $a=6$, $b=4$, $c_1=1$, $c_2=2$ we could see the region of stability in Fig. (2).

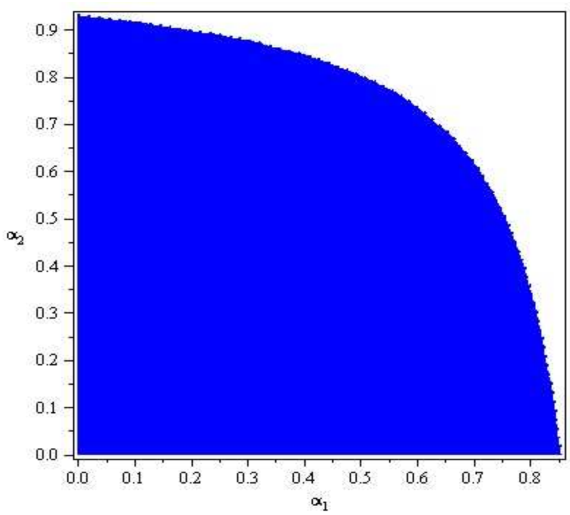

Fig. 1: Region of stability of Nash equilibrium in the plane of the speed of adjustments at $c_1 = 0.01, c_2 = 0.2$.

These figures reveal that the stable area decreases in the direction of (α_1,α_2) when the two firms increase the values of costs. So the higher values of costs and the market capacity a make the region stability area for the market small. Figures (3 & 4) present a bifurcation diagram of system (7) in $(\alpha_1-q_1q_2)$ plane for varies

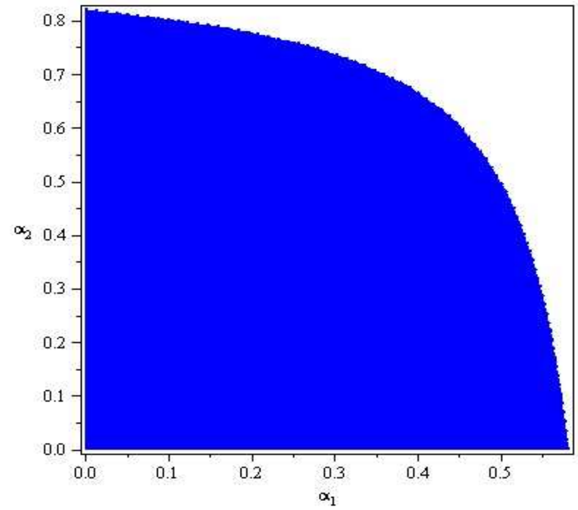

Fig. 2: Region of stability of Nash equilibrium in the plane of the speed of adjustments at $c_1 = 1, c_2=2$.

parameters. Fig. (3) depicts the bifurcation diagram ofthe two-dimensional map when $a=6$, $b=4$, $c_1=1$, $c_2=2$, such we see that the Nash equilibrium approaches to the stable fixed point for $\alpha_1<0.548$ $(\alpha_2=0.3)$.

Then if $\alpha_1>0.548$ the Nash equilibrium becomes unstable. A flip bifurcation (period doubling) takes place at the market when $\alpha_1=0.548$. Fig. (4) shows the bifurcation diagram for $a=4$, $b=3$, $c_1=0.01$, $c_2=2$,since the Period doubling bifurcations appears at $\alpha_1=0.658$. As long as the parameter α_1 increases, the Nash equilibrium $E_4(q_1^*,q_2^*)$ becomes unstable and the bifurcation scenario occurs and ultimately leads to unpredictable (chaotic) motions that make decision in the future for the two firms to their outputs very difficult.

Figures (5 & 6) show the bifurcation diagram with respect to the parameter α_2 (speed of adjustment of bounded rational firm 2). From Fig. (5) we can see the orbit with initial values (0.01,0.02) while the other parameters $a=6$, $b=4$, $c_1=1$, $c_2=2$ are fixed approaches to the stable fixed point $E_4(\frac{5}{8}e^{\frac{5}{8}},\frac{3}{8}e^{\frac{5}{8}})$ for $\alpha_2<0.50$. Then a further increase in the rate of adjustment implies to a stable 2-period cycle emerges for $\alpha_2=0.50$. As long as the parameter α_2 increases a 4-period cycle, cycles of highly periodicity and cascade of flip bifurcations that lead to chaotic motion. It means for a large values of speed of adjustment of bounded rational firm 2, the market converge always to complex dynamics.

Fig. (6) illustrates the behavior of the outputs of the two firms at the parameters $a=4$, $b=3$, $c_1=0.01$, $c_2=2$ with fixed α_1 = 0.85, we see that the system (7) exhibits a chaotic behavior to the market. In fact, an increase of α_1 and/or α_2, starting from a set of parameters which ensures the local stability of the Nash equilibrium, can bring the two firm's quantities out of the stability region, crossing

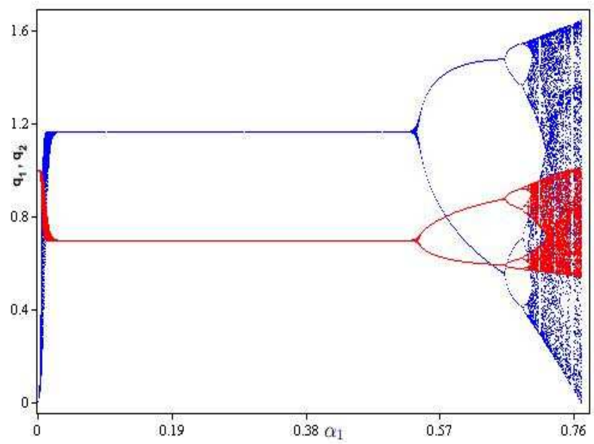

Fig. 3: Bifurcation diagram of the model (7) with respect to α_1 when $\alpha_2 = 0.3$.

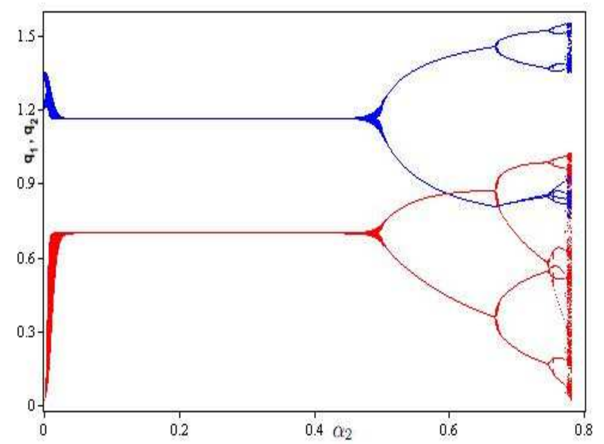

Fig. 5: Bifurcation diagram of the model (7) with respect to α_2 when $\alpha_1 = 0.5$.

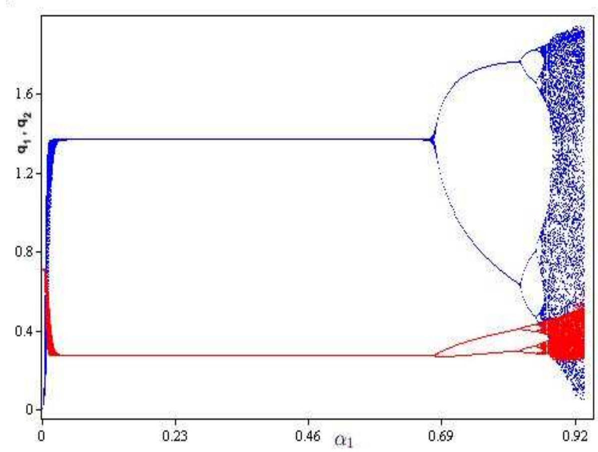

Fig. 4: Bifurcation diagram of the model (7) with respect to α_1 when $c_1 = 0.01$, $c_2=2$.

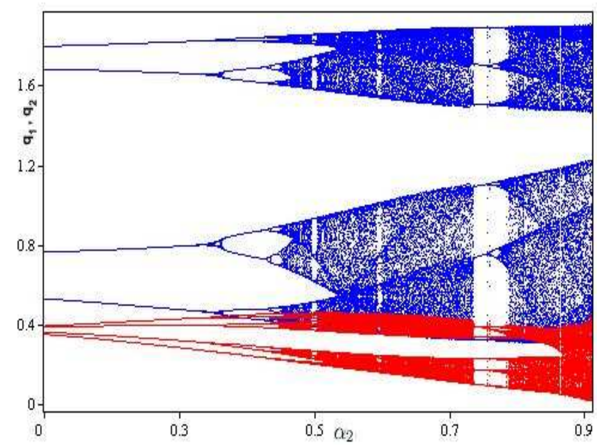

Fig. 6: Bifurcation diagram of the model (7) with respect to α_2 when $c_1 = 0.01$, $c_2 = 2$.

the flip bifurcation curve. We will study the phase portrait of the model (7) when the parameters (α_1, α_2) varied, one can consider the initial condition $(q_{1,0}, q_{2,0})$ situated in the basin of attraction of the Nash point E_4. An attractor fixed point takes place for $\alpha_1 = 0.3$, $\alpha_2 = 0.4$, $a = 6$, $b = 4$, $c_1 = 1$, $c_2 = 2$, which means the market orbit is a stable attractor, as showing in Fig. (7). In Fig. (8) shows that the trajectories of the two firms converge to a stable 2- period cycle when $\alpha_1 = 0.5$, $\alpha_2 = 0.6$. Fig. (9) shows that the firms output undergoes a 4- period cycle emerges for $\alpha_1 = 0.55$, $\alpha_2 = 0.6$, $a = 6$, $b = 4$, $c_1 = 0.1$, $c_2 = 0.2$. A further increase in the speed of adjustment of bounded rational firms implies to a highly periodicity and strange attractors in the market.

In Figures (10 - 12) show the graphs of the strange attractors for different values of (α_1, α_2). The phase portrait of Figures (10 & 11) depict the strange attractor

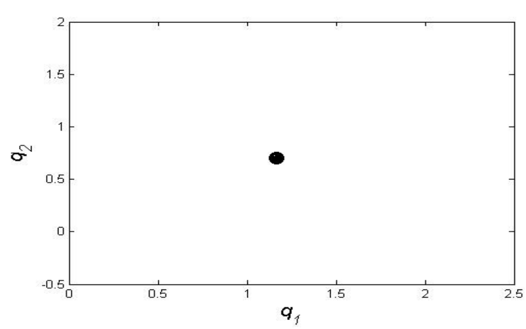

Fig. 7: An attractor fixed point of the model (7).

of the two players for $a = 4$, $b = 3$, $c_1 = 0.1$, $c_2 = 2$. We show the graph of the strange attractor for the parameters

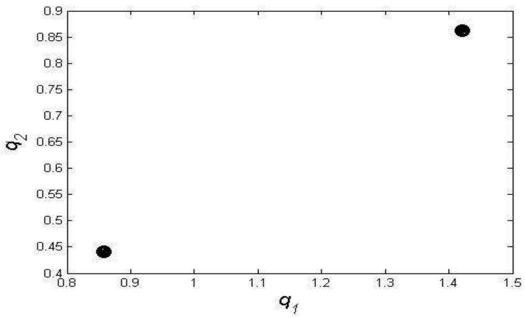

Fig. 8: A stable two-period cycle for $\alpha_1 = 0.5$, $\alpha_2 = 0.6$.

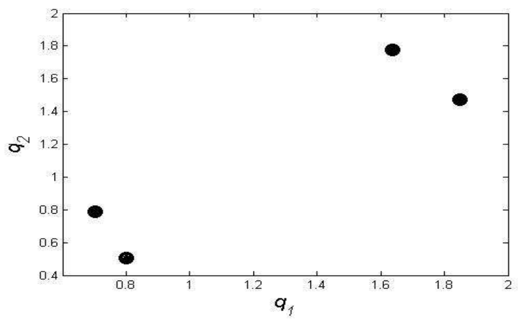

Fig. 9: A four-period cycle for $\alpha_1 = 0.55$, $\alpha_2 = 0.6$.

$(\alpha_1, \alpha_2, a, b, c_1, c_2) = (0.93, 0.7, 4, 3, 0.1, 0.2)$ in Fig. (12). From these results the structure of the market of duopoly game becomes complicated through period doubling bifurcations and more complex attractors are created a round the Nash point so, the complexity of players dynamic output competition can be described a chaotic phenomena.

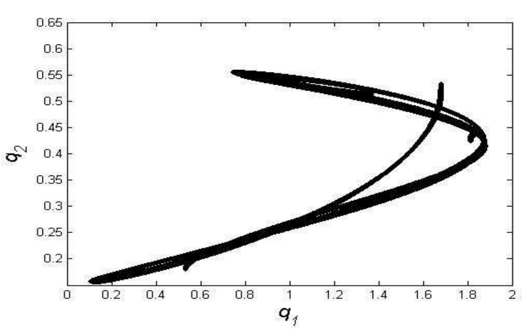

Fig. 10: Strange attractor of the model (7) for $\alpha_1 = 0.9$, $\alpha_2 = 0.6$.

As known, the sensitivity dependence on initial conditions is a characteristic of deterministic chaos. In order to show the sensitivity dependence on initial conditions of system Eq. (7), we have computed two orbits of the variables q_1 and q_2 which coordinates of initial conditions differ by 0.001. Figures (13,14) depict the orbits of q_1, q_2 with initial conditions $q_{1,0} = 0.1$ and $q_{2,0} = 0.02$, $q_{1,0} = 0.101$ and $q_{2,0} = 0.02$ at $(\alpha_1, \alpha_2, a, b, c_1, c_2) = (0.9, 0.7, 4, 3, 0.01, 0.2)$. As expected the orbits rapidly separate each other, thus suggesting the existence of deterministic chaotic " i.e. complex dynamics behaviors occur in the market" .

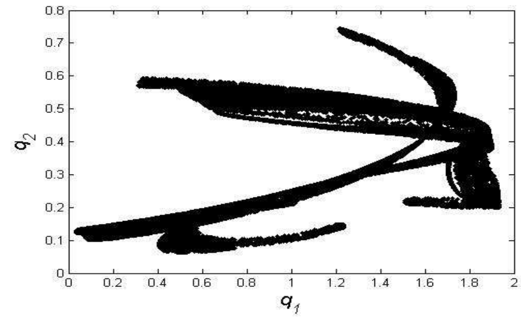

Fig. 11: Strange attractor of the model (7) for $\alpha_1 = 0.9$, $\alpha_2 = 0.7$.

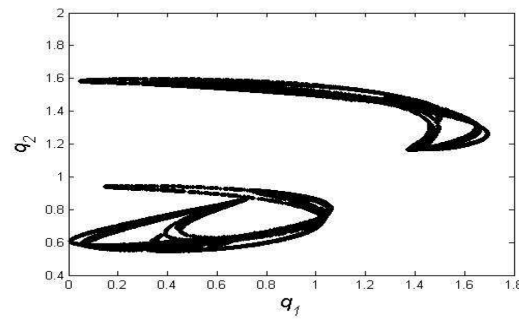

Fig. 12: Strange attractor of the model (7) for $\alpha_1 = 0.93$, $\alpha_2 = 0.7$.

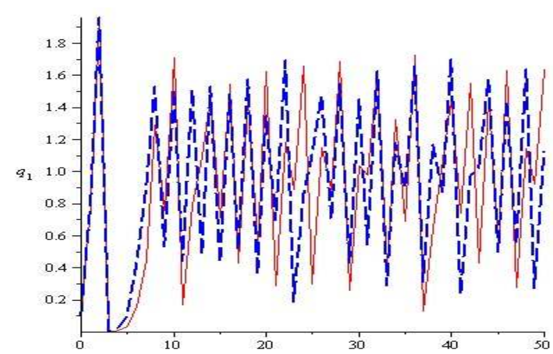

Fig. 13: Sensitive dependence on initial conditions for q_1.

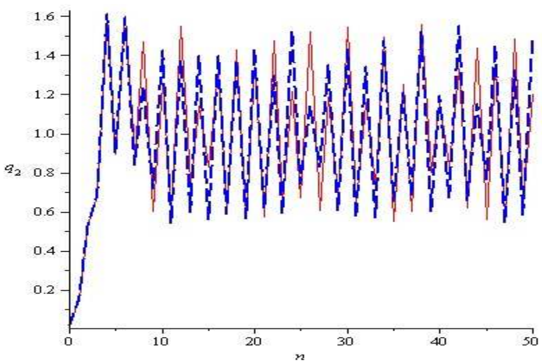

Fig. 14: Sensitive dependence on initial conditions for q_2.

The largest Lyapunov exponents corresponding to Fig. (4) are calculated and plotted in Fig. (15). In the range $0 < \alpha_1 < 0.658$ the Lyapunov exponents are negative, corresponding to a stable coexistence of the system. When $0.658 < \alpha_1 < 0.92$ most Lyapunov exponents are non-negative, and few are negative. This means that there exist stable fixed points or periodic windows in the chaotic band.

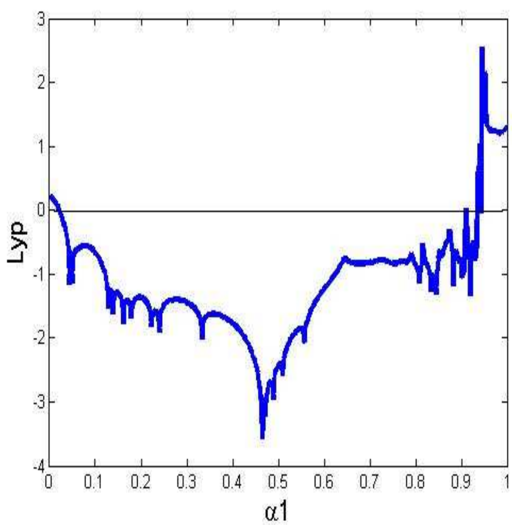

Fig. 15: Maximal Lyapunov exponent versus α_1 corresponding to Fig.4.

We conclude from the numerical experiments, that the adjustment speeds (α_1, α_2) may change the stability of the equilibrium and cause a market structure to behave chaotically.

5 Chaos control

Chaos in Cournot game means that if one player changes its output even slightly then, in the long run large unpredictable changes will occur in the outputs of all players. A producer can use a feedback of his decision-making variable to control the adjustment magnitude. (see Agiza and Elsadany 2013, [23]) have considered such a feedback control in their duopoly game. In order to control chaotic behavior of economic system (7), we apply Pyragas' method. In Pyragas' method [24], control input is based on the difference between the T- time delayed state and the current state,where T denotes a period of the stabilized orbits. So the controlled system is given by

$$q(t+1) = f(q(t), u(t)) \tag{21}$$

where $u(t)$ is the input signal, $q(t)$ is the state variable, and f is a nonlinear vector field. Pyragas proposed the following feedback in order to stabilize a $T-$periodic orbit:

$$u(t) = K(q(t+1-T) - q(t+1)); \;\; t > T \tag{22}$$

where T is the time delay and K is the controlling parameter. We apply this technique to control chaotic behavior for the dynamic game (7).We set $T = 1$; then the controlled system can be expressed as follows:

$$\begin{aligned} q_1(t+1) &= q_1(t) + \frac{\alpha_1}{k+1} q_1(t)\,[a - b(\frac{q_1}{q_1+q_2} \\ &\quad + \ln(q_1+q_2)) - c_1] \\ q_2(t+1) &= q_2(t) + \alpha_2 q_2(t)\,[a - b(\frac{q_2}{q_1+q_2} \\ &\quad + \ln(q_1+q_2)) - c_2] \end{aligned} \tag{23}$$

Then the Jacobian matrix of the controlled system (23) is given by

$$J(q_1,q_2) = \begin{bmatrix} J_{11}(q_1,q_2) & J_{12}(q_1,q_2) \\ J_{21}(q_1,q_2) & J_{22}(q_1,q_2) \end{bmatrix}, \text{ where} \tag{24}$$

$$\begin{aligned} J_{11}(q_1,q_2) &= 1 + \frac{\alpha_1}{k+1}(a-c_1) - (\frac{\alpha_1}{k+1}) * \\ &\quad b\left[\frac{3q_1}{q_1+q_2} + \ln(q_1+q_2) - \frac{q_1^2}{q_1+q_2}\right], \\ J_{12}(q_1,q_2) &= \frac{-\alpha_1 b q_1 q_2}{(k+1)(q_1+q_2)^2}, \\ J_{21}(q_1,q_2) &= \frac{-\alpha_2 b q_1 q_2}{(q_1+q_2)^2}, \\ J_{22}(q_1,q_2) &= 1 + \alpha_2(a-c_2) - \alpha_2 b * \\ &\quad \left[\frac{3q_2}{q_1+q_2} + \ln(q_1+q_2) - \frac{q_2^2}{q_1+q_2}\right]. \end{aligned}$$

Substituting by the Nash equilibrium point into (24) and using the values of parameters

$(\alpha_1, \alpha_2, a, b, c_1, c_2) = (0.9, 0.7, 5, 3, 0.1, 0.2)$ which chaos exists in the system (7). and the Nash equilibrium point becomes,

$E_3(q_1, q_2) = (1.578238432, 1.476416598)$ Then the Jacobian matrix (24) has the form:

$$J(q_1, q_2) = \begin{bmatrix} 1 - \frac{2.069249999}{(k+1)} & \frac{-0.6742500002}{(k+1)} \\ -0.5244166669 & -0.539416665 \end{bmatrix}. \quad (25)$$

By applying Jury conditions (18) on the matrix (25) has eigenvalues with an absolute less than one when $k > 0.4185$. Hence when $k > 0.4185$, all absolute values of eigenvalues are less than one, which means that the system is stable around the Nash equilibrium point.

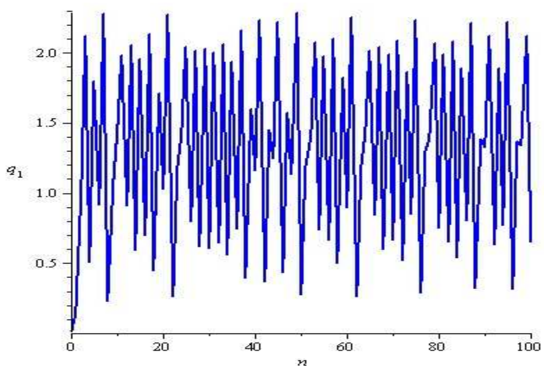

Fig. 16: Uncontrolled chaotic orbit of the state variable q_1.

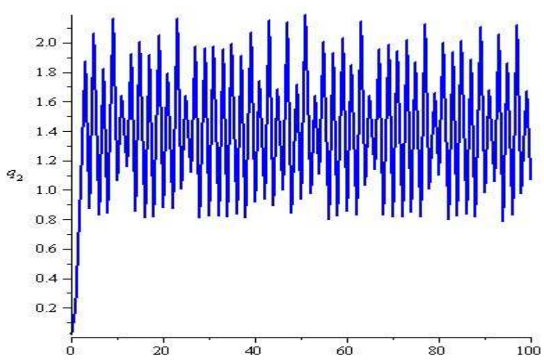

Fig. 17: Uncontrolled chaotic orbit of the state variable q_2.

From figures(16 & 17) the graphs represent the chaotic orbits of uncontrolled system starts from initial values $(q_{1,0}, q_{2,0}) = (0.01, 0.02)$, we see that the map stabilizes by the control parameter $k = 0.56$ starts from initial values $(q_{1,0}, q_{2,0}) = (0.01, 0.02)$ in figures (18 & 19). These figures show that a controlled behavior converges to the fixed point. So the feedback control method is able to control chaos if the two firms of bounded rational players utilize this adjustment method, the market game can switch from a chaotic trajectory to a regular periodic orbit or equilibrium state.

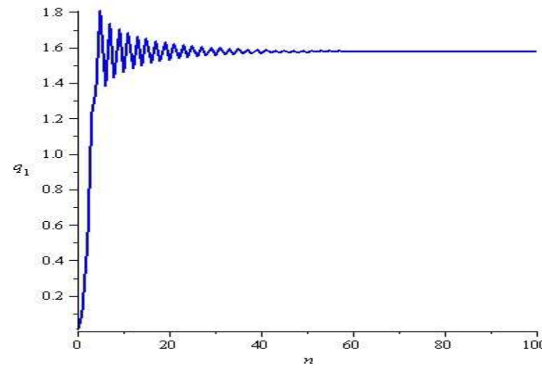

Fig. 18: Controlled chaotic orbit of the state variable q_1 for k = 0.56.

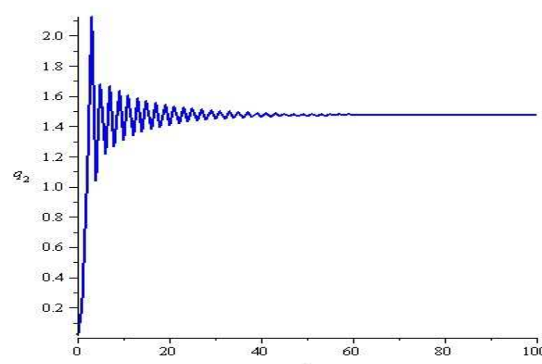

Fig. 19: Controlled chaotic orbit of the state variable q_2 for k = 0.56.

6 synchronization

In this section, we study the mechanisms which can lead to the synchronization of the trajectories in duopoly game model. Obviously, we achieve synchronization when on the diagonal Δ there exists a transversely stable orbit (in the sense that it attracts points not belonging to the diagonal itself). Such an attractor can be also coexisting with non synchronizing trajectories and in such a case it becomes important to know the initial conditions leading to synchronization. The phenomenon of synchronization of a two-dimensional discrete dynamical system:

$$T : (q_1(t), q_2(t)) \to (q_1(t+1), q_2(t+1)) \quad (26)$$

defined by the iteration of a map of the form $q' = T(q_1, q_2)$ and "$'$" denote the unit time advancement operator. The possibility of synchronization arises when an invariant one-dimensional sub- manifold of R^2 exists. For several properties of synchronization in two coupled maps ([25], [26]). The invariant subset on which the synchronized dynamics is the diagonal

$$\Delta = \{(q_1, q_2) | q_1 = q_2\} \quad (27)$$

In this case, the synchronized trajectories are characterized by

$$(q_1(t), q_2(t)) = \{T^t(q_1(0), q_2(0)) | q_1(t) = q_2(t)\}, \quad \forall t \geq 0 \quad (28)$$

These trajectories are governed by the restriction of T to the invariant sub-manifold on which the synchronized dynamics occur, given by the one-dimensional map:

$$f = T|_\Delta \Delta \to \Delta \quad (29)$$

A trajectory of T starting outside of is said to synchronize if $|q_1(t) - q_2(t)| \to 0$ as $t \to \infty$. In this model, the producer labeled by i is characterized by the two parameters α_i and c_i, representing the relative speeds of adjustment and the marginal costs, respectively. In the case, of identical producers characterized by the same value of the parameters:

$$\alpha_1 = \alpha_2 = \alpha \text{ and } c_1 = c_2 = c \quad (30)$$

The map has the symmetry property remaining the same after a reflection through the diagonal Δ of equation $q_1 = q_2$. Under the assumption Eq. (30), the map Eq. (7) becomes:

$$T_s : \begin{cases} q_1' = q_1 + \alpha q_1 \left[a - b\left(\frac{q_1}{q_1+q_2} + \ln(q_1+q_2)\right) - c\right] \\ q_2' = q_2 + \alpha q_2 \left[a - b\left(\frac{q_2}{q_1+q_2} + \ln(q_1+q_2)\right) - c\right] \end{cases} \quad (31)$$

The equilibrium points of Eq. (9) and Eq.(10) become:

$$E_{1s}(0,0),\ E_{2s}\left(0, e^{\frac{a-b-c}{b}}\right),\ E_{3s}\left(e^{\frac{a-b-c}{b}}, 0\right)$$

and the unique Nash equilibrium $E_{4s}(q_{1s}^*, q_{2s}^*)$ where,

$$q_{1s}^* = q_{2s}^* = \frac{e^{\frac{2(a-c)-b}{2b}}}{2}$$

The restriction $T_s|_\Delta$ of T_s to Δ is given by,

$$q' = f(q) = q + \alpha q \left[a - b\left(\frac{1}{2} + \ln(2q)\right) - c\right]$$

which conjugate with one-dimensional map:

$$q'(t+1) = f(q) = q(t) + \alpha q(t) \left[a - c - b\left(\frac{1}{2} + \ln(2q(t))\right)\right] \quad (32)$$

Thus, the dynamical behavior of the restriction of T to the invariant manifold Δ, where synchronized dynamics of the identical players take place, can be obtained from the well-known behavior of the map Eq. (32).

In order to study the transverse stability of the attractor of the synchronized system, we consider the Jacobian matrix:

$$J_{T_s(q_1,q_2)} = \begin{bmatrix} J_{11}(T_s) & J_{12}(T_s) \\ J_{21}(T_s) & J_{22}(T_s) \end{bmatrix}, \text{ where} \quad (33)$$

$$J_{11}(T_s) = 1 + \alpha(a-c) - (\alpha b) * \left[\frac{3q_1}{q_1+q_2} + \ln(q_1+q_2) - \frac{q_1^2}{q_1+q_2}\right]$$

$$J_{12}(T_s) = \frac{-\alpha b q_1 q_2}{(q_1+q_2)^2}$$

$$J_{21}(T_s) = \frac{-\alpha b q_1 q_2}{(q_1+q_2)^2}$$

$$J_{22}(T_s) = 1 + \alpha(a-c) - (\alpha b) * \left[\frac{3q_2}{q_1+q_2} + \ln(q_1+q_2) - \frac{q_2^2}{q_1+q_2}\right]$$

that computed on the line Δ assume the structure

$$J_{T_s(q,q)} = \begin{bmatrix} l(q) & m(q) \\ m(q) & l(q) \end{bmatrix} \quad (34)$$

with $l(q) = 1 + \alpha(a-c) - \alpha b\left[\frac{3-q}{2} + \ln(2q)\right]$, and $m(q) = \frac{-\alpha b}{4}$.

The eigenvalues of matrix (34) are

$$\lambda_\parallel = l(q) + m(q) = 1 + \alpha(a-c) - \alpha b\left[\frac{7}{4} + \ln(2q) - \frac{q}{2}\right] \quad (35)$$

$$\lambda_\perp = l(q) - m(q) = 1 + \alpha(a-c) - \alpha b\left[\frac{5}{4} + \ln(2q) - \frac{q}{2}\right]$$

For the point $E_{4s}(q^*, q^*)$ the transverse eigenvalue is,

$$\lambda_\perp^{E_{4s}} = 1 - \frac{\alpha b}{4}\left(3 - e^{\frac{2(a-c)-b}{2b}}\right). \quad (36)$$

So it is transversely attracting for all that give bounded dynamics on Δ, i.e. for $0 < \alpha b\left(3 - e^{\frac{2(a-c)-b}{2b}}\right) < 4$.

The Nash equilibrium E_{4s} is asymptotically stable node for

$0 < \alpha b\left(3 - e^{\frac{2(a-c)-b}{2b}}\right) < 4$, and a saddle point for $4 < \alpha b\left(3 - e^{\frac{2(a-c)-b}{2b}}\right) < 8$, with unstable set orthogonal to it. Notice that at $\alpha b\left(3 - e^{\frac{2(a-c)-b}{2b}}\right) = 4$, we have

$\lambda_{\perp}(E_{4s}) = 0$. For sufficiently small values of $\alpha b\left(3 - e^{\frac{2(a-c)-b}{2b}}\right)$ any attractor of the restriction $T_s|_{\Delta}$ is also asymptotically stable attractor also for the two-dimensional map T.

When chaotic synchronization is considered, in this case the stability condition of a chaotic attractor depends on the natural transverse Lyapunov exponent

$$\Lambda_{\perp} = \lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N} \ln|\lambda_{\perp}(q(t))| = \lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N} \ln\left|1 + \alpha(a-c) - \alpha b\left[\frac{5}{4} + \ln(2q) - \frac{q}{2}\right]\right| \tag{37}$$

where $q(t)$ is the corresponding trajectory generated by the map $f = T|_{\Delta}$. If $\Lambda_{\perp} < 0$ for each trajectory starting inside attractor, then the attractor is asymptotically stable. In this case, the fact that $\Lambda_{\perp} < 0$ for the generic aperiodic trajectory in the attractor means, that the attractor is transversely attracting on the average.

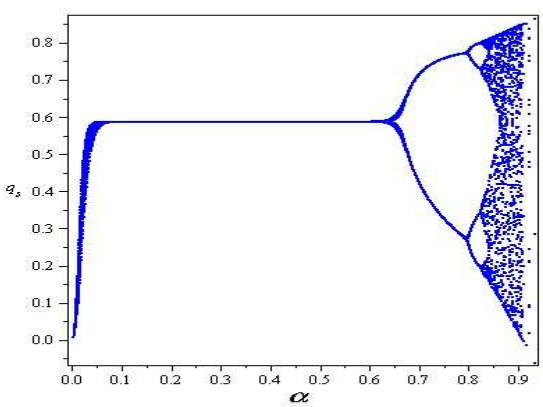

Fig. 20: Bifurcation diagram of synchronized system (32).

A numerical computation of map (32) performed with $a = 4$, $b = 3$ and $c = 2$, is shown in Fig. (20) as shows that the trajectories converges to the Nash equilibrium when $\alpha < 0.84$, for $\alpha > 0.84$, the Nash equilibrium becomes unstable, period doubling bifurcations appear and finally chaotic behaviors occur. Fig. (21) shows that the natural transverse Lyapunov exponent for the chaotic synchronized depends on it.

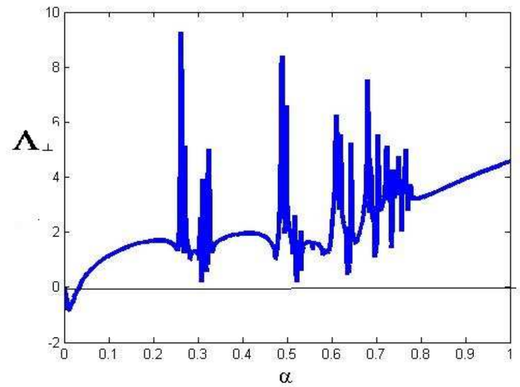

Fig. 21: The natural transverse Lyapunov exponent of the synchronized system (32).

7 Conclusions

This paper discussed the complex dynamic behavior of a Cournot game occurred between two bounded rationality firms which are used a logarithmic inverse demand functions. We showed that two boundary equilibria are unstable and we mainly addressed the problems of the locally asymptotic stability of the unique Nash equilibrium. some complex dynamic features such as region of stability for the unique Nash equilibrium, period doubling bifurcations, strange attractors, Lyapunov exponent and sensitive dependence on initial conditions. We demonstrated that the fast increasing of the rates of adjustment cause the Nash equilibrium becomes unstable through period doubling bifurcations and the market structure to behave chaotically. We have stabilized the chaotic behavior of the model to a stable fixed point by the delay feedback control method. the paper concerned the global behavior of the map out of the invariant manifold where synchronization occurs.

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