Using Finite Volume-Element Method for Solving Space Fractional Advection-Dispersion Equation

Allahbakhsh Yazdani¹,⁎, Navid Mojahed¹, Afshin Babaei¹ and Elena Vazquez Cendon²

¹ Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran
² Faculty of Mathematics, University of Santiago de Compostela, Santiago de Compostela, Spain

Received: 2 Sep. 2018, Revised: 27 May. 2019, Accepted: 27 Jul. 2019
Published online: 1 Jan. 2020

Abstract: In this paper, the numerical solution for space fractional advection-dispersion problem in one-dimension is proposed by B-spline finite volume element method. The fractional derivative is Grunwald-Letnikov in the proposed scheme. The stability and convergence of the proposed numerical method are studied, and the numerical results support the exact results.

Keywords: Finite volume-element method, Advection-dispersion equation, Grunwald-Letnikov derivative, space fractional, fractional calculus.

1 Introduction

Fractional calculus (FC) has been applied in different fields of engineering and science, including electro-magnetics, visco-elasticity, optics, electro-chemistry, fluid mechanics, and signals processing [1–9]. This method has been used in modeling contaminant flow as well [2, 3, 10–13]. Moreover, a wide range of physical phenomena can be modelled by FC to be described more precisely. Furthermore, the fractional derivative-based models are perfect in analysing the damping systems. Because of the wide range of FC applications, most of the analytical and numerical methods, which are recently proposed, are inapplicable [14–23].

Many people have recently worked on solving fractional partial differential equations (FPDEs). Some have addressed the analytical solution of FPDE [2, 24–32], others have explored numerical solutions [24, 33–79]. To solve the differential equations, the three following approaches are adopted: Finite Difference Methods (FDM), Finite Volume Methods (FVM), and Finite Element Methods (FEM) [80, 81].

Finite element volume methods (FEVM) are linked to finite element methods. Precisely, FVEMs are the Petrov-Galerkin form of FEMs, which are developed using two types of partitions; a primal partition and its dual, on a domain Ω. The primal mesh approximates the exact solution, and the equations are discretized over the control volumes by its dual. The two main advantages of FVEM are the accuracy of the method, dependent only on the degree of the approximation polynomial, and flexibility of the control volumes. It is advantageous to handle complicated domains. Badr et al. [82] investigated FVEM for solving a time-fractional advection-diffusion equation and proved the stability of this method.

Transport activity enclosed by complex and non-homogenous conditions sometimes leads to non-classical diffusion that is not completely matched by Fick’s law or pedesis theory [2, 3, 10–13, 83]. Fractional calculus helps overcome such a challenge. If a random walk model takes place as continuous time, it results in a fractional advection-dispersion equation [6]. Space advection-dispersion equation is obtained by putting the fractional derivative term in classical diffusion equation [12]. In the present paper, we will work on space fractional homogenous advection-dispersion equation.

The present paper is outlined as follows. In section two, we apply the FVEM to approximate the numerical solution of the initial value fractional advection-dispersion equation. Stability and convergence of this method are discussed in section three. Some numerical results are illustrated in section four. Section five is devoted to some conclusion.

⁎ Corresponding author e-mail: yazdani@umz.ac.ir
2 Statement of the problem and method of solution

Definition 1. Right and left Riemann-Liouville fractional derivative respectively on Ω is defined as follows [44]:

\[
\frac{\partial^{\gamma} f(x,t)}{\partial x^{\gamma}} = \frac{1}{\Gamma(n - \gamma)} \frac{\partial^n}{\partial x^n} \int_{x-\xi}^{x} f(\xi, t) \xi^{-n+1} \xi^{n-\gamma-1} d\xi,
\]

\[
\frac{\partial^{\gamma} f(x,t)}{\partial (-x)^{\gamma}} = \frac{(-1)^n}{\Gamma(n - \gamma)} \frac{\partial^n}{\partial (-x)^n} \int_{x}^{b} f(\xi, t) \xi^{-n+1} \xi^{n-\gamma-1} d\xi,
\]

where \( n \) is the ceil of \( \gamma \).

Suppose that we discrete the domain \([a, b]\) to \(\mathbb{N}+1\) equal parts, so we can define \(h = (b-a)/\mathbb{N}\) and \(x_i = a + ih, i = 0, \ldots, N\).

Definition 2. Shifted Grunwald formulas on \([a, b]\) for \(0 \leq p \leq 1\) are defined as follows [44]:

\[
\frac{\partial^\alpha f(x,t)}{\partial x^\alpha} \approx \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\lfloor \frac{\alpha}{h} \rfloor + p} (-1)^j \left( \begin{array}{c} \alpha \\ j \end{array} \right) f(x - (j - p)h,t),
\]

\[
\frac{\partial^\alpha f(x,t)}{\partial (-x)^\alpha} \approx \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\lfloor \frac{\alpha}{h} \rfloor + p} (-1)^j \left( \begin{array}{c} \alpha \\ j \end{array} \right) f(x + (j - p)h,t).
\]

For smooth function \( f \), two equations (3) and (4) are equivalent with two equations (1) and (2) respectively [22].

Thus, we can write equations (3) and (4) as:

\[
\frac{\partial^\alpha f(x,t)}{\partial x^\alpha} \approx \frac{1}{h^\alpha} \sum_{j=0}^{\lfloor \frac{\alpha}{h} \rfloor + p} (-1)^j \left( \begin{array}{c} \alpha \\ j \end{array} \right) f(x - (j - p)h,t),
\]

\[
\frac{\partial^\alpha f(x,t)}{\partial (-x)^\alpha} \approx \frac{1}{h^\alpha} \sum_{j=0}^{\lfloor \frac{\alpha}{h} \rfloor + p} (-1)^j \left( \begin{array}{c} \alpha \\ j \end{array} \right) f(x + (j - p)h,t).
\]

In this part, we consider the homogeneous space fractional advection-dispersion equation (7) with initial condition and boundary conditions as,

\[
\begin{cases}
\frac{\partial C(x,t)}{\partial t} + \frac{\partial (VC(x,t))}{\partial x} = K \left( \beta \frac{\partial^\gamma C(x,t)}{\partial x^{\gamma}} + (1 - \beta) \frac{\partial^\gamma C(x,t)}{\partial (-x)^{\gamma}} \right) + S(x,t) & \text{in } \Omega \times [0,T], \\
C(x,t) = 0 & \text{on } \partial \Omega \times [0,T], \\
C(x,0) = f(x) & \text{in } \Omega,
\end{cases}
\]

that considered \( V > 0, K > 0, 0 \leq \beta \leq 1 \) and \( 1 < \gamma \leq 2 \) are constants and \( \Omega = [a,b] \).

Putting \( \alpha = \gamma - 1 \) in equation (7) we will have [83, 84]:

\[
\frac{\partial C}{\partial t} + \frac{\partial (VC)}{\partial x} = \frac{\partial}{\partial x} \left[ K(\beta \frac{\partial^\alpha C(x,t)}{\partial x^{\alpha}} - (1 - \beta) \frac{\partial^\alpha C(x,t)}{\partial (-x)^{\alpha}}) \right] + S(x,t).
\]

According to this agreement, some introductions are obtained around FVEM and Sobolev spaces. Let \( \Omega = (a,b) \subset \mathbb{R} \) be a domain, we define the Sobolev space \( H^m(\Omega), m \geq 0 \), to be the space of all functions \( \psi \) such that weak derivative \( D^{\alpha} \psi \in L_2(\Omega) \) for all \( |\alpha| \leq 1 \) and equipped with the norm and seminorm

\[
||\psi||_k = \left( \sum_{|\alpha| \leq k} ||D^{\alpha}||^2 \right)^{\frac{1}{2}}, \quad |\psi|_k = \left( \sum_{|\alpha| = k} ||D^{\alpha}||^2 \right)^{\frac{1}{2}}.
\]

Interrelated to the Sobolev space \( H^m(\Omega) \), we defined the space \( H^1_0(\Omega) \in H^1(\Omega) \) including functions, vanished on the boundary of \( \Omega \), i.e.,

\[
H^1_0 = \{ \psi \in H^1(\Omega), \psi|_{\partial \Omega} = 0 \}.
\]
To solve the equation (8) using our proposed numerical method, the domain $\Omega$ is divided into finite elements $\xi_i = [x_{i-1}, x_i]$ and $h$ is set equal to $x_i - x_{i-1}$, where the nodes are represented by $x_i$ and $i = 0, 1, 2, \ldots, M$. In addition, $\Omega_h = \{\xi_i\}$ represents the primal partition of $\Omega$ with elements $\xi_i$. For each node, the domain $V_i = [x_{i-1}, x_i]$ is identified as the control volume. The $\Omega_h^*$ is defined as the set of control volumes, $\{V_i\}$. Using piecewise space $S_h^*$ on $\Omega_h^*$, which was first introduced in [85], we established a variational FVE form for equation (8) as follows:

$$
(\frac{\partial}{\partial t} v^*_h) + (\frac{\partial}{\partial t} (VC), v^*_h) = K\beta \left( \frac{\partial}{\partial t} (\frac{\partial C}{\partial x}), v^*_h \right) - K(1 - \beta) \left( \frac{\partial}{\partial x} (\frac{\partial a}{\partial (-x)}), v^*_h \right) + (S, v^*_h) v^*_h \in S_h^* \tag{9}
$$

that $(\ldots)$ is inner product in $L_2(\Omega)$. An estimation of $C \in H^1(\Omega \times (0, T))$ in $S_h$ for (9) is taken by the discrete FVEM. $S_h$ is defined on $\Omega_h$ as

$$
S_h(x) = \{v_h(x) \in C(\Omega) : v_h \big|_{\partial \Omega} \text{ is linear and } v_h \big|_{\partial \Omega} = 0 \}.
$$

Different elections for solution spaces, test spaces, and dual partitions cause different FVEMs [85]. In this paper, we have shown $S_h = \text{span}\{\phi_i : 0 \leq i \leq M\}$ and $S_h^* = \text{span}\{\chi_i : 0 \leq i \leq M\}$ where $\phi_i$s (linear B-spline functions) defined by

$$
\phi_i(x) = \begin{cases} 
\frac{x_i - x}{h}, & x_i - 1 \leq x \leq x_i, \\
\frac{x_{i+1} - x}{h}, & x_i \leq x \leq x_{i+1}, \\
0, & \text{otherwise},
\end{cases}
$$

and $\chi_i$s (characteristic functions) related by the control volume $V_i$ defined by

$$
\chi_i(x) = \begin{cases} 
1, & x \in V_i, \\
0, & \text{otherwise}.
\end{cases}
$$

Each $C_h(x, t) \in S_h$ may be written as

$$
C_h(x, t) = \sum_{i=0}^{M} \delta_i(t) \phi_i(x),
$$

where the coefficients $\delta_i(t)$ should be calculated from the initial conditions and boundary conditions using the FVEM. The discrete FVEM is defined as: Find $C_h(t) = C_h(., .)$ depending on $S_h$, for each $t > 0$, in order that, for any $V_i, i = 1, 2, \ldots, M - 1$

$$
\int_{\Omega} \frac{\partial C_h}{\partial t} \chi_{x} dx + \int_{\Omega} \frac{\partial C_h}{\partial x} \chi_{x} dx = K\beta \int_{\Omega} \frac{\partial^2 C_h}{\partial x^2} \chi_{x} dx - K(1 - \beta) \int_{\Omega} \frac{\partial a C_h}{\partial (-x)} \chi_{x} dx + (S, v^*_h) v^*_h \in S_h^*, t > 0, \tag{13}
$$

to statement of the FVEM for (8), it is verified that

$$
\int_{\Omega} \frac{\partial C_h}{\partial t} \chi_{x} dx + \int_{\Omega} \frac{\partial C_h}{\partial x} \chi_{x} dx = K\beta \int_{\Omega} \frac{\partial^2 C_h}{\partial x^2} \chi_{x} dx - K(1 - \beta) \int_{\Omega} \frac{\partial a C_h}{\partial (-x)} \chi_{x} dx + (S, \chi_{x}) dx.
$$

According to definition of $\chi_i, i = 1, 2, \ldots, M - 1$, we have

$$
\int_{V_i} \frac{\partial C_h}{\partial t} dx + \int_{V_i} \frac{\partial C_h}{\partial x} dx = K\beta \int_{V_i} \frac{\partial^2 C_h}{\partial x^2} dx - K(1 - \beta) \int_{V_i} \frac{\partial a C_h}{\partial (-x)} dx + (S, \chi_{x}) dx.
$$

Let $\delta_i = \delta_i(t), w^\alpha_j = (-1)^j \left( \frac{\alpha}{j} \right)$ and $\delta^j = \delta_i(t_j)$ where $t_j = j \ast \tau$. 

© 2020 NSP
Natural Sciences Publishing Co.
Using shifted and simple Grunwald-Letnikov fractional derivatives on \([a, b]\) results in:

\[
\frac{\partial^{\alpha}}{\partial x^{\alpha}} C(x_{i+\frac{1}{2}}, t) \approx \frac{1}{h^\alpha} \sum_{j=0}^{i+1} w^\alpha_j C(x_{i-j+1}, t),
\]

\[
\frac{\partial^{\alpha}}{\partial (-x)^{\alpha}} C(x_{i-\frac{1}{2}}, t) \approx \frac{1}{h^\alpha} \sum_{j=0}^{M-i} w^\alpha_j C(x_{i+j}, t),
\]

\[
\frac{\partial^{\alpha}}{\partial x^{\alpha}} C(x_{i-\frac{1}{2}}, t) \approx \frac{1}{h^\alpha} \sum_{j=0}^{i} w^\alpha_j C(x_{i-j}, t),
\]

\[
\frac{\partial^{\alpha}}{\partial (-x)^{\alpha}} C(x_i, t) \approx \frac{1}{h^\alpha} \sum_{j=0}^{M-i} w^\alpha_j C(x_{i+j}, t),
\]

(15)

Putting relation (2) in equation (13) results in:

\[
\frac{d}{dt} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \sum_{i=0}^{M} \delta \phi(x) dx = -V \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{\partial}{\partial x} \left( \sum_{i=0}^{M} \delta \phi(x) \right) dx
\]

\[+ K \beta \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{\partial}{\partial x} \left( \frac{\partial^\alpha}{\partial x^\alpha} \sum_{i=0}^{M} \delta \phi(x) \right) dx
\]

\[- K(1 - \beta) \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{\partial}{\partial x} \left( \frac{\partial^\alpha}{\partial (-x)^\alpha} \sum_{i=0}^{M} \delta \phi(x) \right) dx
\]

\[+ \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} S(x, t) dx.
\]

(16)

We need to calculate each term of relation (16).

First term is:

\[
\frac{d}{dt} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \sum_{i=0}^{M} \delta \phi(x) dx = \frac{d}{dt} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left( \delta_{i-1} \phi_{i-1}(x) + \delta_i \phi_i(x) + \delta_{i+1} \phi_{i+1}(x) \right) dx,
\]

\[= \frac{d}{dt} \left( \delta_{i-1} \times \frac{h}{8} + \delta_i \times \frac{3h}{4} + \delta_{i+1} \times \frac{h}{8} \right),
\]

\[= \frac{h \delta_{i-1}^{j+1} - \delta_{i-1}^j}{\tau} + \frac{3h \delta_{i+1}^{j+1} - \delta_{i+1}^j}{\tau} + \frac{h \delta_{i+1}^{j+1} - \delta_{i+1}^j}{\tau}.
\]

(17)

Second term is:

\[
\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{\partial}{\partial x} \sum_{i=0}^{M} (\delta \phi(x)) dx = \left. \left( \delta_{i-1} \phi_{i-1}(x) + \delta_i \phi_i(x) \right) \right|_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} + \left. \left( \delta_{i+1} \phi_{i+1}(x) \right) \right|_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}},
\]

\[= \delta_{i-1} \left( 0 - \frac{1}{2} \right) + \delta_i \left( 1 - \frac{1}{2} \right) + \delta_{i+1} \left( \frac{1}{2} - 1 \right) + \delta_{i+1} \left( \frac{1}{2} - 0 \right),
\]

\[= -\frac{1}{2} \delta_{i-1} + \frac{1}{2} \delta_{i+1}.
\]

(18)
Third term is:

\[
\int_{\frac{x_i}{2}}^{\frac{x_{i+1}}{2}} \frac{\partial}{\partial x} \left( \frac{\partial}{\partial (-x)^{a}} \left( \sum_{i=0}^{M} \delta \phi(x) \right) \right) dx = \frac{\partial}{\partial (-x)^{a}} \left( \sum_{i=0}^{M} \delta \phi(x) \right) \bigg|_{x_i}^{x_{i+1}} \cdot \frac{\partial}{\partial x} + \frac{\partial}{\partial x^{a}} \left( \sum_{i=0}^{M} \delta \phi(x) \right) \bigg|_{x_i}^{x_{i+1}} ,
\]

\[
= \frac{\partial}{\partial (-x)^{a}} \left( \delta \phi(x) \right) \bigg|_{x_i}^{x_{i+1}} + \frac{\partial}{\partial x^{a}} \left( \delta \phi(x) \right) \bigg|_{x_i}^{x_{i+1}} ,
\]

\[
= \frac{1}{h^{a}} \left[ \delta_{i-1} \left( \sum_{j=0}^{i} \left( w_{j}^{a} \phi_{i-1}(x_{i-j}) \right) - \sum_{j=0}^{i} \left( w_{j}^{a} \phi_{i}(x_{i-j}) \right) \right) \right] + \delta_{i} \left( \sum_{j=0}^{i} \left( w_{j}^{a} \phi_{i}(x_{i-j}) \right) - \sum_{j=0}^{i} \left( w_{j}^{a} \phi_{i+1}(x_{i-j}) \right) \right) ,
\]

\[
= 0 + \frac{1}{h^{a}} \left[ \delta_{i} \left( w_{i}^{a} - w_{0}^{a} \right) + \delta_{i+1} \left( w_{0}^{a} - 0 \right) \right] ,
\]

\[
= \frac{1}{h^{a}} \left[ \delta_{i} \left( -\alpha - 1 \right) + \delta_{i+1} \left( 1 \right) \right] ,
\]

and fourth term is:

\[
\int_{\frac{x_i}{2}}^{\frac{x_{i+1}}{2}} \frac{\partial}{\partial x} \left( \frac{\partial}{\partial (-x)^{a}} \left( \sum_{i=0}^{M} \delta \phi(x) \right) \right) dx = \frac{\partial}{\partial (-x)^{a}} \left( \sum_{i=0}^{M} \delta \phi(x) \right) \bigg|_{x_i}^{x_{i+1}} + \frac{\partial}{\partial x^{a}} \left( \sum_{i=0}^{M} \delta \phi(x) \right) \bigg|_{x_i}^{x_{i+1}} ,
\]

\[
= \frac{\partial}{\partial (-x)^{a}} \left( \delta \phi(x) \right) \bigg|_{x_i}^{x_{i+1}} + \frac{\partial}{\partial x^{a}} \left( \delta \phi(x) \right) \bigg|_{x_i}^{x_{i+1}} ,
\]

\[
= \frac{1}{h^{a}} \left[ \delta_{i-1} \left( \sum_{j=0}^{M-i} \left( w_{j}^{a} \phi_{i-1}(x_{i-j}) \right) - \sum_{j=0}^{M-i} \left( w_{j}^{a} \phi_{i+j}(x_{i-j}) \right) \right) \right] + \delta_{i} \left( \sum_{j=0}^{M-i} \left( w_{j}^{a} \phi_{i+j}(x_{i-j}) \right) - \sum_{j=0}^{M-i} \left( w_{j}^{a} \phi_{i-j}(x_{i-j}) \right) \right) ,
\]

\[
= 0 + \frac{1}{h^{a}} \left[ \delta_{i} \left( w_{i}^{a} - w_{0}^{a} \right) + \delta_{i+1} \left( w_{0}^{a} - 0 \right) \right] ,
\]

\[
= \frac{1}{h^{a}} \left[ \delta_{i} \left( -\alpha - 1 \right) + \delta_{i+1} \left( 1 + \alpha \right) \right] .
\]

Let

\[
\overline{S}_{i}(t) = \frac{1}{h} \int_{\frac{x_i}{2}}^{\frac{x_{i+1}}{2}} S(x,t) dx .
\]

We know \( \overline{S}_{i}(t) \approx S(x_i, t) \) [44].
Replacing (17), (18), (19), (20) and (21) in equation (16) and using backward method result in:

\[
\frac{h}{8} \left( \frac{\delta_{i+1}^j - \delta_i^j}{\tau} \right) + \frac{3h}{4} \left( \frac{\delta_i^{j+1} - \delta_i^j}{\tau} \right) + \frac{h}{8} \left( \frac{\delta_i^j - \delta_{i-1}^j}{\tau} \right) = -\frac{V}{2} (-\delta_i^{j+1} + \delta_i^j) + \frac{K\beta}{h^{\alpha}} [\delta_i^{j+1} (-\alpha - 1) + \delta_i^j (1)] - \frac{K(1 - \beta)}{h^{\alpha}} [\delta_i^{j+1} (-1) + \delta_i^j (1 + \alpha)] + hS_i^{j+1}.
\]

We define matrix \(A, B, \Delta^j\) and \(D^j\) as follows

\[
A_{(M-1)\times(M-1)} = \begin{bmatrix}
6 & 1 & 0 & 0 & \cdots \\
1 & 6 & 1 & 0 & \cdots \\
0 & 1 & 6 & 1 & 0 \\
0 & 0 & \cdots & \cdots & \cdots \\
\cdots & 0 & 1 & 6 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix},
\]

\[
B_{(M-1)\times(M-1)} = \begin{bmatrix}
a_M & b_M & 0 & 0 & \cdots \\
0 & c_M & a_M & b_M & 0 \\
0 & 0 & \cdots & c_M & b_M \\
\cdots & 0 & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix},
\]

\[
\Delta^j = \begin{bmatrix}
\delta_i^j & \delta_{i+1}^j & \cdots & \delta_{M-2}^j & \delta_{M-1}^j \\
\end{bmatrix}^T,
\]

\[
D^j = \begin{bmatrix}
S_i^j & S_{i+1}^j & \cdots & S_{M-2}^j & S_{M-1}^j \\
\end{bmatrix}^T,
\]

where

\[
c_M = -\frac{h}{8\tau} + \frac{V}{2} + \frac{K(1 - \beta)}{h^{\alpha}},
\]

\[
a_M = -\frac{3h}{4\tau},
\]

\[
b_M = \frac{K\beta}{h^{\alpha}} (1 + \alpha),
\]

\[
hS_i^{j+1} = -\frac{h}{8\tau} A \times \Delta^j = B \times \Delta^{j+1} + D^{j+1}, \forall j \geq 0, \forall j \in \mathbb{Z}.
\]

3 Stability and convergence analysis

Implementing the analysis of Von-Neumann stability [75, 86] and the mathematical inductions, the stability conditions of the proposed numerical method is investigated.

**Theorem 1.** If \(h\) is small enough, then the method explained in equation (22) is stable.

**Proof.** Let we define \(C^j = [C(x_0, t_j) C(x_1, t_j) \ldots C(x_{M-2}, t_j) C(x_{M-1}, t_j)]\). We used Von-Neumann processes. We can write according to equation (23):

\[
\Delta^j = -\frac{h}{8\tau} (B^{-1}A) \Delta^{j-1} - B^{-1}D^j = \left(-\frac{h}{8\tau}\right)^2 (B^{-1}A)^2 \Delta^{j-2} - \frac{h}{8\tau} B^{-1} ABD^{j-1} - B^{-1} D^j.
\]

Proceeding with this method, we get the following result

\[
\Delta^j = \left(-\frac{h}{8\tau}\right)^j (B^{-1}A)^j \Delta^0 + \left(-\frac{h}{8\tau}\right)^{j-1} (B^{-1}A)^{j-1} B^{-1} D^j + \left(-\frac{h}{8\tau}\right)^{j-2} (B^{-1}A)^{j-2} B^{-1} D^2 + \ldots + B^{-1} D^j.
\]
and for exact solution

$$C^j = \left(-\frac{h}{8\tau}\right)^j(B^{-1}A)^iC^0 + \left(-\frac{h}{8\tau}\right)^{j-1}(B^{-1}A)^{j-1}B^{-1}D^j$$

+ \left(-\frac{h}{8\tau}\right)^{j-2}(B^{-1}A)^{j-2}B^{-1}D^2 + \cdots + B^{-1}D^j. \quad (26)$$

We define the error $e^j = \Delta^j - C^j$ so

$$e^j = \left(-\frac{h}{8\tau}\right)^j(B^{-1}A)^i e^0.$$  

For compatible matrix

$$\|e^j\| \leq \left\|\left(-\frac{h}{8\tau}B^{-1}A\right)^j\right\| \cdot \|e^0\|.$$  

If $M > 0$ such that $\left\|\left(-\frac{h}{8\tau}B^{-1}A\right)^j\right\| \leq M$ the difference scheme is stable [7]. We know

$$\left\|\left(-\frac{h}{8\tau}B^{-1}A\right)^j\right\| \leq \left\|\left(-\frac{h}{8\tau}B^{-1}A\right)^{j-1}\right\| \leq \cdots \leq \left\|\left(-\frac{h}{8\tau}B^{-1}A\right)\right\|.$$  

As a result, Lax definition of stability is made certain if [87]

$$\left\|\left(-\frac{h}{8\tau}B^{-1}A\right)\right\| \leq 1.$$  

We know $h$ is small enough. Therefore,

$$\left\|\frac{h}{8\tau}A\right\|_\infty = \left(\frac{h}{8\tau}(1 + 6 + 1)\right) = \frac{h}{\tau} < 1,$$  

and again for small enough $h$

$$\left\|B\right\|_\infty = \left|\frac{h}{8\tau}\right| + \frac{V}{2} + K(1 - \beta) + \frac{3h}{4\tau} + \frac{K(1 - \beta)(1 + \alpha)}{h^\alpha} + \frac{K\beta(1 + \alpha)}{h^\alpha} + \frac{h}{8\tau}\left|\frac{V}{2} + K_{\beta}\right|$$

$$= \left(\frac{h}{8\tau} + \frac{V}{2} + \frac{K(1 - \beta)}{h^\alpha}\right) + \left(\frac{3h}{4\tau} + \frac{K(1 - \beta)(1 + \alpha)}{h^\alpha} + \frac{K\beta(1 + \alpha)}{h^\alpha}\right) + \left(\frac{h}{8\tau} + \frac{K_{\beta}}{h^\alpha}\right)$$

$$= \frac{h}{2\tau} + \frac{K(2 + \alpha)}{h^\alpha} + 1 \Rightarrow \left\|B\right\|_\infty < 1.$$  

Using (31) and (32), the relation (30) is established and the method is stable for small enough $h$.

**Theorem 2.** The method presented in (22) is convergent and the order of the scheme is one in space and time.

**Proof.** Replacing exact solution with numerical solution in equation (22), we will have

$$\frac{h(\alpha + 1)}{8\tau}(C_{i-1}^{j+1} - C_{i-1}^j) + \frac{3h(\alpha + 1)}{4\tau}(C_{i+1}^{j+1} - C_i^j) + \frac{h(\alpha + 1)}{8\tau}(C_{i+1}^{j+1} - C_i^{j+1})$$

$$= -\frac{V}{2}\alpha K\tau C_{i-1}(x_i,t_j) + C_i^{j+1} + K\beta(C_{i+1}^{j+1}(-\alpha - 1) + C_i^{j+1})$$

$$- K(1 - \beta)(-C_{i+1}^j + C_i^{j+1}(\alpha + 1)) + h^{1+\alpha}S(x_i,t_{j+1}). \quad (33)$$

Hence, local truncation error is calculated using Taylor series as follows:

$$T_{ij} = -h^{1+\alpha}S(x_i,t_j) + K\alpha C(x_i,t_j) + h^{1+\alpha}C_i(x_i,t_j) + \frac{1}{2}h^{1+\alpha}\tau C_{xx}(x_i,t_j)$$

$$+ \frac{1}{2}h^{1+\alpha}\tau^2 C_{xxt}(x_i,t_j) + h\alpha C(x_i,t_j) + h^{2+\alpha}V C(x_i,t_j) - 2hK\beta C_i(x_i,t_j)$$

$$+ h\alpha C(x_i,t_j) + h^{1+\alpha}V C(x_i,t_j) - 2hK\beta C_x(x_i,t_j) + \frac{1}{2}hK\tau^2 C_{xxt}(x_i,t_j)$$

$$+ \frac{1}{8}h^{3+\alpha}C_{xxt}(x_i,t_j) + \frac{1}{2}h^2K\tau C_{xxt}(x_i,t_j)$$

$$+ \frac{1}{16}h^{3+\alpha}C_{xxt}(x_i,t_j) - \frac{1}{4}h^2K\tau^2 C_{xxtt}(x_i,t_j) + \cdots. \quad (34)$$
Therefore, it is noticeable that the local truncation error is of order $O(h) + O(\tau)$.

4 Numerical examples

In this section, we will compare the numerical solution resulted in FVEM with the exact solution the following example:

\[
\begin{aligned}
\frac{\partial C(x,t)}{\partial t} + \frac{\partial (VC(x,t))}{\partial x} &= K(\beta \frac{\partial^3 C(x,t)}{\partial x^3} + (1 - \beta) \frac{\partial^3 C(x,t)}{\partial (-x)^3}), \quad \text{in} \quad [0,400] \times [0,T], \\
C(0,t) &= C(400,t) = 0, \\
C(x,0) &= \delta(x - 200). 
\end{aligned}
\]  

(35)

From [14], the Fourier transform of the exact solution of (35) on infinite domain is:

\[
\hat{C}(k,t) = \exp\left[\frac{1}{2}(2 - 2\beta)(-ik)^7Kt + \beta(ik)^7 - ikVt\right].
\]  

(36)

Hejazi et al. [44] proved that domain $[0,400]$ is large enough to use the above-mentioned solution as exact solution of (35).

We will use $h = 1, \tau = 0.5$ for all examples. In Figure (1), we solved equation (35) for $V = 0, K = 1, \beta = 0$ and $t = 100$ for several amounts of $\alpha$. In Figure (2) we solved equation (35) for $V = 0, K = 1, \alpha = 0.4$ and $t = 50$ for several different amount of $\beta$. In Figure (3), we solved equation (35) for $V = 0.5, K = 1, \beta = 0.5$ and $\alpha = 0.4$ numerous of $t$.

In Table (1) and Table (2), we calculate relationship between error with $h$ and $\tau$. 

---

Fig. 1: Comparison numerical (symbols) and exact (solid) solution for several $\alpha$

Fig. 2: Comparison numerical (symbols) and exact (solid) solution for several $\beta$
Fig. 3: Comparison numerical (symbols) and exact (solid) solution for several $t$

Table 1: calculating error for $V = 0.5, K = 1, \beta = 0.5, \alpha = 0.4$ and $\tau = 0.5$

<table>
<thead>
<tr>
<th>$h$</th>
<th>Error</th>
<th>rate of convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$8.01 \times 10^{-4}$</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>$3.08 \times 10^{-4}$</td>
<td>1.38</td>
</tr>
<tr>
<td>0.25</td>
<td>$1.14 \times 10^{-4}$</td>
<td>1.43</td>
</tr>
<tr>
<td>0.125</td>
<td>$4.81 \times 10^{-5}$</td>
<td>1.35</td>
</tr>
<tr>
<td>0.0625</td>
<td>$1.73 \times 10^{-5}$</td>
<td>1.37</td>
</tr>
</tbody>
</table>

Table 2: calculating error for $V = 0.5, K = 1, \beta = 0.5, \alpha = 0.4$ and $h = 1$

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>Error</th>
<th>rate of convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>$8.01 \times 10^{-4}$</td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>$3.98 \times 10^{-4}$</td>
<td>1.01</td>
</tr>
<tr>
<td>0.125</td>
<td>$2.03 \times 10^{-4}$</td>
<td>0.97</td>
</tr>
<tr>
<td>0.0625</td>
<td>$1.02 \times 10^{-4}$</td>
<td>0.99</td>
</tr>
</tbody>
</table>

Note: Order of accuracy can be estimated as $[81]$

$$p_k := \frac{\log\left(\frac{\varepsilon_k}{\varepsilon_{k+1}}\right)}{\log(2)}.$$  

5 Conclusion

In the present study, a FVEM has been successfully used to introduce a solution for a space fractional advection-dispersion equation. The fractional derivative is considered in the Grunwald form. We proved that when the mesh grid size is small enough, the full-discretization is stable. Based on the presented results, FVEM is an accurate numerical solution for space fractional advection-dispersion equations. The accuracy of our model improves through increasing the degree of basis function $\phi_i$. The above-mentioned example demonstrates that the results of our numerical method are compatible with those of the theoretical method.

References


