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Approximate Scheme for Fractional Differential Equation of Order $1 < \alpha < 2$ Using Finite Difference Method

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Abstract: In this article, we develop composite Simpson's $(\frac{1}{3})^{rd}$ rule for fractional differential equation with Caputo derivative of order α, $1<\alpha<2$ using finite difference method. We use forward, central and backward difference formulae for approximating the derivatives. We give comparison of numerical solution given by Simpson's $(\frac{1}{3})^{rd}$ rule and Trapezoidal rule. Result shows that Simpson's rule gives better approximation. Furthermore, as an application of the scheme we solve test problems using Mathematical software.

Keywords: Fractional differential equation, composite Simpson's $(\frac{1}{3})^{rd}$ rule, finite difference method, Matlab.

1 Introduction

The fractional calculus is a name for the theory of integrals and derivatives of arbitrary real or complex order which have been focus of many studies due to their application in various areas of physics and engineering [1,2,3,4,5,6,7,8]. There are efficient methods for exact and numerical solutions of fractional differential equations. Particularly, Exact solutions of fractional differential equations can not be obtained easily. Some of these numerical methods include: Variational Iteration Method [9], Extrapolation Method [10], General (n+1)-explicit finite difference formulas [11] and fractional finite difference method (FFDM).

In this article, we consider linear fractional differential equation containing Caputo fractional derivative. We develop a new technique by which fractional derivatives of u(t) are approximated by finite difference method. The composite Simpson's $(\frac{1}{3})^{rd}$ rule [12] is applied to approximate the integral part in Caputo definition of order $1 < \alpha < 2$, then FFDM is used to approximate first, second and third order derivatives. Our method efficiency is demonstrated by comparing approximate solutions for fractional differential equations with the exact solutions or with other approximate solutions those are obtained by other solvers and methods. Moreover, MATLAB is used to solve these differential equations.

1.1 Preliminaries

In this section, we will introduce definitions of fractional derivative. Also, we will provide commonly used finite difference formulae and composite Simpson's rule for definite integral.

Definition 1: The Riemann – Liouville fractional derivative [13] is defined as

$$(D^{\alpha}u)(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_{a}^{t}(t-x)^{n-\alpha-1}u(x)dx, \quad t \in [a,b], \tag{1}$$

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where n is a positive integer satisfying $n-1<\alpha\leq n$, $\alpha\in R^{+}$.

Definition 2: The Caputo fractional derivative [13] for $n-1<\alpha\leq n$, $\alpha\in R^{+}$ is defined as

$$(D^{\alpha}u)(t)=\frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}(t-x)^{n-\alpha-1}u^{n}(x)dx,\quad t\in[a,b]. \tag{2}$$

1.1.1 Finite difference formulae

Finite difference formulae are commonly used to solve differential equations numerically. Suitable finite difference formulae are used to replace derivative terms on a discrete domain. The accuracy is increased if the number of mesh points are increased. In general, Taylor series is used to obtained finite difference formulae.

For $u^{''}(x)$, the finite difference formula is

$$u^{''}(x)=\frac{u(x+2h)-2u(x+h)+u(x)}{h^{2}}+O(h). \tag{3}$$

The forward finite difference formula for $u^{'''}(x)$ is

$$u^{'''}(x)=\frac{-u(x)+3u(x+h)-3u(x+2h)+u(x+3h)}{h^{3}}+O(h). \tag{4}$$

The central finite difference formula for $u^{'''}(x)$ is

$$u^{'''}(x)=\frac{-u(x-2h)+2u(x-h)-2u(x+h)+u(x+2h)}{2h^{3}}+O(h^{2}). \tag{5}$$

The backward finite difference formula for $u^{'''}(x)$ is

$$u^{'''}(x)=\frac{u(x)-3u(x-h)+3u(x-2h)-u(x-3h)}{h^{3}}+O(h). \tag{6}$$

However, there are number of methods for estimating definite integrals, we will use the following composite Simpson's rule [12].

Theorem 1: Let $u\in C^{4}$ [a,b], n be even, $h=\frac{b-c}{n}$, and $x_{i}=a+ih$, for $i=0,1,2,\ldots.n$. There exists a γ ∈(a,b) for which the composite Simpson's rule for n subintervals can be written as

$$\int_{a}^{b}u(x)dx=\frac{h}{3}\left[u(a)+2\sum_{i=1}^{\frac{n}{2}-1}u(x_{2i})+4\sum_{i=1}^{\frac{n}{2}}u(x_{2i-1})+u(b)\right]-\frac{b-a}{180}h^{4}u^{4}(\gamma). \tag{7}$$

2 FFDM for the fractional derivative

In this section, we introduce method of approximating the Caputo fractional derivative. We used Caputo definition of fractional derivatives because it has a benefit of dealing properly with the initial value problem which is given in terms of their integer order and the field variables.

For $a=0,n=1,t\geq 0$ and $1<\alpha<2$, the Caputo definition for fractional derivative equation (2) is defined as

$$(D^{\alpha}u)(t)=\frac{1}{\Gamma(2-\alpha)}\int_{0}^{t}(t-x)^{1-\alpha}u^{''}(x)dx. \tag{8}$$

In right hand side of equation (8), we apply integration by parts and get,

$$(D^{\alpha}u)(t)=\frac{1}{(2-\alpha)\Gamma(2-\alpha)}\left[(t)^{2-\alpha}u^{''}(0)+\int_{0}^{t}(t-x)^{2-\alpha}u^{'''}(x)dx\right]. \tag{9}$$

The Composite Simpson's rule is used to approximate the last integral in (9), we get,

$$\int_0^t (t-x)^{2-\alpha} u^{'''}(x)dx \approx \frac{h}{3}\Bigg[(t)^{2-\alpha}u^{'''}(0) + 2\sum_{i=1}^{\frac{n}{2}-1}(t-x_{2i})^{2-\alpha}u^{'''}(x_{2i}) + 4\sum_{i=1}^{\frac{n}{2}-1}(t-x_{2i-1})^{2-\alpha}u^{'''}(x_{2i-1}) + (t-b)^{2-\alpha}u^{'''}(b)\Bigg], \tag{10}$$

where $h = \frac{b-a}{n}$ and $x_i = a + ih$, for $i = 0, 1, 2, \ldots n$.

Now, substituting value of equation (10) in equation (9), we get

$$(D^{\alpha}u)(t) = \frac{1}{(2-\alpha)\Gamma(2-\alpha)}\Bigg[(t)^{2-\alpha}u^{''}(0) + \frac{h}{3}\Bigg[(t)^{2-\alpha}u^{'''}(0) + 2\sum_{i=1}^{\frac{n}{2}-1}(t-x_{2i})^{2-\alpha}u^{'''}(x_{2i}) + 4\sum_{i=1}^{\frac{n}{2}-1}(t-x_{2i-1})^{2-\alpha}u^{'''}(x_{2i-1}) + (t-b)^{2-\alpha}u^{'''}(b)\Bigg]\Bigg]. \tag{11}$$

For small value of h, we approximate $u^{''}(0)$ and $u^{'''}(0)$ using forward finite difference formula for 2^{rd} order derivative, equation (3), and 3^{rd} order derivative, equation (4) respectively. Also, $u^{'''}(x_{2i})$ and $u^{'''}(x_{2i-1})$ can be approximated using central finite difference formula for 3^{rd} order derivative, equation (5). Finally, backward finite difference formula for 3^{rd} order derivative in equation (6) is used to approximate $u^{'''}(b)$.

By substituting values of $u^{''}(0)$, $u^{'''}(0)$, $u^{'''}(x_{2i})$, $u^{'''}(x_{2i-1})$ and $u^{'''}(b)$ in equation (11), we obtain

$$(D^{\alpha}u)(t) = \frac{1}{(2-\alpha)\Gamma(2-\alpha)}\left[P + \frac{h}{3}\left[Q + 2R + 4S + T\right]\right]. \tag{12}$$

where

$$P = (t)^{2-\alpha}\frac{u(2h) - 2u(h) + u(0)}{h^2},$$

$$Q = (t)^{2-\alpha}\frac{-u(0) + 3u(h) - 3u(2h) + u(3h)}{h^3},$$

$$R = \sum_{i=1}^{\frac{n}{2}-1}(t-x_{2i})^{2-\alpha}\frac{u(x_{2i}-2h) + 2u(x_{2i}-h) - 2u(x_{2i}+h) + u(x_{2i}+2h)}{2h^3},$$

$$S = \sum_{i=1}^{\frac{n}{2}}(t-x_{2i-1})^{2-\alpha}\frac{-u(x_{2i-1}-2h) + 2u(x_{2i-1}-h) - 2u(x_{2i-1}+h) + u(x_{2i-1}+2h)}{2h^3},$$

$$T = (t-b)^{2-\alpha}\frac{u(b) - 3u(b-h) + 3u(b-2h) - u(b-3h)}{h^3}.$$

Now, we can find $D^{\alpha}u(t)$ approximately using equation (12) which depends on the value of h. If we decrease value of h, then approximate value of $D^{\alpha}u(t)$ will be more accurate.

3 Applications

The following examples will be solved using in Matlab for fractional differential equations of order $1 < \alpha < 2$ and compare with the exact solutions. We will use equation (12) for the approximation of $D^{\alpha}u(t)$. **Example 3.1** [14] Consider

Table 1: Numerical solution of example 3.1 with h = 0.1

T	Exact	Approximate (Trapezoidal Rule)	Abs. Error	Approximate (Simpson's (1/3)rd Rule)	Abs. Error
0.1	0.01	0.00999999999999998	2.26E-17	0.0100000000000000	0.00E+00
0.2	0.04	0.0399999999999980	2.08E-16	0.0399999999999999	1.11E-16
0.3	0.09	0.0899999999999940	5.97E-16	0.0900000000000000	0.00E+00
0.4	0.16	0.1599999999999900	1.03E-15	0.1599999999999990	1.03E-15
0.5	0.25	0.2499999999999800	2.00E-15	0.2500000000000000	0.00E+00
0.6	0.36	0.3599999999999700	3.00E-15	0.3600000000000000	0.00E+00
0.7	0.49	0.4899999999999700	2.94E-15	0.4900000000000000	0.00E+00
0.8	0.64	0.6399999999999600	4.11E-15	0.6400000000000000	0.00E+00
0.9	0.81	0.8099999999999400	6.11E-15	0.8100000000000000	0.00E+00
1	1	0.9999999999999300	6.99E-15	1.0000000000000000	0.00E+00

Table 2: Numerical solution of example 3.2 with h = 0.1

T	Exact	Approximate (Trapezoidal Rule)	Abs. Error	Approximate (Simpson's (1/3)rd Rule)	Abs. Error
0.1	0.01	0.0100000000000320	3.20E-15	0.00999999999999845	1.55E-15
0.2	0.04	0.0400000000000930	9.29E-15	0.0400000000000150	1.49E-15
0.3	0.09	0.0900000000001480	1.48E-14	0.0900000000000420	4.20E-15
0.4	0.16	0.1600000000001900	1.90E-14	0.1599999999998800	1.20E-14
0.5	0.25	0.250000000000023000	2.30E-14	0.2500000000000000	0.00E+00
0.6	0.36	0.3600000000002800	2.80E-14	0.3600000000000000	0.00E+00
0.7	0.49	0.4900000000003300	3.31E-14	0.4899999999998500	1.49E-14
0.8	0.64	0.6400000000003800	3.79E-14	0.6400000000000000	1.11E-16
0.9	0.81	0.8100000000004300	4.30E-14	0.8099999999996400	3.61E-14
1	1	1.0000000000004000	4.00E-14	1.0000000000000000	0.00E+00

the following fractional differential equation:

$$D^{\alpha}u(t) = t^{\alpha}u(t) + 4\sqrt{\frac{t}{\pi}} - t^{\alpha+2}, \tag{13}$$

with given conditions $u(0) = 0, u^{'}(0) = 0$.

For $\alpha = 1.5$, the exact solution is $u(t) = t^2$. The results are obtained using (12) for $h = 0.1$ are given in Table 1. We compare results with solution given in [15] using trapezoidal method and also with exact solutions.

Example 3.2 [14] For the fractional differential equation:

$$D^2u(t) + D^{1.5}u(t) + u(t) = 4\sqrt{\frac{t}{\pi}} - t^2 + 2, \tag{14}$$

with given condition $u(0) = 0$, the exact solution is $u(t) = t^2$. The results are for $h = 0.1$ are given in Table 2.

4 Conclusion

In this work, we derived composite Simpson's $(\frac{1}{3})^{rd}$ rule for fractional differential equation with Caputo derivative of order α, $1<\alpha<2$ using finite difference method. We used forward, central and backward difference formulae for approximating the derivatives. Finally, numerical examples are simulated to demonstrate efficiency of given method. We observed that the obtained results using Matlab software gave very less absolute error compared with the exact solution. We gave comparison of numerical solutions given by Simpson's $(\frac{1}{3})^{rd}$ rule and Trapezoidal rule. Result shows that Simpson's rule gives better approximation.

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