Generalized Perfect Numbers Connected with Arithmetic Functions

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1 Introduction

For a natural number $n$, we denote the sum of positive divisors of $n$ by $\sigma(n) = \sum_{d|n} d$ and the sum of proper positive divisors of $n$ by $\rho(n) = \sigma(n) - n$. A natural number $n$ is called a perfect number if $\rho(n) = n$ or equivalently $\sigma(n) = 2n$. The first few perfect numbers are 6, 28, 496, 8128, ... (Sloanes A000396 [13]). Euclid discovered that the first four perfect numbers are generated by the formula $2^{p-1}(2^p - 1)$ about 300 B.C. [8]. He also noticed that $2^p - 1$ is a prime number for every instance, and in Proposition IX.36 of Elements gave the proof, that the discovered formula gives an even perfect number whenever $2^p - 1$ is prime. In order for $2^p - 1$ to be a prime, $n$ must itself to be a prime. A Mersenne prime is a prime number of the form $2^p - 1$, where $p$ must also be a prime number. Any even perfect number $n$ is of the form $n = 2^{p-1}(2^p - 1)$, where $2^p - 1$ is a Mersenne prime. Perfect numbers are intimately connected with these primes, since there is a concrete one-to-one association between even perfect numbers and Mersenne primes. The fact that Euclid's formula gives all possible even perfect numbers was proved by Euler two millennia after the formula was discovered. There are only 48 known Mersenne primes (2013 [12]) and hence only 48 even perfect numbers are known. There is a conjecture that there are infinitely many perfect numbers. The search for new ones is to keep on going by search program via the Internet; named GIMPS (Great Internet Mersenne Prime Search). It is not known if any odd perfect number exists, although numbers up to $10^{300}$ have been checked without success [2]. Recently T. Goto, Y. Ohno [3], D. Ianucci [5, 6], K. G. Hare [4] established several results on odd perfect numbers. A positive integer $n$ is called superperfect number if $\sigma(\sigma(n)) = 2n$. The notion of these numbers was introduced by D. Suryanarayana [11] in 1969. Even superperfect numbers are of the form $2^{p-1}$, where $2^p - 1$ is a Mersenne prime. The first few superperfect numbers are 2, 4, 16, 64, 4096, 65536, 262144,... It is not known whether there are any odd superperfect numbers. An odd superperfect number $n$ would have to be a square number such that either $n$ or $\sigma(n)$ is divisible by at least three distinct primes [7].

A positive integer $n$ is called $k$-hyperperfect number if $\sigma(n) = \frac{k+1}{k}n + \frac{k-1}{k}$. On can remark that a number is perfect iff it is 1-hyperperfect. The concept of $k$-hyperperfect number was given by Minoli and Bear [10] and they also conjecture that there are $k$-hyperperfect numbers for every $k$. All hyperperfect numbers less than $10^{11}$ have been computed by J.S. Craine [9]. Bege and Fogarasi [1] introduced the concept of super-hyperperfect number. A positive integer $n$ is called super-hyperperfect number if $\sigma(\sigma(n)) = \frac{k+1}{k}n + \frac{k-1}{k}$. They have conjectured that all super-hyperperfect numbers are of the form $3^{p-1}$, where $p$ and $\frac{3^p - 1}{2}$ are primes. For any natural number $n$, Eulers phi-function and Dedekinds Arithmetic function are given by $\phi(n) = n\Pi_{p|n}(1 - \frac{1}{p})$. 

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and \( \psi(n) = n \prod_{p \mid n} (1 + \frac{1}{p}) \) respectively, where \( p \) runs through the distinct prime divisors of \( n \).

2 Main Results

**Theorem 2.1.** If \( k > 1, 1 + 2 + 4 + 8 + \ldots + k = 2k - 1 \), where \( 2k - 1 \) is a prime and \( k(2k - 1) \) is a perfect number, then \( (\phi \circ \psi \circ \sigma)(k(2k - 1)) = 2(\phi \circ \psi)(k(2k - 1)) \).

**Proof:** It is clear that \( (k, 2k - 1) = 1 \), and \( k \) is even and of the form \( 2^n, n \geq 1 \). Since both \( \phi \) and \( \psi \) are multiplicative functions,

\[
(\phi \circ \psi \circ \sigma)(k(2k - 1)) = \phi(\psi(\sigma(k(2k - 1))))
\]

\[
= \phi(\psi(2k(2k - 1)))
\]

\[
= \phi(\psi(2k)) \psi(2k - 1)
\]

\[
= \phi(2k(1 + 1/2)2k)
\]

\[
= 2k^2(1 - 1/2)
\]

\[
= 2k^2
\]

(1) and (2) give the result.

**Theorem 2.2.** If \( k > 1, 1 + 2 + 4 + 8 + \ldots + k = 2k - 1 \), where \( 2k - 1 \) is a prime and \( k(2k - 1) \) is a perfect number, then \( (2\phi)^n(\phi(\psi(k(2k - 1)))) = k^2 \), \( n \geq 0 \).

**Proof:** We prove the result by applying induction on \( n \). By equation (2), the result is true for \( n = 0 \). Let's assume the result is true for any positive integer \( n > 0 \). Then

\[
(2\phi)^n(\phi(\psi(k(2k - 1)))) = k^2
\]

Now,

\[
(2\phi)^{n+1}(\phi(\psi(k(2k - 1)))) = 2\phi((2\phi)^n(\phi(\psi(k(2k - 1)))))
\]

\[
= 2\phi(2k^2) \quad \text{(By hypothesis)}
\]

\[
= 2k^2(1 - 1/2) = k^2
\]

Hence the theorem follows.

**Theorem 2.3.** If \( k > 1, 1 + 2 + 4 + 8 + \ldots + k = 2k - 1 \), where \( 2k - 1 \) is a prime and \( k(2k - 1) \) is a perfect number, then \( (\phi \circ 2\psi)^n \circ (\phi \circ \psi)(k(2k - 1)) = k^2, n \geq 0 \).

**Proof:** We prove the result by applying induction on \( n \). By equation (2), the result is true for \( n = 0 \). Let's assume the result is true for \( n > 0 \). Then

\[
(\phi \circ 2\psi)^n \circ (\phi \circ \psi)(k(2k - 1)) = k^2
\]

Now

\[
(\phi \circ 2\psi)^{n+1} \circ (\phi \circ \psi)(k(2k - 1))
\]

\[
= (\phi \circ 2\psi)((\phi \circ 2\psi)^n \circ (\phi \circ \psi)(k(2k - 1)))
\]

\[
= \phi(2\psi(2k^2)) \quad \text{(By hypothesis)}
\]

\[
= \phi(3k^2) = k^2
\]

Hence the theorem follows.

**Corollary 2.4.** If \( k > 1, 1 + 2 + 4 + 8 + \ldots + k = 2k - 1 \), where \( 2k - 1 \) is a prime and \( k(2k - 1) \) is a perfect number, then

\[
(2\phi)^n \circ (\phi \circ \psi)(k(2k - 1)) = (\phi \circ 2\psi)^n \circ (\phi \circ \psi)(k(2k - 1)),
\]

\( n \geq 0 \)

**Theorem 2.5.** If \( k > 1, 1 + 2 + 4 + 8 + \ldots + k = 2k - 1 \), where \( 2k - 1 \) is a prime and \( k(2k - 1) \) is a perfect number, then \( \psi^n(k(2k - 1)) = 3.2^{-n-1}k^2 \), \( m \geq 1 \). Since \( \psi \) is multiplicative function,

\[
\psi(k(2k - 1)) = \psi(k)\psi(2k - 1) = 3k^2
\]

Thus the result is true for \( n = 1 \).

**Theorem 2.6.** For any even perfect number \( n \),

\[
(\phi \circ \rho)(n) = \frac{(\phi \circ \sigma)(n)}{2}
\]

**Proof** An even perfect number \( n \) is of the form \( 2^{p-1}(2^p - 1) \), where \( 2^p - 1 \) is a Mersenne prime.

Now

\[
(\phi \circ \rho)(n) = \phi(\rho(n)) = \phi(n) = \phi(2^{p-1}(2^p - 1))
\]

\[
= 2^{p-1}(1 - \frac{1}{2})(2^p - 2)
\]

\[
= 2^{p-1}(2^{p-1} - 1)
\]

Again,

\[
(\phi \circ \sigma)(n) = \phi(\sigma(n)) = \phi(2n) = \phi(2^p(2^p - 1))
\]

\[
= 2^p(1 - \frac{1}{2})(2^p - 2)
\]

\[
= 2^p(2^{p-1} - 1)
\]

Thus \( (\phi \circ \rho)(n) = \frac{(\phi \circ \sigma)(n)}{2} \)

**Theorem 2.7.** If \( n = p^{k-1}(p^k - p + 1) \) where \( p \) and \( p^k - p + 1 \) are primes, then \( n \) is \((p-1)\)-hyperperfect number.
\[ \sigma(n) = \sigma(p^{k-1}) \sigma(p^k - p + 1) \]
\[ = \frac{p^k - 1}{p - 1} (p^k - p + 2) \]
\[ = \frac{p^k(p^k - p + 1) + p - 2}{p - 1} \]
\[ = \frac{p^{k+1} - p^k + p^k - p + 1}{p - 1} + \frac{p - 2}{p - 1} \]
\[ = \frac{p}{p - 1} p^{-1} (p^k - p + 1) + \frac{p - 2}{p - 1} \]

Hence \( n \) is \( (p - 1) \)-hyperperfect number.

We generalize conjecture 2 introduced by Antal Bege and Kinga Fagarasi in [1] as follows:

**Conjecture 1.** All \((p - 1)\)-hyperperfect numbers are of the form \( n = p^{k-1}(p^k - p + 1) \), where \( p \) and \( p^k - p + 1 \) are primes.

We proof the next theorem by assuming the conjecture 1 to be true.

**Theorem 2.8.** If \( n \) is a \((p - 1)\)-hyperperfect number and \( m = p^k - p + 1 \), is a a positive integer, then

(i) \( \psi(n) = (1 - \frac{1}{p}) \left( \frac{m + p - 1}{m + p - 2} \right) \sigma(n) \)

(ii) \( \phi(n) = \frac{(p - 1)(m - 1)}{(p + 1)(m + 1)} \psi(n) \)

(iii) \( \sigma(n) = (\frac{p}{p - 1}) \left( \frac{m + 1}{m - 1} \right) \left( \frac{m + p - 2}{m + p - 1} \right) \phi(n) \)

**Proof:** By assuming the conjecture 1 to be true, we can write \( n = p^{k-1}(p^k - p + 1) \), where \( p^k - p + 1 \) is a prime. Thus \( n = p^{k-1}m = \frac{m(m + p - 1)}{p} \) with \( m = p^k - p + 1 \) is a prime. Now,

\[ \sigma(n) = \sigma(p^{k-1}) \sigma(m) = \frac{p^k - 1}{p - 1} (m + 1) \]
\[ = \frac{(m + 1)(m + p - 2)}{p - 1} \quad (3) \]

\[ \phi(n) = \phi(p^{k-1}) \phi(m) \]
\[ = \frac{p^k - 1}{p} (m - 1) \]
\[ = \frac{(p - 1)(m - 1)(m + p - 1)}{p^2} \quad (4) \]

\[ \psi(n) = \psi(p^{k-1}) \psi(m) = \frac{p^k - 1}{p} (m + 1) \]
\[ = \frac{(p + 1)(m + 1)(m + p - 1)}{p^2} \quad (5) \]

(3) and (5) give,

\[ \psi(n) = (1 - \frac{1}{p^2}) \left( \frac{m + p - 1}{m + p - 2} \right) \sigma(n) \quad (6) \]

(4) and (5) give,

\[ \phi(n) = \frac{(p - 1)(m - 1)}{(p + 1)(m + 1)} \psi(n) \quad (7) \]

(6) and (7) give,

\[ \phi(n) = \frac{(p - 1)(m - 1)}{(p + 1)(m + 1)} (1 - \frac{1}{p^2}) \left( \frac{m + p - 1}{m + p - 2} \right) \sigma(n) \]

This gives,

\[ \sigma(n) = \left( \frac{p}{p - 1} \right)^2 \left( \frac{m + 1}{m - 1} \right) \left( \frac{m + p - 2}{m + p - 1} \right) \phi(n) \quad (8) \]

For \( 2 \)-hyperperfect number \( n \), and \( m = 3^k - 2 \), one can easily obtain the following result as a particular case of the theorem 2.8.

(i) \( \psi(n) = \frac{8}{9} \left( \frac{m + 2}{m + 1} \right) \sigma(n) \)

(ii) \( \phi(n) = \frac{2}{9} \left( \frac{m - 1}{m + 1} \right) \psi(n) \)

(iii) \( \sigma(n) = \frac{9}{4} \left( \frac{m + 1}{m + 2} \right)^2 \phi(n) \)

**Theorem 2.9.** If \( n \) is a \((p - 1)\)-hyperperfect number, then

(i) \( \phi(n^2) = n \phi(n) \)

(ii) \( \psi(n^2) = n \psi(n) \)

**Proof:** Since \( n \) is a \((p - 1)\)-hyperperfect number,

\[ n = p^{k-1}(p^k - p + 1) \]

Where \( p^k - p + 1 \) is a prime.

Now

\[ \phi(n) = \phi(p^{k-1}(p^k - p + 1)) \]
\[ = \phi(p^{k-1}) \phi(p^k - p + 1) \]
\[ = p^{k-1} (1 - \frac{1}{p}) (p^k - p) \]
\[ = p^{k-1} (p - 1) (p^{k-1} - 1) \]

Also

\[ \psi(n) = \psi(p^{k-1}(p^k - p + 1)) \]
\[ = \psi(p^{k-1}) \psi(p^k - p + 1) \]
\[ = p^{k-1} (1 + \frac{1}{p}) (p^k - p + 2) \]
\[ = p^{k-2} (p + 1) (p^k - p + 2) \]

(i) \( \phi(n^2) = \phi(p^{2k-2}) \phi((p^k - p + 1)^2) \]
\[ = p^{2k-2} (1 - \frac{1}{p}) (p^k - p + 1)^2 (1 - \frac{1}{p^k - p + 1}) \]
\[ = p^{k-1} (p^k - p + 1) \phi(n) \]
\[ = n \phi(n) \]
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(ii)
\[\psi(n^2) = \psi(p^{2k-2})\psi((p^k - p + 1)^2) = p^{2k-2}(1 + \frac{1}{p^k})^2(1 + \frac{1}{p^k - p + 1}) = p^{k-1}(p^k - p + 1)(p^{k-1}(p^k - p + 2) = n\psi(n)\]

Theorem 2.10. If \( n \) is a super perfect number, then
(i) \( \psi(n) = 3\phi(n) \)
(ii) \( \sigma(n) = 4\phi(n) - 1 \)

Proof: Let \( n \) be an even super perfect number. Then \( n = 2^{p-1} \), where \( 2^p - 1 \) is Mersenne prime.

Now
\[\psi(n) = \frac{3}{2}n \quad (9)\]
\[\phi(n) = \frac{1}{2}n \quad (10)\]
\[\sigma(n) = 2^p - 1 = 2n - 1 \quad (11)\]

(9) and (10) give \( \psi(n) = 3\phi(n) \)
Also, (10) and (11) give \( \sigma(n) = 4\phi(n) - 1 \)

Theorem 2.11. If \( n \) is a super hyperperfect number, then
(i) \( \psi(n) = 2\phi(n) \)
(ii) \( \sigma(n) = \frac{3}{4}(3n - 1)\phi(n) \)

Proof: Let \( n \) be a super hyperperfect number. Then \( n = 3^{p-1} \), where \( p \) and \( \frac{3^p - 1}{2} \) are prime numbers.

Now
\[\psi(n) = \frac{3}{2}n \quad (12)\]
\[\phi(n) = 2.3^{p-2} \quad (13)\]
\[\sigma(n) = \frac{3p - 1}{2} = \frac{3n - 1}{2} \quad (14)\]

From (12) and (13), we obtain \( \psi(n) = 2\phi(n) \)
Also, from (13) and (14), we obtain \( \sigma(n) = \frac{3}{4}(3n - 1)\phi(n) \).

Proposition 2.12 If \( n \) is 2-hyperperfect number then 
\( n(\sigma(n) - 1) \) is \( n \)-th pentagonal number.

Proof: Let \( n \) be a 2-hyperperfect number. Then \( \sigma(n) = \frac{3}{2^n} + \frac{1}{2} \)

Now
\[n(\sigma(n) - 1) = n\left(\frac{3}{2^n} + \frac{1}{2} - 1\right) = \frac{n(3n - 1)}{n} = P_n \]

Hence the result follows.

Proposition 2.13. If \( n \) is super hyperperfect number then 
\( n\sigma(n) \) is \( n \)-th pentagonal number.

Proof: Let \( n \) be a super hyperperfect number. Then \( n = 3^{p-1} \), where \( p \) and \( \frac{3^p - 1}{2} \) are prime numbers.

Now
\[\sigma(n) = \frac{3p - 1}{2} = \frac{3n - 1}{2} \]
\[\Rightarrow n\sigma(n) = \frac{n(3n - 1)}{2} = P_n \]

Hence the result follows.

References

http://www.gimps.org
http://www.research.att.com/njas/sequences/
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