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Elzaki Decomposition Method and its Applications in Solving Linear and Nonlinear Schrodinger Equations

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Abstract: In this paper, the Elzaki transform and Adomian decomposition method are coupled and used to determine the analytical solution of both linear and nonlinear Schrodinger differential equations in what is termed as Elzaki decomposition method. The proposed method worked perfectly without any need of linearization or discretization in comparison with other methods. The solutions obtained for the problems considered are in full agreement with their corresponding exact solutions in literature.

Keywords: Elzaki Transform; Adomian Decomposition Method; Schrodinger Equations

1 Introduction

Linear and nonlinear Schrodinger equations often arise in many branches of physics and engineering science such as in quantum mechanics, optics and plasma physics among others. The study of these equations and their solutions has become of great interest to many researchers due to its various applications. To cite a few, Wazwaz [1] utilized the Adomian decomposition method as a reliable technique for treatment of Schrodinger equations. In Wazwaz [2], the variational iteration method was used to determine the exact solutions for both linear and nonlinear Schrodinger equations. Also, Zhang et al [3] used the He's frequency formulation as a method to search for the solutions of Schrodinger equations, and the solutions determined turn out to be in good agreement with the results determined in [1,2]. However, we intend to couple the Elzaki transform established recently by Elzaki [4] with the celebrated method of the 80^{th}; the Adomian decomposition method [5,6]. The Elzaki transform is known for its effectiveness in solving linear ordinary differential equations, linear partial differential equation and integral equations among its competing transforms as demonstrated in [7,8,9] While on the other hand, the Adomian decomposition method [5,6] is a well-known method for solving linear and nonlinear, homogeneous and nonhomogeneous differential and partial differential equations, integro-differential and fractional differential equations that gives exact solutions in form of a convergent series. Further, the Adomian decomposition method is also proven be to an effective and powerful method for treating the afore mentioned equations after the successes recorded by many researches such as in [10,11,12,13,14,15,16].

It is expected in the end of this study that this coupling, the Elzaki decomposition method would give exact solutions for the linear and nonlinear Schrodinger equations under consideration in relation to other decompositions methods that also work perfectly in other settings such as in Laplace decomposition method [17], Sumudu decomposition method [18], Natural decomposition method [19], Aboodh decomposition method [20] and other couplings available in the literature as the effectiveness of both the Elzaki transform and the Adomian decomposition method cannot be overestimated. perspective.

2 Elzaki Transform

The Elzaki transform of the functions belonging to a class A, where
$A = \{u(t) : \exists M, k_1, k_2 > 0 \; such\; that \; |u(t)| < Me^{|t|/k_j}, if \; t \in (-1)^j \times [0,\infty)\}$ where $u(t)$ is denoted by $E[u(t)] = U(v)$ and defined as

$$E[u(t)] = v\int_0^{\infty} u(t)e^{-\frac{t}{v}}dt = U(v), \; v \in (k_1, k_2). \quad (1)$$

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Following are some of the properties of Elzaki transform

1. $E\{t^n\} = n!\, v^{n+2}, \quad n \geq 0$
2. $E\{e^{-at}\} = \frac{v^2}{1+av}$
3. $E\{sinat\} = \frac{av^3}{1+a^2v^2}$
4. $E\{u(t)\} = \frac{U(v)}{v} - vu(0)$
5. $E\{u^n(t)\} = \frac{U(v)}{v^n} - \sum_{k=0}^{n-1} v^{2-n+k} u^k(0)$,

3 Elzaki Adomian Decomposition Method

We consider the more general form of the nonlinear Schrodinger differential equation given in the complex valued function u of the form

$$iu_t + u_{xx} + \lambda |u|^{2r} u = 0,\ r \geqslant 1,\ i = \sqrt{-1} \tag{2}$$

with the following initial and boundary conditions given by

$$u(x,0) = f(x) \text{ and } u(0,t) = g(x) \ \&\ u_x(0,t) = h(x). \tag{3}$$

Clearly, when $\lambda = 0$ in Eq.(2) we then obtain the linear version of the Schodinger equation Eq.(2), i.e,

$$iu_t + u_{xx} = 0. \tag{4}$$

Now, to present the Elzaki decomposition method on the more general Schrodinger equation given in Eq.(2), we first rewrite Eq.(2) as

$$u_t = iu_{xx} + i\lambda |u|^{2r} u. \tag{5}$$

We then use the Elzaki transform defined in Eq.(1) of both sides of Eq.(5);

$$E[u_t] = E[iu_{xx}] + E[i\lambda |u|^{2r} u]. \tag{6}$$

Using the differentiation property of Elzaki transform and the initial condition we get

$$\frac{1}{v} E[u] - vf(x) = iE[u_{xx}] + i\lambda E[|u|^{2r} u]. \tag{7}$$

$$E[u] = v^2 f(x) + ivE[u_{xx}] + i\lambda vE[|u|^{2r} u]. \tag{8}$$

Next step is to replace the unknown function u by an infinite series given by

$$u = \sum_{m=0}^{\infty} u_m(x,t), \tag{9}$$

and we replace the nonlinear term $Nu = |u|^{2r} u$ by the series

$$Nu = \sum_{m=0}^{\infty} A_m(u_0, u_1, \ldots), \tag{10}$$

where $A_m(u_0, u_1, \ldots)'$s are the Adomian polynomials [5,6] to be determined recurrently by the formula

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N\left(\sum_{i=0}^{n} \lambda^i u_i \right) \right]_{\lambda=0}, \quad n = 0, 1, 2, \ldots \tag{11}$$

Thus, on substituting Eq.(9) , Eq.(10) and Eq.(11) into Eq.(8) we get

$$E\left[\sum_{m=0}^{\infty} u_m(x,t) \right] = v^2 f(x) + ivE\left[\sum_{m=0}^{\infty} u_{m_{xx}} \right] + i\lambda vE\left[\sum_{m=0}^{\infty} A_m \right] \tag{12}$$

$$\sum_{m=0}^{\infty} E[u_m(x,t)] = v^2 f(x) + ivE\left[\sum_{m=0}^{\infty} u_{m_{xx}} \right] + i\lambda vE\left[\sum_{m=0}^{\infty} A_m \right]. \tag{13}$$

Thus, on comparing both sides of Eq.(13) and then taking the inverse Elzaki transform; we finally obtain the general solution of Eq.(2) given recursively as:

$$\begin{aligned}
u_0(x,t) &= f(x), \\
u_1(x,t) &= iE^{-1}\left[vE[u_{0_{xx}}] \right] + i\lambda E^{-1}\left[vE[A_0] \right], \\
u_2(x,t) &= iE^{-1}\left[vE[u_{1_{xx}}] \right] + i\lambda E^{-1}\left[vE[A_1] \right], \\
u_3(x,t) &= iE^{-1}\left[vE[u_{2_{xx}}] \right] + i\lambda E^{-1}\left[vE[A_2] \right], \\
u_4(x,t) &= iE^{-1}\left[vE[u_{3_{xx}}] \right] + i\lambda E^{-1}\left[vE[A_3] \right], \\
&\vdots
\end{aligned} \tag{14}$$

and so on. Where, $f(x)$ is the prescribed initial condition, and A_n's are the Adomian polynomials to be determined from Eq.(11). Thus, Eq.(14) can be written in compact form as

$$\begin{aligned}
u_0(x,t) &= f(x), \\
u_{n+1}(x,t) &= iE^{-1}\left[vE[u_{n_{xx}}] \right] + i\lambda E^{-1}\left[vE[A_n] \right], n \geqslant 0.
\end{aligned}$$

4 Application of the Method

Here, we consider the following examples in order to demonstrate the effectiveness of the method described above.

4.1 Example One

Consider the linear Schrodinger differential equation with $\lambda = 0$

$$iu_t + u_{xx} = 0, \tag{15}$$

with the initial condition given by

$$u(x,0) = ae^{ikx}, \text{ with } a \ \&\ k \text{ constants}. \tag{16}$$

Applying th Elzaki transform to Eq.(15)

$$\frac{1}{v}E[u(x,t)] - vu(x,0) = iE[u_{xx}] \tag{17}$$

$$E[u(x,t)] = v^2u(x,0) + ivE[u_{xx}]. \tag{18}$$

On using the initial condition given in Eq.(16),

$$E[u(x,t)] = v^2ae^{ikx} + ivE[u_{xx}]. \tag{19}$$

Now, applying the inverse Elzaki transform to Eq.(19), we get

$$u(x,t) = ae^{ikx} + E^{-1}\left[ivE[u_{xx}]\right]. \tag{20}$$

Assuming the infinite series solution of the unknown function u and comparing both sides of Eq.(20) as described above, we thus obtain the general solution recursively as

$$\begin{aligned} u_0(x,t) &= ae^{ikx}, \\ u_{n+1}(x,t) &= iE^{-1}\left[vE[u_{n_{xx}}], n \geqslant 0.\right. \end{aligned} \tag{21}$$

The first few components of $u(x,t)$ are given by

$$u_0(x,t) = ae^{ikx}, \tag{22}$$

$$\begin{aligned} u_1(x,t) &= iE^{-1}\left[vE[u_{0_{xx}}]\right] \\ &= iE^{-1}\left[-ak^2v^3e^{ikx}\right], \\ &= -aik^2te^{ikx}, \end{aligned} \tag{23}$$

$$\begin{aligned} u_2(x,t) &= iE^{-1}\left[vE[u_{1_{xx}}]\right] \\ &= E^{-1}\left[-ak^4v^4e^{ikx}\right], \\ &= \frac{-ak^4t^2e^{ikx}}{2!}, \end{aligned} \tag{24}$$

$$\begin{aligned} u_3(x,t) &= iE^{-1}\left[vE[u_{2_{xx}}]\right] \\ &= iE^{-1}\left[ak^6v^5e^{ikx}\right], \\ &= \frac{iak^6t^3e^{ikx}}{3!}, \end{aligned} \tag{25}$$

and so on. Thus, on taking the sum of the above iterations we obtain

$$u(x,t) = ae^{ikx}\left(1-(ik^2t)+\frac{(ik^2t)^2}{2!}-\frac{(ik^2t)^3}{3!}+\dots\right), \tag{26}$$

which is leading to the exact solution

$$u(x,t) = ae^{ik(x-kt)}. \tag{27}$$

4.2 Example Two

Let us consider the linear Schrodinger differential equation with $\lambda = 0$ again

$$iu_t + u_{xx} = 0, \tag{28}$$

but with the initial condition given by

$$u(x,0) = cosh(3x). \tag{29}$$

Applying th Elzaki decomposition method to Eq.(28) and initial condition given in Eq.(29), we get the recurrence relation given by:

$$\begin{aligned} u_0(x,t) &= cosh(3x), \\ u_{n+1}(x,t) &= iE^{-1}\left[vE[u_{n_{xx}}]\right], n \geqslant 0. \end{aligned} \tag{30}$$

We express few components as

$$u_0(x,t) = cosh(3x), \tag{31}$$

$$\begin{aligned} u_1(x,t) &= iE^{-1}\left[vE[u_{0_{xx}}]\right] \\ &= iE^{-1}\left[9v^3cosh(3x)\right], \\ &= 9icosh(3x), \end{aligned} \tag{32}$$

$$\begin{aligned} u_2(x,t) &= iE^{-1}\left[vE[u_{1_{xx}}]\right] \\ &= iE^{-1}\left[81iv^4cosh(3x)\right], \\ &= -\frac{81}{2!}t^2cosh(3x), \end{aligned} \tag{33}$$

$$\begin{aligned} u_3(x,t) &= iE^{-1}\left[vE[u_{2_{xx}}]\right] \\ &= iE^{-1}\left[\frac{729}{2!}i^2v^5cosh(3x)\right], \\ &= -i\frac{729}{3!}t^3cosh(3x), \end{aligned} \tag{34}$$

and so on. Thus, on summing the above iterations we obtain

$$u(x,t) = cosh(3x)\left(1+(9it)+\frac{(9it)^2}{2!}+\frac{(9it)^3}{3!}+\dots\right), \tag{35}$$

which is leading to the exact solution

$$u(x,t) = cosh(3x)e^{9it}. \tag{36}$$

4.3 Example Three

We consider the nonlinear Schrodinger differential equation with $\lambda = -2$ and $r = 1$

$$iu_t + u_{xx} - 2|u|^2u = 0, \tag{37}$$

with the initial condition given by

$$u(x,0) = e^{ix}. \tag{38}$$

As explained above in the Elzaki decomposition method, Eq.(37) with the initial condition in Eq.(38) have the general solution given by recursively as

$$\begin{aligned} u_0(x,t) &= e^{ix},\\ u_{n+1}(x,t) &= iE^{-1}\Big[vE[u_{n_{xx}}]\Big] - 2iE^{-1}\Big[vE[A_n]\Big], n \geqslant 0, \end{aligned} \tag{39}$$

where A_n's are the Adomian polynomials to be determined from the nonlinear term

$$Nu = |u|^2 u = u^2\overline{u}, \tag{40}$$

where $\overline{u}$ is the conjugate of u, with few terms using the formula in Eq.(11) expressed as:

$$\begin{aligned} A_0 &= u_0^2\overline{u}_0,\\ A_1 &= 2u_0u_1\overline{u}_0 + u_0^2\overline{u}_1,\\ A_2 &= 2u_0u_2\overline{u}_0 + u_1^2\overline{u}_0 + 2u_0u_1\overline{u}_1 + u_0^2\overline{u}_2,\\ &\ldots \end{aligned} \tag{41}$$

and so on. Now, the few components are as follows:

$$u_0(x,t) = e^{ix}, \tag{42}$$

$$\begin{aligned} u_1(x,t) &= iE^{-1}\Big[vE[u_{0_{xx}}]\Big] - 2iE^{-1}\Big[vE[A_0]\Big]\\ &= iE^{-1}\Big[i^2v^3e^{ix}\Big] - 2iE^{-1}\Big[v^3e^{ix}\Big],\\ &= -3ite^{ix}, \end{aligned} \tag{43}$$

$$\begin{aligned} u_2(x,t) &= iE^{-1}\Big[vE[u_{1_{xx}}]\Big] - 2iE^{-1}\Big[vE[A_1]\Big]\\ &= iE^{-1}\Big[3iv^4e^{ix}\Big] - 2iE^{-1}\Big[-3iv^4e^{ix}\Big],\\ &= \frac{-9t^2e^{ix}}{2!}, \end{aligned} \tag{44}$$

$$\begin{aligned} u_3(x,t) &= iE^{-1}\Big[vE[u_{2_{xx}}]\Big] - 2iE^{-1}\Big[vE[A_2]\Big]\\ &= iE^{-1}\Big[9v^5e^{ix}\Big] - 2iE^{-1}\Big[-9v^5e^{ix}\Big],\\ &= \frac{27it^3e^{ix}}{3!}, \end{aligned} \tag{45}$$

and so on. Thus, on summing up the above iterations we get

$$u(x,t) = e^{ix}\Big(1 - (3it) + \frac{(3it)^2}{2!} - \frac{(3it)^3}{3!} + \ldots\Big), \tag{46}$$

which is leading to the exact solution

$$u(x,t) = e^{i(x-3t)}. \tag{47}$$

4.4 Example Four

Let us again consider the nonlinear Schrodinger differential equation with $\lambda = 2$ and $r = 1$

$$iu_t + u_{xx} + 2|u|^2u = 0, \tag{48}$$

with the initial condition given by

$$u(x,0) = e^{ix}. \tag{49}$$

As in the above example, we obtain the general solution recursively given by

$$\begin{aligned} u_0(x,t) &= e^{ix},\\ u_{n+1}(x,t) &= iE^{-1}\Big[vE[u_{n_{xx}}]\Big] + 2iE^{-1}\Big[vE[A_n]\Big], n \geqslant 0, \end{aligned} \tag{50}$$

where A_n's are the Adomian polynomials for the nonlinear term given in Eq.(41).
We now express the few components as follows:

$$u_0(x,t) = e^{ix}, \tag{51}$$

$$\begin{aligned} u_1(x,t) &= iE^{-1}\Big[vE[u_{0_{xx}}]\Big] + 2iE^{-1}\Big[vE[A_0]\Big]\\ &= iE^{-1}\Big[i^2v^3e^{ix}\Big] + 2iE^{-1}\Big[v^3e^{ix}\Big],\\ &= ite^{ix}, \end{aligned} \tag{52}$$

$$\begin{aligned} u_2(x,t) &= iE^{-1}\Big[vE[u_{1_{xx}}]\Big] + 2iE^{-1}\Big[vE[A_1]\Big]\\ &= iE^{-1}\Big[i^3v^4e^{ix}\Big] + 2iE^{-1}\Big[iv^4e^{ix}\Big],\\ &= \frac{-t^2e^{ix}}{2!}, \end{aligned} \tag{53}$$

$$\begin{aligned} u_3(x,t) &= iE^{-1}\Big[vE[u_{2_{xx}}]\Big] + 2iE^{-1}\Big[vE[A_2]\Big]\\ &= iE^{-1}\Big[v^5e^{ix}\Big] + 2iE^{-1}\Big[-v^5e^{ix}\Big],\\ &= \frac{-it^3e^{ix}}{3!}, \end{aligned} \tag{54}$$

and so on. Thus, summing the above iterations we obtain

$$u(x,t) = e^{ix}\Big(1 + (it) + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \ldots\Big), \tag{55}$$

which is leading to the exact solution

$$u(x,t) = e^{i(x+t)}. \tag{56}$$

5 Conclusion

In conclusion, the Elzaki transform and Adomian decomposition method are coupled and utilized to treat linear and nonlinear Schrodinger differential equations in what is known as Elzaki decomposition method. The method works perfectly as the solutions obtained yield remarkable exact solutions for all the four numerical problems considered; and in all the four problems, the solutions obtained turn out to be in full agreement with the famous results in the literature.

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