Mathematical Sciences Letters
An International Journal

http://dx.doi.org/10.12785/msl/030318

Generalized Perfect Numbers Connected with Arithmetic Functions

Azizul Hoque and Himashree Kalita*

Department of Mathematics, Gauhati University, Guwahati, 781014, India

Received: 13 Apr. 2014, Revised: 24 May 2014, Accepted: 26 May 2014
Published online: 1 Sep. 2014

Abstract: In this paper, we obtain some new results for perfect numbers and generalized perfect numbers connected with the relationship among arithmetic functions σ, ϕ and ψ. These arithmetic functions and their compositions play vital role in this work.

Keywords: Perfect number, Superperfect number, k-hyperperfect number, Super-hyperperfect number, Arithmetic functions.

1 Introduction

For a natural number n, we denote the sum of positive divisors of n by $\sigma(n) = \sum_{d|n} d$ and the sum of proper positive divisors of n by $\rho(n) = \sigma(n) - n$. A natural number n is called a perfect number if $\rho(n) = n$ or equivalently $\sigma(n) = 2n$. The first few perfect numbers are 6, 28, 496, 8128..... (Sloanes A000396 [13]). Euclid discovered that the first four perfect numbers are generated by the formula $2^{n-1}(2^n - 1)$ about 300 B.C. [8]. He also noticed that $2^n - 1$ is a prime number for every instance, and in Proposition $IX.36$ of *Elements* gave the proof, that the discovered formula gives an even perfect number whenever $2^n - 1$ is prime. In order for $2^n - 1$ to be a prime, n must itself to be a prime. A Mersenne prime is a prime number of the form $2^p - 1$, where p must also be a prime number. Any even perfect number n is of the form $n = 2^{p-1}(2^p - 1)$, where $2^p - 1$ is a Mersenne prime. Perfect numbers are intimately connected with these primes, since there is a concrete one-to-one association between even perfect numbers and Mersenne primes. The fact that Euclids formula gives all possible even perfect numbers was proved by Euler two millennia after the formula was discovered. There are only 48 known Mersenne primes (2013 [12]) and hence only 48 even perfect numbers are known. There is a conjecture that there are infinitely many perfect numbers. The search for new ones is to keep on going by search program via the Internet; named GIMPS (Great Internet Mersenne Prime Search). It is not known if any odd perfect number exists, although numbers up to 10^{300} have been checked without success [2]. Recently T. Goto, Y. Ohno [3], D. Ianucci [5,6], K. G. Hare [4] established several results on odd perfect numbers. A positive integer n is called superperfect number if $\sigma(\sigma(n)) = 2n$. The notion of these numbers was introduced by D. Suryanarayana [11] in 1969. Even superperfect numbers are of the form 2^{p-1}, where $2^p - 1$ is a Mersenne prime. The first few superperfect numbers are 2, 4, 16, 64, 4096, 65536, 262144..... It is not known whether there are any odd superperfect numbers. An odd superperfect number n would have to be a square number such that either n or $\sigma(n)$ is divisible by at least three distinct primes [7].

A positive integer n is called k-hyperperfect number if $\sigma(n) = \frac{k+1}{k}n + \frac{k-1}{k}$. On can remark that a number is perfect iff it is 1-hyperperfect. The concept of k-hyperperfect number was given by Minoli and Bear [10] and they also conjecture that there are k-hyperperfect numbers for every k. All hyperperfect numbers less than 10^{11} have been computed by J.S. Craine [9]. Bege and Fogarasi [1] introduced the concept of super-hyperperfect number. A positive integer n is called super-hyperperfect number if $\sigma(\sigma(n)) = \frac{k+1}{k}n + \frac{k-1}{k}$. They have conjectured that all super-hyperperfect numbers are of the form 3^{p-1}, where p and $\frac{3^p - 1}{2}$ are primes. For any natural number n, Eulers phi-function and Dedekinds Arithmetic function are given by $\phi(n) = n\prod_{p|n}(1 - \frac{1}{p})$

* Corresponding author e-mail: ahoque.ms@gmail.com

and $\psi(n) = n\prod_{p|n}(1+\frac{1}{p})$ respectively, where p runs through the distinct prime divisors of n.

2 Main Results

Theorem 2.1. If $k > 1, 1+2+4+8+...+k = 2k-1$, where $2k-1$ is a prime and $k(2k-1)$ is a perfect number, then $(\phi\circ\psi\circ\sigma)(k(2k-1)) = 2(\phi\circ\psi)(k(2k-1))$.

Proof: It is clear that $(k, 2k-1) = 1$, and k is even and of the form $2^n, n \geq 1$. Since both ϕ and ψ are multiplicative functions,

$$\begin{aligned}(\phi\circ\psi\circ\sigma)(k(2k-1)) &= \phi(\psi(\sigma(k(2k-1))))\\ &= \phi(\psi(2k(2k-1)))\\ &= \phi(\psi(2k)\psi(2k-1))\\ &= \phi(2k(1+1/2)2k)\\ &= 2.2k^2(1-1/2)\\ &= 2k^2 \qquad (1)\end{aligned}$$

Also

$$\begin{aligned}(\phi\circ\psi)(k(2k-1)) &= \phi(\psi(k(2k-1)))\\ &= \phi(\psi(k)\psi(2k-1))\\ &= \phi(k(1+1/2)2k)\\ &= \phi(3k^2)\\ &= \phi(3\phi(k^2)) = k^2 \qquad (2)\end{aligned}$$

(1) and (2) give the result.

Theorem 2.2. If $k > 1, 1+2+4+8+...+k = 2k-1$, where $2k-1$ is a prime and $k(2k-1)$ is a perfect number, then $(2\phi)^n(\phi(\psi(k(2k-1)))) = k^2,\ n \geq 0$.

Proof: We prove the result by applying induction on n. By equation (2), the result is true for $n = 0$. Lets assume the result is true for any positive integer $n > 0$. Then $(2\phi)^n(\phi(\psi(k(2k-1)))) = k^2$
Now,

$$\begin{aligned}(2\phi)^{n+1}(\phi(\psi(k(2k-1)))) &= 2\phi((2\phi)^n(\phi(\psi(k(2k-1)))))\\ &= 2\phi(k^2) \qquad (By\ hypothesis)\\ &= 2k^2(1-1/2) = k^2\end{aligned}$$

Hence the theorem follows.

Theorem 2.3. If $k > 1, 1+2+4+8+...+k = 2k-1$, where $2k-1$ is a prime and $k(2k-1)$ is a perfect number, then $(\phi\circ 2\psi)^n\circ(\phi\circ\psi)(k(2k-1)) = k^2,\ n \geq 0$.

Proof: We prove the result by applying induction on n. By equation (2), the result is true for $n = 0$. Lets assume the result is true for $n > 0$. Then
$(\phi\circ 2\psi)^n\circ(\phi\circ\psi)(k(2k-1)) = k^2$
Now

$$\begin{aligned}&(\phi\circ 2\psi)^{n+1}\circ(\phi\circ\psi)(k(2k-1))\\ &= (\phi\circ\ 2\psi)((\phi\circ 2\psi)^n\circ(\phi\circ\psi)(k(2k-1)))\\ &= \phi(2\psi(k^2)) \qquad (By\ hypothesis)\\ &= \phi(3k^2) = k^2\end{aligned}$$

Hence the theorem follows.

Corollary 2.4. If $k > 1, 1+2+4+8+...+k = 2k-1$, where $2k-1$ is a prime and $k(2k-1)$ is a perfect number, then

$$(2\phi)^n\circ(\phi\circ\psi)(k(2k-1)) = (\phi\circ 2\psi)^n\circ(\phi\circ\psi)(k(2k-1)),$$

$n \geq 0$

Theorem 2.5. If $k > 1, 1+2+4+8+...+k = 2k-1$, where $2k-1$ is a prime and $k(2k-1)$ is a perfect number, then $\psi^n(k(2k-1)) = 3.2^{n-1}k^2,\ \ n \geq 1$.

Proof: It is clear that $(k, 2k-1) = 1$, and k is even and of the form $2^m,\ \ m \geq 1$. Since ψ is multiplicative function,

$$\psi(k(2k-1)) = \psi(k)\psi(2k-1) = 3k^2$$

Thus the result is true for $n = 1$.
Suppose the result is true for $n > 1$. Then
$\psi^n(k(2k-1)) = 3.2^{n-1}k^2$
Now

$$\begin{aligned}\psi^{n+1}(k(2k-1)) &= \psi(3.2^{n-1}k^2)\\ &= \psi(3.2^{2m+n-1}),\\ &= \psi(3)\psi(2^{2m+n-1})\\ &= 4.2^{2m+n-1}(1+1/2)\\ &= 3.2^{2m+n} = 3.2^nk^2\end{aligned}$$

Hence the theorem follows.

Theorem 2.6. For any even perfect number n,
$(\phi\circ\rho)(n) = \dfrac{(\phi\circ\sigma)(n)}{2}$

Proof An even perfect number n is of the form $2^{p-1}(2^p - 1)$, where $2^p - 1$ is a Mersenne prime.
Now

$$\begin{aligned}(\phi\circ\rho)(n) = \phi(\rho(n)) = \phi(n) &= \phi(2^{p-1}(2^p-1))\\ &= 2^{p-1}(1-\frac{1}{2})(2^p-2)\\ &= 2^{p-1}(2^{p-1}-1)\end{aligned}$$

Again,

$$\begin{aligned}(\phi\circ\sigma)(n) = \phi(\sigma(n)) = \phi(2n) &= \phi(2^p(2^p-1))\\ &= 2^p(1-\frac{1}{2})(2^p-2)\\ &= 2^p(2^{p-1}-1)\end{aligned}$$

Thus $(\phi\circ\rho)(n) = \dfrac{(\phi\circ\sigma)(n)}{2}$

Theorem 2.7. If $n = p^{k-1}(p^k - p + 1)$ where p and $p^k - p + 1$ are primes, then n is $(p-1)$-hyperperfect number.

Proof: From definition and basic results of the divisor function σ, it follows that:

$$\begin{aligned}\sigma(n) &= \sigma(p^{k-1})\sigma(p^k-p+1)\\ &= \frac{p^k-1}{p-1}(p^k-p+2)\\ &= \frac{p^k(p^k-p+1)+p-2}{p-1}\\ &= \frac{p}{p-1}p^{k-1}(p^k-p+1)+\frac{p-2}{p-1}\\ &= \frac{p}{p-1}n+\frac{p-2}{p-1}\end{aligned}$$

Hence n is $(p-1)$-hyperperfect number.

We generalize conjecture 2 introduced by Antal Bege and Kinga Fogarasi in [1] as follows:

Conjecture 1. All $(p-1)$-hyperperfect numbers are of the form $n = p^{k-1}(p^k-p+1)$, where p and p^k-p+1 are primes.

We proof the next theorem by assuming the conjecture 1 to be true.

Theorem 2.8. If n is a $(p-1)$- hyperperfect number and $m = p^k-p+1$,k is a a positive integer, then

(i) $\psi(n) = (1-\frac{1}{p^2})(\frac{m+p-1}{m+p-2})\sigma(n)$

(ii) $\phi(n) = \frac{(p-1)(m-1)}{(p+1)(m+1)}\psi(n)$

(iii) $\sigma(n) = (\frac{p}{p-1})^2(\frac{m+1}{m-1})(\frac{m+p-2}{m+p-1})\phi(n)$

Proof: By assuming the conjecture 1 to be true, we can write $n = p^{k-1}(p^k-p+1)$, where p^k-p+1 is a prime. Thus $n = p^{k-1}m = \frac{m(m+p-1}{p}$ with $m = p^k-p+1$ is a prime.

Now,

$$\begin{aligned}\sigma(n) = \sigma(p^{k-1})\sigma(m) &= \frac{p^k-1}{p-1}(m+1)\\ &= \frac{(m+1)(m+p-2)}{p-1}\end{aligned} \tag{3}$$

$$\begin{aligned}\phi(n) &= \phi(p^{k-1})\phi(m)\\ &= p^{k-1}(1-\frac{1}{p})(m-1)\\ &= \frac{(p-1)(m-1)(m+p-1)}{p^2}\end{aligned} \tag{4}$$

$$\begin{aligned}\psi(n) = \psi(p^{k-1})\psi(m) &= p^{k-1}(1+\frac{1}{p})(m+1)\\ &= \frac{(p+1)(m+1)(m+p-1)}{p^2}\end{aligned} \tag{5}$$

(3) and (5) give,

$$\psi(n) = (1-\frac{1}{p^2})(\frac{m+p-1}{m+p-2})\sigma(n) \tag{6}$$

(4) and (5) give,

$$\phi(n) = \frac{(p-1)(m-1)}{(p+1)(m+1)}\psi(n) \tag{7}$$

(6) and (7)give,

$$\phi(n) = \frac{(p-1)(m-1)}{(p+1)(m+1)}(1-\frac{1}{p^2})(\frac{m+p-1}{m+p-2})\sigma(n)$$

This gives,

$$\sigma(n) = (\frac{p}{p-1})^2(\frac{m+1}{m-1})(\frac{m+p-2}{m+p-1})\phi(n) \tag{8}$$

For 2-hyperperfect number n, and $m = 3^k-2$, one can easily obtain the following result as a particular case of the theorem 2.8.

(i) $\psi(n) = \frac{8}{9}(\frac{m+2}{m+1})\sigma(n)$

(ii) $\phi(n) = \frac{1}{2}(\frac{m-1}{m+1})\psi(n)$

(iii) $\sigma(n) = \frac{9}{4}\frac{(m+1)^2}{(m-1)(m+2)}\phi(n)$

Theorem 2.9. If n is a $(p-1)$- hyperperfect number, then

(i) $\phi(n^2) = n\phi(n)$

(ii) $\psi(n^2) = n\psi(n)$

Proof: Since n is a $(p-1)$- hyperperfect number,

$$n = p^{k-1}(p^k-p+1)$$

Where p^k-p+1 is a prime.

Now

$$\begin{aligned}\phi(n) &= \phi(p^{k-1}(p^k-p+1))\\ &= \phi(p^{k-1})\phi(p^k-p+1)\\ &= p^{k-1}(1-\frac{1}{p})(p^k-p)\\ &= p^{k-1}(p-1)(p^{k-1}-1)\end{aligned}$$

Also

$$\begin{aligned}\psi(n) &= \psi(p^{k-1}(p^k-p+1))\\ &= \psi(p^{k-1})\psi(p^k-p+1)\\ &= p^{k-1}(1+\frac{1}{p})(p^k-p+2)\\ &= p^{k-2}(p+1)(p^k-p+2)\end{aligned}$$

(i)

$$\begin{aligned}\phi(n^2) &= \phi(p^{2k-2})\phi((p^k-p+1)^2)\\ &= p^{2k-2}(1-\frac{1}{p})(p^k-p+1)^2(1-\frac{1}{p^k-p+1})\\ &= p^{k-1}(p^k-p+1).p^{k-1}(p-1)(p^{k-1}-1)\\ &= n\phi(n)\end{aligned}$$

(ii)

$$\begin{aligned}\psi(n^2) &= \psi(p^{2k-2})\psi((p^k-p+1)^2)\\ &= p^{2k-2}(1+\frac{1}{p})(p^k-p+1)^2(1+\frac{1}{p^k-p+1})\\ &= p^{k-1}(p^k-p+1).p^{k-1}(p+1)(p^k-p+2)\\ &= n\psi(n)\end{aligned}$$

Theorem 2.10. If n is a super perect number, then
(i) $\psi(n) = 3\phi(n)$
(ii) $\sigma(n) = 4\phi(n) - 1$

Proof: Let n be an even super perfect number. Then $n = 2^{p-1}$, where $2^p - 1$ is Mersenne prime.
Now

$$\psi(n) = 3.2^{p-2} = \frac{3}{2}n \tag{9}$$

$$\phi(n) = 2^{p-2} = \frac{1}{2}n \tag{10}$$

$$\sigma(n) = 2^p - 1 = 2n - 1 \tag{11}$$

(9) and (10 give $\psi(n) = 3\phi(n)$
Also, (10) and (11) give $\sigma(n) = 4\phi(n) - 1$

Theorem 2.11. If n is a super hyperperect number, then
(i) $\psi(n) = 2\phi(n)$
(ii)$\sigma(n) = \frac{3}{4}(\frac{3n-1}{n})\phi(n)$

Proof: Let n be a super hyperperfect number. Then $n = 3^{p-1}$, where p and $\frac{3^p-1}{2}$ are prime numbers.
Now

$$\psi(n) = 4.3^{p-2} \tag{12}$$

$$\phi(n) = 2.3^{p-2} \tag{13}$$

$$\sigma(n) = \frac{3^p-1}{2} = \frac{3n-1}{2} \tag{14}$$

From (12) and (13), we obtain $\psi(n) = 2\phi(n)$
Also, from(13)and(14), we obtain $\sigma(n) = \frac{3}{4}(\frac{3n-1}{n})\phi(n)$.

Proposition 2.12 If n is 2-hyperperfect number then $n(\sigma(n) - 1)$ is n-th pentagonal number.

Proof: Let n be a 2-hyperperfect number. Then
$\sigma(n) = \frac{3}{2}n + \frac{1}{2}$
Now
$n(\sigma(n) - 1) = n(\frac{3}{2}n + \frac{1}{2} - 1) = \frac{n(3n-1)}{n} = P_n$
Hence the result follows.

Proposition 2.13. If n is super hyperperfect number then $n\sigma(n)$ is n-th pentagonal number.

Proof: Let n be a super hyperperfect number. Then $n = 3^{p-1}$, where p and $\frac{3^p-1}{2}$ are prime numbers.
Now
$\sigma(n) = \frac{3^p-1}{2} = \frac{3n-1}{2}$
$\Rightarrow n\sigma(n) = \frac{n(3n-1)}{2} = P_n$
Hence the result follows.

Proposition 2.14. If n is an even super perfect number then $\frac{n}{2}(\sigma(n) + n)$ is n-th pentagonal number.

Proof: Let n be an even super perfect number. Then $n = 2^{p-1}$, where $2^p - 1$ is Mersenne prime.
Now
$\sigma(n) = 2^p - 1 = 2n - 1$
$\Rightarrow \frac{n}{2}(\sigma(n) + n) = \frac{n(3n-1)}{2} = P_n$
Hence the result follows.

Acknowledgement

The first Author acknowledges UGC for JRF Fellowship.

References

[1] A. Bege, K. Fogarasi, Generalized perfect numbers, Acta Univ.Sapientiae, Mathematica **1**, 1, 73-82(2009).
[2] R. P. Brent, G. L. Cohen, H. J. J. te Riele, Improved techniques for lower bounds for odd perfect numbers, Math. Comp. **57**, 857868(1991).
[3] T. Goto, Y. Ohno, Odd perfect numbers have a prime factor exceeding 108, Math. Comp. **77**,18591868(2008).
[4] K. G. Hare, New techniques for bounds on the total number of prime factors of an odd perfect number, Math. Comput.**74**, 10031008(2005).
[5] D. Ianucci, The second largest prime divisors of an odd perfect number exceeds ten thousand, Math. Comp.**68**, 17491760(1999).
[6] D. Ianucci, The third largest prime divisors of an odd perfect number exceeds one hundred, Math. Comp.**69**,867879(2000).
[7] H. J. Kanold, Uber Superperfect numbers, Elem. Math. **24**, 61-62(1969).
[8] Oliver Knill, The oldest open problem in mathematics, NEU Math Circle, December2(2007).
[9] J. S. McCranie, A study of hyperperfect numbers, J. Integer Seq.**3**,Article 00.1.3(2000).
[10] D. Minoli, R. Bear, Hyperperfect numbers, Pi Mu Epsilon J. **6**, 153157(1975).
[11] D. Suryanarayana, Superperfect numbers, Elem. Math.**14**, 1617(1969).
[12] Great Internet Mersenne Prime Search (GIMPS). $http://www.gimps.org$
[13] The on-line encyclopedia of integer sequences. $http://www..research.att.com/njas/sequences/$

Azizul Hoque was born in 1984 in Dhubri, Assam, India. He got his M.Sc. degree in Pure Mathematics in 2008 from Gauhati University, India and M.Tech. degree in Computational Seismology in 2011 from Tezpur University, India. He has published some research articles in Arithmetic functions, Class numbers of number fields, Ring Theory and Graph Theory. He is a Junior Research Fellow in the Department of Mathematic, Gauhati University, India.

Himashree Kalita was born in 1985 in Guwahati, Assam, India. She received her M.Sc. degree in Pure Mathematics in 2008 and M.Phil. degree in 2010 from Gauhati University, India. She also received PGDCA in 2011 from Gauhati University, India. She is working in Ring Theory, Module Theory and Arithmetic functions. She is a Research Scholar in the Department of Mathematics, Gauhati University, India.