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Detection of new dynamical features of two coupled spins with an antiferromagnetic environment

A. S. F. Obada [1], *D. A. M. Abo-Kahla* [2,*] *and M. Abdel-Aty* [3]

[1] Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City 11884, Cairo, Egypt
[2] Department of Mathematics, Faculty of Education, Ain Shams University, Roxy, Cairo, Egypt
[3] Department of Mathematics, Faculty of Science, Sohag University, 82524 Sohag, Egypt

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Abstract: We find a class of spin environment at finite temperature in the thermodynamics limit models where individual quantum trajectories may depend on parameters that are undetermined by the full open system evolution. Our criteria are functions of the given quantum state and detect genuine multipartite entanglement, purity and information entropy that had not been identified so far. They are experimentally accessible without quantum state tomography and are easily computable as no optimization or eigenvalue evaluation is needed.

Keywords: Dynamical features, Coupled spins, An antiferromagnetic environment.

1 Introduction

In several recent papers (see e.g. [1]- [4]), the quantum correlation dynamics in open quantum systems have been studied. It was shown that the quantum correlation measured by quantum discord is more resistant against the environment than quantum entanglement [3]. Entanglement was first recognized as a curiosity of quantum mechanics because it gives rise to seemingly nonlocal correlations of measurement results of distant observers. Whereas the central role of many-body entanglement for various applications of quantum information processing [5] is undoubted, its role in quantum phase transitions [6] or antiferromagnetic environment is still debated, and questions concerning its potential assistance to the astonishing transport efficiency of inhomogeneous magnetic field and the anisotropy are still essentially open.

To answer such questions we need reliable techniques to characterize different dynamical features and entanglement properties of the considered system. It is usually addressed by means of concurrence criteria, information entropy (Shannon entropy) and purity, which work very well in two qubits case or even multipartite case [7, 8]. However, vast areas of the considered state-spaces are still widely unexplored due to the lack of suitable tools for detecting and characterizing entanglement. In fact, any realistic quantum systems interact inevitably with their surrounding environments will introduce quantum noise into the systems. Thus it is of fundamental importance to know the influence of the environment due to an external magnetic field on quantum entanglement and classical correlation.

Antiferromagnets subjected to an external magnetic field attracted considerable attention over the years [9]. One of the interesting phenomena in antiferromagnetic materials under an applied magnetic field is the magnetic-field-induced spin-flop transition. When the applied magnetic field is increased to the critical field point, the antiferromagnetic polarization flips into the direction perpendicular to the field. This is called the spin-flop transition, a first-order phase transition. Investigating entanglement in quantum spin systems with Heisenberg interactions [9]- [15] has been of the considerable interest, as a simple but realistic solid-state system, not only have been used to simulate a quantum computer, as well as quantum dots [17], nuclear spins [18], electronic spins [19], and optical lattice [20], but also display useful applications in quantum state transfer [21]. However, vast areas of the considered state-spaces are still widely unexplored due to the lack of suitable tools for detecting and characterizing general

* Corresponding author e-mail: doaa_abukahla@ymail.com

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dynamical features of coupled spins with an antiferromagnetic environment.

In this paper we study the most popular model of quantum open system based on inhomogeneous magnetic field and the anisotropy, but we keep the spin environment unchanged. Using a unifying approach in terms of canonical transformations we obtain an analytical solution of the system. The influence of the temperature and the crystal anisotropy field on the entanglement, information entropy and purity will be investigated.

The paper is organized as follows. We start in Sect. 2 by reviewing some basic concepts and formulae concerning the entanglement measure of our choice. After that, in Sect. 3, we present an equation of motion for the entanglement of an initially pure state of two two-level systems when either one of them undergoes some evolution (as described by a completely positive map); extensions of this setup are also introduced. Finally, in Sect. 4, we comment on the generalization of our framework to higher dimensional systems.

2 The model

The Hamiltonian of our model is the extension of the Hamiltonian of the model in Ref. [20]. We considered here the magnetic field and the anisotropy are inhomogeneous, but we keep the spin environment unchanged. The Hamiltonian of the whole system can be written as [21],

$$H = H_S + H_{SB} + H_B,$$

$$H_S = (\mu_0+\zeta)S_{01}^z + (\mu_0-\zeta)S_{02}^z + \Omega(S_{01}^+S_{02}^- + S_{01}^-S_{02}^+) + \Delta S_{01}^z S_{02}^z,$$

$$H_{SB} = \frac{g_0}{\sqrt{N}}[(S_{01}^+ + S_{02}^+)\sum_{i=1}^N S_i^-] + \frac{g_0}{\sqrt{N}}[(S_{01}^- + S_{02}^-)\sum_{i=1}^N S_i^+],$$

$$H_B = \frac{g}{N}\sum_{i\neq j}^N (S_i^+S_j^- + S_i^-S_j^+),$$

where H_S, H_{SB} and H_B are the Hamiltonian of the system, system-bath interaction and bath, respectively. The external magnetic fields are assumed to be along the Z-direction, μ_0 describes the uniformity of the field while ζ measures the degree of the inhomogeeity of the field. Ω is the coupling constant between any two-qubit spins and Δ is the anisotropy parameter. S_{0i}^+ and S_{0i}^- $(i=1,2)$ are the spin-flip operators of the spin system. S_i^+ and S_i^- are the corresponding of the ith qubit spin in the bath. N is the number of the bath atoms. g_0 is the coupling constant between the qubit system spins and bath spins, while g is that between the bath spins [22-25].

By using the collective angular momentum operator $J_\pm = \sum_{i=1}^N S_i^\pm$, the Hamiltonian in eqs. (3) and (4) becomes as follows:

$$H_{SB} = \frac{g_0}{\sqrt{N}}[(S_{01}^+ + S_{02}^+)J_-] + \frac{g_0}{\sqrt{N}}[(S_{01}^- + S_{02}^-)J_+], \quad (1)$$

$$H_B = \frac{g}{N}\sum_{i\neq j}^N (J_+J_- + J_-J_+) - g. \quad (2)$$

After using the Holstein-Primakoff transformation which is defined as:

$$J_+ = b^\dagger(\sqrt{N - b^\dagger b}), J_- = (\sqrt{N-b^\dagger b})b, with\, [b^\dagger, b] = 1, \quad (3)$$

the Hamiltonian in eqs. (5) and (6) can be written as:

$$H_{SB} = g_0[(S_{01}^+ + S_{02}^+)(\sqrt{1-\tfrac{b^\dagger b}{N}})b] + g_0[(S_{01}^- + S_{02}^-)b^\dagger(\sqrt{1-\tfrac{b^\dagger b}{N}})],$$

$$H_B = g[b^\dagger(1-\frac{b^\dagger b}{N})b + (\sqrt{1-\frac{b^\dagger b}{N}})bb^\dagger(\sqrt{1-\frac{b^\dagger b}{N}})] - g. \quad (4)$$

If $N \longrightarrow \infty$ (in the thermodynamic limit) and we have finite temperature, we can write, from eqs. (8) and (9), H_{SB} and H_B as follows:

$$H_{SB} = g_0[(S_{01}^+ + S_{02}^+)b + (S_{01}^- + S_{02}^-)b^\dagger], \quad (5)$$

$$H_B = 2gb^\dagger b, \quad (6)$$

since the energy of the excitations original from the interaction between the system and the bath is very low, we bought the approximation that $\frac{b^\dagger b}{N}$ tends to zero.

In the following, we deducted the density matrix evolution of the spin system. Since the Hamiltonian is time independent, we can write the density matrix as follows:

$$\rho(t) = \exp(-iHt)\rho(0)\exp(iHt). \quad (7)$$

We assume that $\rho(0) = \rho_S(0)\otimes\rho_B$, i.e. is separable between the system and the bath, where $\rho_S(0)$ describe the density initial state of the spin system and $\rho_B = \frac{\exp(\frac{-H_B}{KT})}{Z}$, where Z is the partition function, $Z = \frac{1}{1-\exp(\frac{-g}{T})}$, and the Boltzmann constant K is equal one here. We can obtain the reduced system density matrix by taking the trace over the environment, i.e. $\rho_S(t) = Tr(\rho(t)),$.

If we take the initial state $|\Psi\rangle = \cos\alpha|01\rangle + \sin\alpha|10\rangle$ where $0\leq\alpha\leq\pi$, $\rho_S(0) = |\Psi\rangle\langle\Psi|$, the reduced density matrix of the system can be written as:

$$\begin{aligned}\rho_S(t) =& \\ &\tfrac{1}{Z}\cos^2\alpha Tr_B[\exp(-iHt)|01\rangle\times\exp(\tfrac{-H_B}{T})\langle 10|\exp(iHt)] \\ &+\tfrac{1}{Z}\sin^2\alpha Tr_B[\exp(-iHt)|10\rangle\times\exp(\tfrac{-H_B}{T})\langle 01|\exp(iHt)] \\ &+\tfrac{1}{Z}\cos\alpha\sin\alpha Tr_B[\exp(-iHt)|01\rangle\times\exp(\tfrac{-H_B}{T})\langle 01|\exp(iHt)] \\ &+\tfrac{1}{Z}\cos\alpha\sin\alpha Tr_B[\exp(-iHt)|10\rangle\times\exp(\tfrac{-H_B}{T})\langle 10|\exp(iHt)],\end{aligned}$$

In order to calculate $\exp(-iHt)|01\rangle$, for example, we suppose

$$\exp(-iHt)|01\rangle = A|00\rangle + B|01\rangle + C|10\rangle + D|11\rangle, \quad (8)$$

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where A,B,C and D are functions of operators b, $b^{\dagger}$, and time t. By using the schrödinger equation one can find A,B,C and D. Then the reduced density matrix of the system can be expressed in the basis $\{|1\rangle = |gg\rangle, |2\rangle = |ge\rangle, |3\rangle = |eg\rangle, |4\rangle = |ee\rangle\}$,

$$\begin{aligned}
\rho_{11} &= \sum_{n=0}^{\infty}(\cos^2\alpha A_1A_1^{\dagger} + \sin^2\alpha E_1E_1^{\dagger} + \cos\alpha\sin\alpha A_1E_1^{\dagger} \\
&\quad + \cos\alpha\sin\alpha E_1A_1^{\dagger})(n+1)\exp(\frac{-2gn}{T}), \\
\rho_{22} &= \sum_{n=0}^{\infty}(\cos^2\alpha B_1B_1^{\dagger} + \sin^2\alpha F_1F_1^{\dagger} + \cos\alpha\sin\alpha B_1F_1^{\dagger} \\
&\quad + \cos\alpha\sin\alpha F_1B_1^{\dagger})\exp(\frac{-2gn}{T}), \\
\rho_{33} &= \sum_{n=0}^{\infty}(\cos^2\alpha C_1C_1^{\dagger} + \sin^2\alpha G_1G_1^{\dagger} + \cos\alpha\sin\alpha C_1G_1^{\dagger} \\
&\quad + \cos\alpha\sin\alpha G_1C_1^{\dagger})\exp(\frac{-2gn}{T}), \\
\rho_{44} &= \sum_{n=0}^{\infty}(\cos^2\alpha D_1D_1^{\dagger} + \sin^2\alpha K_1K_1^{\dagger} + \cos\alpha\sin\alpha D_1K_1^{\dagger} \\
&\quad + \cos\alpha\sin\alpha K_1D_1^{\dagger})n\exp(\frac{-2gn}{T}), \\
\rho_{23} &= \sum_{n=0}^{\infty}(\cos^2\alpha B_1C_1^{\dagger} + \sin^2\alpha F_1G_1^{\dagger} + \cos\alpha\sin\alpha B_1G_1^{\dagger} \\
&\quad + \cos\alpha\sin\alpha F_1C_1^{\dagger})\exp(\frac{-2gn}{T}). \qquad (9)
\end{aligned}$$

Depending on the pervious treatments, We calculated the concurrence, the purity and the information entropy $H(\sigma_z)$ and $H(\sigma_x)$.

3 Concurrence

In two-qubit system, the concurrence [26] can be considered to quantify the quantum correlation. In the mixed state, the concurrence is defined to be the minimum average value of the measure over all ensemble decompositions of ρ

$$C(\rho) = \min_{\{p_j,|\psi_k\rangle\}}\{\sum p_jC(\psi_j)\rho = \sum p_j|\psi_j\rangle\langle\psi_j|\}, \quad (10)$$

which also indicates that the concurrence is convex, i.e., $C(\lambda_1\rho_1 + \lambda_2\rho_2) \le \lambda_1C(\rho_1) + \lambda_2C(\rho_2)$, where $\lambda_1, \lambda_2 \ge 0$ and $\lambda_1 + \lambda_2 = 1$. In the case of an arbitrary joint state of two qubits, Wootters derived a simple formula for concurrence which is very useful for analysis and calculation. The explicit form of concurrence C can be expressed as $C(\rho) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}$, where λ_k are the eigenvalues of the matrix $\xi = \rho\tilde{\rho} = \rho(\sigma_y^1 \otimes \sigma_y^2)\rho^{\intercal}(\sigma_y^1 \otimes \sigma_y^2)$ arranged in decreasing order. Here $\rho^{\intercal}$ denotes the transpose of density matrix ρ. It is clear that the concurrence lies between zero and one.

4 The information entropies and the purity

By using the atomic reduced density operator $\rho_{12}(t)$, we obtain the information entropies of the atomic operators σ_x, σ_y and σ_z in the form [27-28]

$$H(\sigma_\gamma) = -\sum_{i=1}^{4}P_i(\sigma_\gamma)\ln P_i(\sigma_\gamma), \gamma = x,y,z. \quad (11)$$

-The evolution of the purity $P_S(t)$ is given by

$$P_S(t) = 1 - Tr_S(\rho_S^2(t)) \quad (12)$$

where $\rho_S(t)$ is the reduced density matrix of bipartite system which is defined by $\rho_S(t) = Tr_F\ \rho(t)$.

It is shown that with the increase of the temperature T, the entanglement decreases, but the range of disentanglement intervals increases. This can be understood that when the temperature is high, the decoherence effect induced by the environment is strong. So, the disentanglement intervals appear and the entanglement decreased. Besides, the purity decreased, but the information entropy $H(\sigma_z)$ and $H(\sigma_x)$ increased (see Fig. 1).

It is shown that with the increase of the anisotropy parameter $\Delta = 50$, the entanglement increases remarkably. On the other hand the disentanglement intervals disappears, i.e. in this case the environment has no effect on the entanglement evolution between the two qubits. Also, the purity increased, while the information entropy $H(\sigma_z)$ and $H(\sigma_x)$ decreased (see Fig. 2).

We consider the case in which the initial state is $\cos\alpha|01\rangle + \sin\alpha|10\rangle$ where $0 \le \alpha \le \pi$. So, when $\alpha = 0$, we have the initial state $|01\rangle$, i.e. the two-qubits are initially in the separate state $|01\rangle$. We find in the pervious case ($\alpha = 0$) the unentangled system evolves into an entangled system, while the purity decreased with the increase of the time, but the information entropy $H(\sigma_z)$ and $H(\sigma_x)$ increased. It is clear the effect of the environment. when $\alpha = \frac{\pi}{4}$, we have the initial state $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$, i.e. the two-qubits are initially entangled. Here, the disentanglement intervals appear, because of the simple effect of the environment. Also, we find the purity increased with the increase of the time, while the information entropy $H(\sigma_z)$ and $H(\sigma_x)$ decreased. when $\alpha = \frac{\pi}{12}$, the entanglement and the purity decreased remarkably, but the range of disentanglement intervals increases. However, the information entropy $H(\sigma_z)$ and $H(\sigma_x)$ increased. when $\alpha = \frac{5\pi}{6}$, we have the initial state $(-\frac{1}{2}|01\rangle + \frac{\sqrt{3}}{2}|10\rangle)$, the environment has no effect on the entanglement evolution between the two qubits, the purity and the information entropy $H(\sigma_z)$ and $H(\sigma_x)$ (see Fig. 3).

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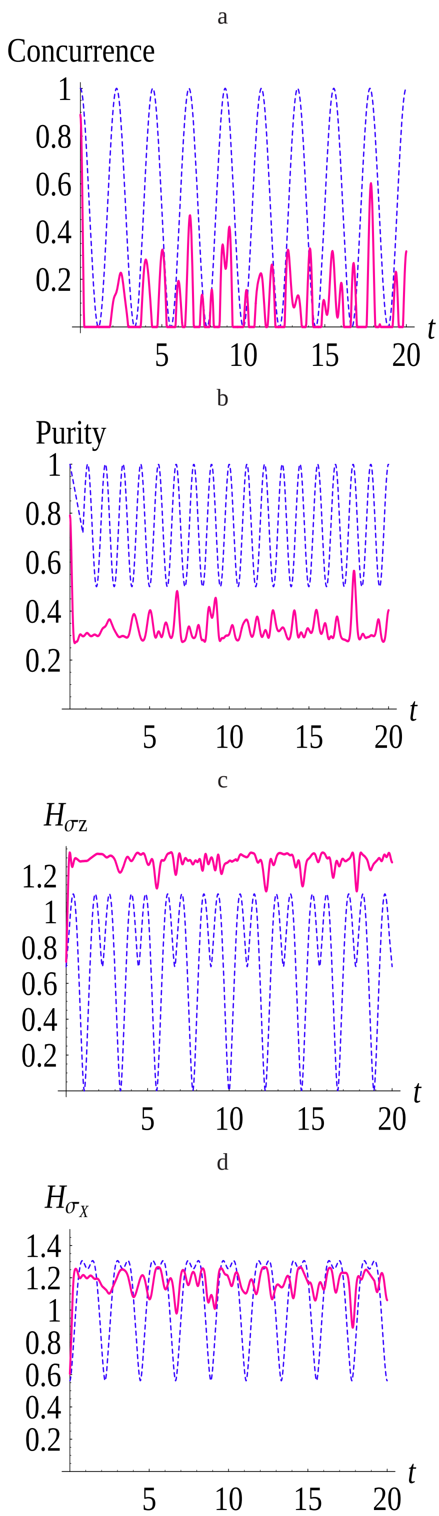

Fig. 1 Figures of the case in which $\mu_0 = 2, \Omega = 0, g = 1, g_0 = 1, \zeta = 0,\ \Delta = 0, \alpha = \frac{\pi}{4}$ where dot blue and solid red curves correspond, respectively, to temperature $T = 0.1$ and 10

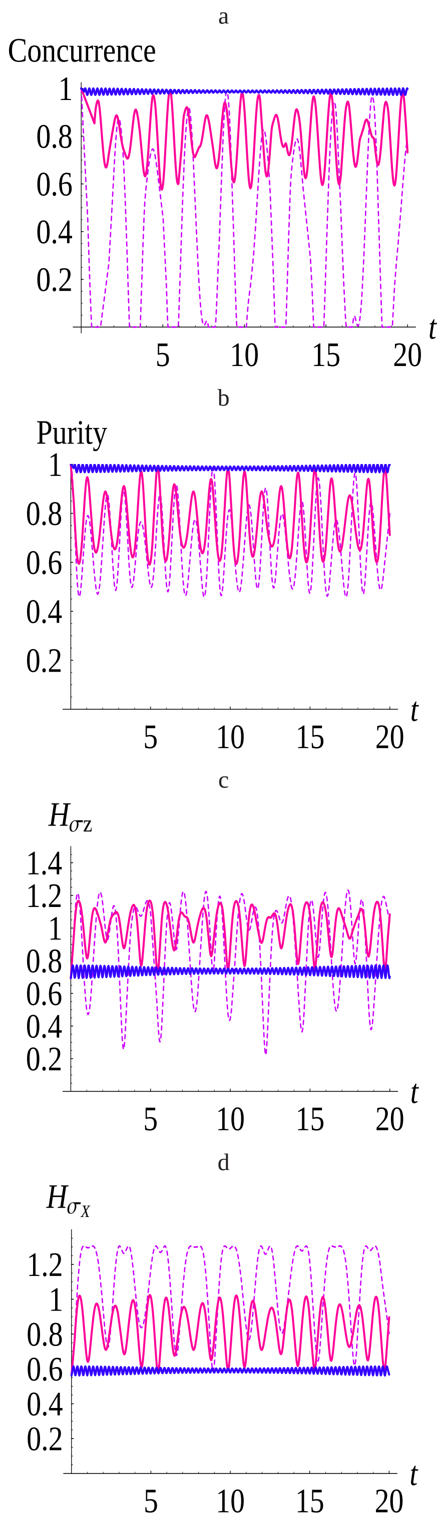

Fig. 2 Figures of the case in which $\mu_0 = 2, \Omega = 0, g = 1, g_0 = 1, \zeta = 0,\ T = 1, \alpha = \frac{\pi}{4}$ where dot blue and solid red curves correspond, respectively, to temperature $\Delta = 0,\ 10$ and 50

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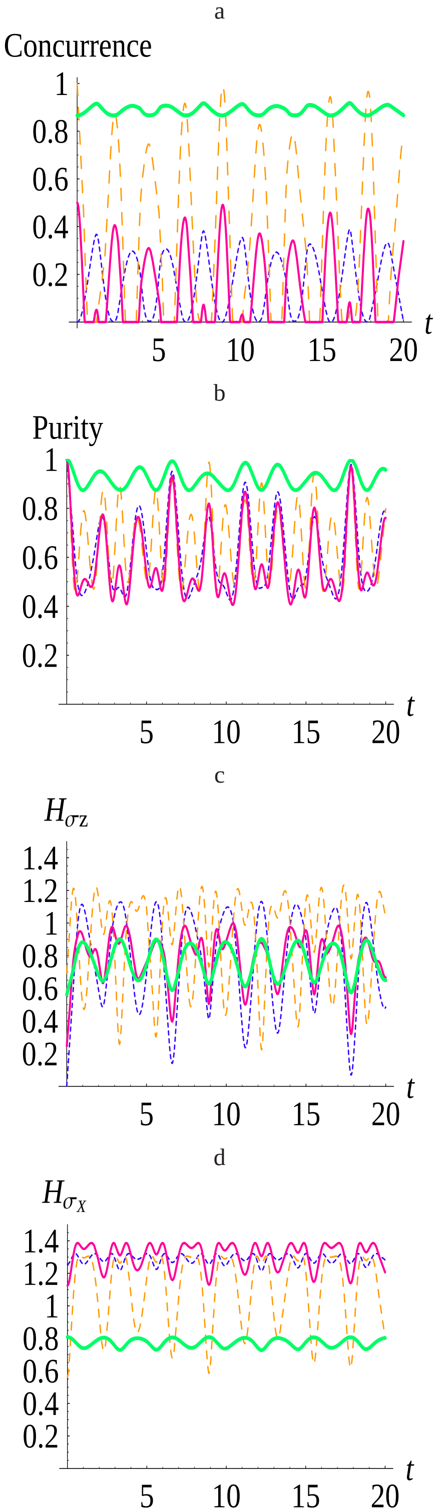

Fig. 3 Figures of the case in which $\mu_0 = 2, \Omega = 0, g = 1, g_0 = 1, \zeta = 0, \Delta = 0, T = 1$ where dot blue, dashed orange,solid red, and solid bold green curves correspond, respectively, to $\alpha = 0, \frac{\pi}{4}, \frac{\pi}{12}$ and $\frac{5\pi}{6}$.

5 conclusion

We studied two-qubit system under a quantum spin environment at finite temperature in the thermodynamics limit. We have the initial state $\cos\alpha\,|01\rangle + \sin\alpha\,|10\rangle$, $0 \leq \alpha \prec \pi$, we calculated the entanglement, the purity and the information entropy $H(\sigma_z)$ and $H(\sigma_x)$. We find with the increase of the temperature T, the entanglement and the purity decreased, but the information entropy $H(\sigma_z)$ and $H(\sigma_x)$ increased. Because of the decoherence effect induced by the environment is strong. with the increase of the anisotropy parameter Δ, the entanglement and the purity increased remarkably, while the information entropy $H(\sigma_z)$ and $H(\sigma_x)$ decreased, i.e. the environment has no effect on the entanglement evolution between the two qubits, the purity and the information entropy. For any α the effect of the environment is clear. However, when $\alpha = \frac{3\pi}{4}(\frac{5\pi}{6})$, the environment has no effect on the entanglement evolution between the two qubits, the purity and the information entropy.

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