Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/110208

Solutions of Singular IVP's of Lane-Emden type by Homotopy analysis method with Genetic Algorithm

*Waleed Al-Hayani**, *Laheeb Alzubaidy and Ahmed Entesar*

Department of Mathematics, Department of Software Engineering, College of Computer Science and Mathematics, University of Mosul, Iraq.

Received: 21 Dec. 2016, Revised: 18 Jan. 2017, Accepted: 21 Jan. 2017
Published online: 1 Mar. 2017

Abstract: In this paper, Homotopy Analysis Method with Genetic Algorithm is presented and used for solving Lane-Emden type singular initial value problems. The method is demonstrated for a variety of problems where approximate-exact solutions are obtained. The results obtained in all cases show the reliability and the efficiency of this method.

Keywords: Homotopy Analysis Method; Genetic Algorithm; Singular IVPs; Lane-Emden type equations; Padé Approximants; Simpson Rule

1 Introduction

Various problems arising in the field of mathematical physics and astrophysics can be distinctively formulated as equations of Lane-Emden type initial value problems [1,2,3], defined in the form

$$y'' + \frac{r}{x}y' + f(y) = 0,\ 0 < x \leq 1, \tag{1}$$

subject to conditions

$$y(0) = A,\ \ y'(0) = B. \tag{2}$$

where r, A and B are constants and $f(y)$ is a real-valued continuous function. This equation was used to model various phenomena such as the theory of stellar structure, the thermal behaviour of a spherical cloud of gas, isothermal gas spheres and the theory of thermionic currents [1,2,3].

On the other hand, another class of singular IVPs of Lane-Emden type can also be given in the form:

$$y'' + \frac{r}{x}y' + f(x,y) = g(x),\ 0 < x \leq 1, \tag{3}$$

subject to conditions given in Eq. (2), $f(x,y)$ is a continuous real valued function, and $g(x) \in [0,1]$. Eq. (3) differs from the classical Lane-Emden type Eq. (1), for the function $f(x,y)$ and for the inhomogeneous term $g(x)$.

Since, Lane-Emden type equations have significant applications in many fields of the scientific and technical world, a variety of forms of $f(y)$ have been investigated by many researchers. A discussion of the formulation of these models and the physical structure of the solutions can be found in the literature. Though the numerical solutions of Lane-Emden type problems can be easily obtained by computer, analytical solutions are much needed for physical understanding.

Various analytical methods have been used to solve Lane-Emden equations, the main difficulty arises in the singularity of the equation at $x = 0$. Actually most techniques in use for handling the Lane-Emden-type problems are based on either series solutions or perturbation techniques.

Recently, Yıldırım and Özis [4] used the variational iteration method (VIM) for solving singular IVPs of Lane-Emden. Dehghan and Shakeri [5] applied an exponential transformation to the Lane-Emden equation to overcome the difficulty of a singular point at $x = 0$ and solved the resulting nonsingular problem by the VIM. Approximate solutions to the above problems were presented by Shawagfeh [6], Wazwaz [7,8] and Wazwaz et. al. [9] by applying the Adomian decomposition method (ADM) which provides a convergent series solution. Ramos [10] presented a series approach to the Lane-Emden equation and gave the comparison with homotopy perturbation method (HPM). Yıldırım and Özis

* Corresponding author e-mail: waleedalhayani@yahoo.es

[11] and also Chowdhury and Hashim [12] gave the solutions of a class of singular second-order IVPs of Lane-Emden type by using HPM.

The main objective of this paper is to apply Homotopy Analysis Method (HAM) to obtain approximate-exact solutions for different models of Lane-Emden type singular IVPs. While the VIM [4,5] requires the determination of Lagrange multiplier in its computational algorithm, HAM is independent of any such requirements, HAM handles linear and nonlinear terms in a simple and straightforward manner without any additional requirements. Also, in this paper we apply Genetic Algorithm (GA) to obtain approximate solution of the same equations.

In what follows, we give a brief review of HAM and GA.

2 Analysis of the Homotopy Analysis Method

To describe the basic ideas of the HAM, we consider the following differential equation:

$$N\left[y(x)\right] = k(x), \tag{4}$$

where N is a nonlinear operator, x denotes the independent variable, $y(x)$ is an unknown function and $k(x)$ is a known analytic function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional Homotopy method, Liao [13] constructs the so called zero-order deformation equation

$$(1-q)L\left[\phi(x;q) - y_0(x)\right] = qh\left[N\left[\phi(x;q)\right] - k(x)\right], \tag{5}$$

where $q \in [0,1]$ is an embedding parameter, h is a nonzero auxiliary parameter, L is an auxiliary linear operator, $y_0(x)$ is an initial guess of $y(x)$ and $\phi(x;q)$ is an unknown function. It is important, that one has great freedom to choose auxiliary objects such as h and L in HAM. Obviously, when $q = 0$ and $q = 1$ it holds

$$\phi(x;0) = y_0(x), \quad \phi(x;1) = y(x), \tag{6}$$

respectively. Thus, as q increases from 0 to 1, the solution $\phi(x;q)$ varies from the initial guess $y_0(x)$ to the solution $y(x)$. Expanding $\phi(x;q)$ in Taylor series with respect to q, we have

$$\phi(x;q) = y_0(x) + \sum_{m=1}^{+\infty} y_m(x)\, q^m, \tag{7}$$

where

$$y_m(x) = \frac{1}{m!}\left.\frac{\partial^m \phi(x;q)}{\partial q^m}\right|_{q=0}. \tag{8}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are so properly chosen, the above series (7) converges at $q = 1$, then we have

$$\phi(x;1) = y_0(x) + \sum_{m=1}^{+\infty} y_m(x), \tag{9}$$

which must be one of the solutions of the original nonlinear equation, as proved by Liao [13]. If $h = -1$, Eq. (5) becomes

$$(1-q)L\left[\phi(x;q) - y_0(x)\right] + q\left[N\left[\phi(x;q)\right] - k(x)\right] = 0, \tag{10}$$

which is used mostly in the HPM [14].

According to Eq. (8), the governing equation can be deduced from the zero-order deformation equation (5). We define the vectors

$$\overrightarrow{y}_i = \left\{y_0(x), y_1(x), \ldots, y_i(x)\right\}. \tag{11}$$

Differentiating Eq. (5) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$L\left[y_m(x) - \chi_m y_{m-1}(x)\right] = hR_m\left(\overrightarrow{y}_{m-1}\right), \tag{12}$$

where

$$R_m\left(\overrightarrow{x}_{m-1}\right) = \frac{1}{(m-1)!}\left.\frac{\partial^{m-1}\left\{N\left[\phi(x;q)\right] - k(x)\right\}}{\partial q^{m-1}}\right|_{q=0}, \tag{13}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \tag{14}$$

It should be emphasized that $y_m(x)$ $(m \ge 1)$ are governed by the linear equation (12) with the linear boundary conditions that come from the original problem, which can be easily solved by symbolic computation softwares such as Maple and Mathematica.

3 Genetic Algorithms

Definition 1. *Genetic Algorithms are search and optimization techniques based on Darwin's Principle of Natural Selection.*

Definition 2. *Genetic Algorithm Operators [15,16]*

The simplest form of genetic algorithm involves three types of operators: *selection*, *crossover*,and *mutation*.

Selection: This operator selects chromosomes in the population for reproduction. The fitter the chromosome ,the more times it is likely to be selected to reproduce.

Crossover: This operator randomly chooses a locus and exchanges the subsequences before and after that locus between two chromosomes to create two offspring. For example, the strings 10000100 and 11111111 could be crossed over after the third locus in each to produce the two offspring 10011111 and 11100100.

The crossover operator roughly mimics biological recombination between two single-chromosome (haploid) organisms.

Mutation: This operator randomly flips some of the bits in a chromosome. For example, the string 00000100 might be mutated in its second position to yield 01000100. Mutation can occur at each bit position in a string with some probability, usually very small (e.g., 0.001).

A Simple Genetic Algorithm [15,16]

Given a clearly defined problem to be solved and a bit string representation for candidate solutions, a simple GA works as follows:

1. Start with a randomly generated population of n l-bit chromosomes (candidate solutions to a problem).
2. Calculate the fitness $f(x)$ of each chromosome x in the population.
3. Repeat the following steps until n offspring have been created:
   (a) Select a pair of parent chromosomes from the current population, the probability of selection being an increasing function of fitness. Selection is done "with replacement," meaning that the same chromosome can be selected more than once to become a parent.
   (b) With probability pc (the "crossover probability" or "crossover rate"), cross over the pair at randomly chosen point (chosen with uniform probability) to form two offspring. If no crossover takes place, form two offspring that are exact copies of their respective parents. (Note that here the crossover rate is defined to be the probability that two parents will crossover in a single point. There are also "multi-point crossover" versions of the GA in which the crossover rate for a pair of parents is the number of points at which a crossover takes place.)
   (c) Mutate the two offspring at each locus with probability pm (the "mutation probability" or "mutation rate"), and place the resulting chromosomes in the new population .If n is odd, one new population member can be discarded at random.
4. Replace the current population with the new population.
5. Go to step 2.

4 Applications of the method

In this work, we apply HAM to obtain approximate-exact solutions of the Lane-Emden type equations and compare them with the solutions obtained by VIM, ADM and HPM. Also, we apply GA to obtain approximate solution of the same equations. To show the high accuracy of the solution results compared with the exact solution, we give the numerical results, maximum absolute error (MAE), $\|\cdot\|_2$, maximum relative error (MRE), maximum residual error (MRR), the estimated order of convergence (EOC) at the point x_i and the global estimated order of convergence (GEOC).

Example 1. Firstly, let us consider the following linear homogeneous Lane-Emden equation

$$y'' + \frac{2}{x}y' - \left(4x^2+6\right)y = 0, \tag{15}$$

subject to initial conditions

$$y(0) = 1,\ y'(0) = 0. \tag{16}$$

The exact solution for this problem is

$$y_{Exact}(x) = e^{x^2}. \tag{17}$$

To solve Eqs. (15)–(16) by means of the standard HAM, we choose the initial approximation

$$y_0(x) = 1,$$

and the linear operator

$$L[\phi(x;q)] = \frac{\partial^2\phi(x;q)}{\partial x^2} + \frac{2}{x}\frac{\partial\phi(x;q)}{\partial x},$$

with the property

$$L\left[-\frac{c_1}{x}+c_2\right] = 0,$$

where c_i $(i = 1,2)$ are constants of integration. Furthermore, Eq .(15) suggests that we define the nonlinear operator as

$$N[\phi(x;q)] = \frac{\partial^2\phi(x;q)}{\partial x^2} + \frac{2}{x}\frac{\partial\phi(x;q)}{\partial x} - \left(4x^2+6\right)\phi(x;q),$$

Using the above definition, we construct the zeroth-order deformation equation as in (5) and (6) and the mth-order deformation equation for $m \geq 1$ is

$$L[y_m(x) - \chi_m y_{m-1}(x)] = hR_m(\vec{y}_{m-1}), \tag{18}$$

with the initial conditions

$$y_m(0) = 0,\ y'_m(0) = 0$$

where

$$R_m(\vec{y}_{m-1}) = y''_{m-1} + \frac{2}{x}y'_{m-1} - \left(4x^2+6\right)y_{m-1}.$$

Now, the solution of the mth-order deformation Eq. (18) for $m \geq 1$ is

$$y_m(x) = \chi_m y_{m-1}(x) + h\int_0^x x^{-2}\int_0^x x^2 R_m(\vec{y}_{m-1}). \tag{19}$$

We now successively obtain

$$y_1(x) = -\frac{1}{5}hx^4 - hx^2,$$

$$y_2(x) = \frac{1}{90}h^2x^8 + \frac{13}{105}h^2x^6 - \left(\frac{1}{5}h - \frac{1}{10}h^2\right)x^4 - \left(h+h^2\right)x^2,$$

$$y_3(x) = -\frac{1}{3510}h^3x^{12} - \frac{59}{11550}h^3x^{10} + \left(\frac{1}{45}h^2 - \frac{1}{210}h^3\right)x^8 + \left(\frac{26}{105}h^2 + \frac{43}{210}h^3\right)x^6$$
$$-\left(\frac{1}{5}h - \frac{1}{5}h^2 - \frac{2}{5}h^3\right)x^4 - \left(h+2h^2+h^3\right)x^2,$$
$$\vdots$$

and so on, in this manner the rest of the iterations can be obtained. Thus, the approximate solution in a series form is given by

$$y(x) = y_0(x) + \sum_{m=1}^{+\infty} y_m(x),$$

Hence, the series solution when $h=-1$ is

$$y(x) = 1 + x^2 + \frac{1}{2!}x^4 + \frac{1}{3!}x^6 + \frac{1}{4!}x^8 + \cdots. \tag{20}$$

This series has the closed form as $n \to \infty$

$$y(x) = e^{x^2},$$

which is exactly the exact solution (17) compatible with VIM, ADM and HPM.

In Table 1 show a comparison of the numerical results applying the HAM $(n=5)$, Padé approximants (PA) of order $[6,6]$, Iteration of the Integral Equation (IIE) (19) and the numerical solution of (19) with the Simpson rule (SIMP) with the exact solution (17). Twenty points have been used in the Simpson rule. In Table 2, we list the MAE, $\|\cdot\|_2$, MRE and MRR obtained by HAM with the exact solution on $[0,1]$. The EOC at the point x_i and the GEOC on $[0,1]$ are given in Table 3.
As it is shown in [17], a necessary condition for the convergence of the method is that $\|y_{n+1}\|_2 < \|y_n\|_2$ for all n. In Figure 1 we represent the plot of $\frac{\|y_{n+1}\|_2}{\|y_n\|_2}$ for $n=0,\ldots,8$.

Also, this example is solved by using GA as follows:

We'll choose six chromosomes represent values of x between $(0,1)$ randomly as in Table 4 where $f(x_i)$ is the series solution of HAM given by (20) and $p(x_i) = \frac{f(x_i)}{\sum_{i=1}^{6} f(x_i)}$ is the probability of each chromosomes with $\sum_{i=1}^{6} f(x_i) = 8.4419$.

In selection GA operation, Roulette method used to obtained the best four chromosomes and then converting them from the decimal format to the binary format:

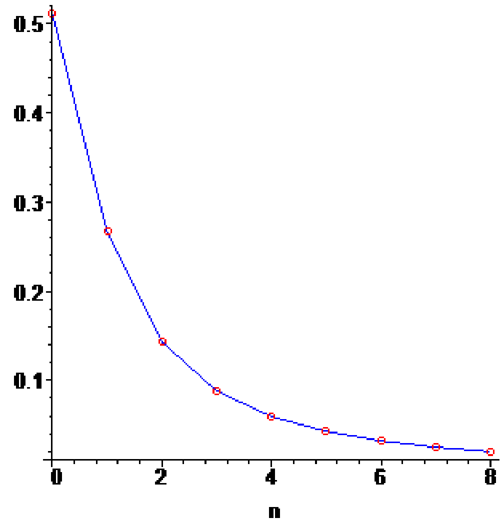

Fig. 1: Plot of $\frac{\|y_{n+1}\|_2}{\|y_n\|_2}$ for $n=0,1,\ldots,8$.

$x_{11} = 0.1 = 0.0001100110011$

$x_{21} = 0.3 = 0.0100110011000$

$x_{51} = 0.4 = 0.0110011001100$

$x_{31} = 0.5 = 0.1000000000000$

In this step of GA, crossover operation (two point) is done as follows:

$cut_1 = 2, \quad cut_2 = 10$

x_{12} : 0.0000110010011

x_{22} : 0.0101100111000

x_{52} : 0.0100000001100

x_{32} : 0.1010011000000

In the last step of GA a mutation operation (bit inverse, $m=5$) and then converting them from the binary format to the decimal format:

x_{13} : 0.0000010010011 = 0.0179

x_{23} : 0.0101000111000 = 0.3193

x_{53} : 0.0100100001100 = 0.2827

x_{33} : 0.1010111000000 = 0.6797

The optimal solution is found after 51 generation to converge to the exact solution, where $x=0$. After execute the Eq. (20) many times by using GA as in Table 5 we found the optimal solution

Table 1. Numerical results for example 1

x	$y_{Exact}(x)$	HAM	PA $[6,6]$	IIE	SIMP
0.0	1.0	1.0	1.0	1.0	1.0
0.1	1.0100501	1.0100501	1.0100501	1.0100501	1.0100501
0.2	1.0408107	1.0408107	1.0408107	1.0408107	1.0408107
0.3	1.0941742	1.0941742	1.0941742	1.0941742	1.0941742
0.4	1.1735108	1.1735108	1.1735108	1.1735108	1.1735108
0.5	1.2840254	1.2840251	1.2840251	1.2840254	1.2840254
0.6	1.4333294	1.4333276	1.4333276	1.4333294	1.4333294
0.7	1.6323162	1.6323070	1.6323067	1.6323160	1.6323160
0.8	1.8964808	1.8964404	1.8964384	1.8964797	1.8964797
0.9	2.2479079	2.2477536	2.2477417	2.2479024	2.2479024
1.0	2.7182818	2.7177547	2.7176936	2.7182575	2.7182575

Table 2. MAE, $\|\cdot\|_2$, MRE and MRR for example 1

n	MAE	$\|\cdot\|_2$	MRE	MRR
3	$8.336E-02$	$2.135E-02$	$3.066E-02$	$4.349E-00$
4	$8.126E-03$	$1.827E-03$	$2.989E-03$	$7.523E-01$
5	$5.270E-04$	$1.067E-04$	$1.938E-04$	$7.599E-02$
6	$2.429E-05$	$4.511E-06$	$8.938E-06$	$5.027E-03$
7	$8.357E-07$	$1.440E-07$	$3.074E-07$	$2.346E-04$
8	$2.225E-08$	$3.594E-09$	$8.186E-09$	$8.134E-06$
9	$4.719E-10$	$7.199E-11$	$1.736E-10$	$2.177E-07$
10	$8.161E-12$	$1.182E-12$	$3.002E-12$	$4.638E-09$

Table 3. EOC and GEOC for example 1

x	EOC, $n=5$
0.1	1.0704
0.2	1.0882
0.3	1.1036
0.4	1.1182
0.5	1.1328
0.6	1.1478
0.7	1.1634
0.8	1.1799
0.9	1.1975
1.0	1.2166
GEOC	1.2318

Table 4. Chromosomes select for example 1.

x_i	0.1	0.3	0.5	0.7	0.4	0.9
$f(x_i)$	1.0101	1.0942	1.2840	1.6323	1.1735	2.2478
$p(x_i)$	0.1197	0.1296	0.1521	0.1934	0.1390	0.2663

Table 5. Optimal solution of Genetic Algorithm for example 1

x	$y_{Exact}(x)$	GA	HAM	PA $[6,6]$	SIMP
0.000	1.0	1.0	1.0	1.0	1.0
0.012	1.0001440	1.0001441	1.0001440	1.0001440	1.0001440
0.017	1.0002890	1.0003048	1.0002890	1.0002890	1.0002890
0.040	1.0016012	1.0015823	1.0016012	1.0016012	1.0016012
0.067	1.0044990	1.0044873	1.0044990	1.0044990	1.0044990

Example 2. We consider the following linear nonhomogeneous Lane-Emden equation

$$y''+\frac{8}{x}y'+xy=x^5-x^4+44x^2-30x, \tag{21}$$

subject to initial conditions

$$y(0)=0,\ y'(0)=0. \tag{22}$$

The exact solution for this problem is

$$y_{Exact}(x)=x^4-x^3. \tag{23}$$

To solve Eqs. (21)–(22) by means of the standard HAM, we choose the initial approximation

$$y_0(x)=0,$$

and the linear operator

$$L[\phi(x;q)]=\frac{\partial^2\phi(x;q)}{\partial x^2}+\frac{8}{x}\frac{\partial\phi(x;q)}{\partial x},$$

with the property

$$L\left[-\frac{c_1}{7x}+c_2\right]=0,$$

where c_i $(i=1,2)$ are constants of integration. Furthermore, Eq. (21) suggests that we define the nonlinear operator as

$$N[\phi(x;q)]=\frac{\partial^2\phi(x;q)}{\partial x^2}+\frac{8}{x}\frac{\partial\phi(x;q)}{\partial x}+x\phi(x;q)-\left(x^5-x^4+44x^2-30x\right),$$

Using the above definition, we construct the zeroth-order deformation equation as in (5) and (6) and the mth-order deformation equation for $m\geq 1$ is

$$L\left[y_m(x)-\chi_m y_{m-1}(x)\right]=hR_m\left(\overrightarrow{y}_{m-1}\right), \tag{24}$$

with the initial conditions (22)

$$y_m(0)=0,\ y_m'(0)=0$$

where

$$R_m\left(\overrightarrow{y}_{m-1}\right)=y''_{m-1}+\frac{8}{x}y'_{m-1}+xy_{m-1}-(1-\chi_m)\left(x^5-x^4+44x^2-30x\right),$$

Now, the solution of the mth-order deformation Eq. (24) for $m\geq 1$ is

$$y_m(x)=\chi_m y_{m-1}(x)+h\int_0^x x^{-8}\int_0^x x^8 R_m\left(\overrightarrow{y}_{m-1}\right). \tag{25}$$

We now successively obtain

$$y_1(x)=-\frac{1}{98}hx^7+\frac{1}{78}hx^6-hx^4+hx^3,$$

$$y_2(x)=-\frac{1}{16660}h^2x^{10}+\frac{1}{11232}h^2x^9-\left(\frac{1}{49}h^2+\frac{1}{98}h\right)x^7+\left(\frac{1}{39}h^2+\frac{1}{78}h\right)x^6$$
$$-\left(h^2+h\right)x^4+\left(h^2+h\right)x^3,$$
$$\vdots$$

and so on, in this manner the rest of the iterations can be obtained. Thus, the approximate solution in a series form is given by

$$y(x)=y_0(x)+\sum_{m=1}^{+\infty}y_m(x),$$

Hence, the series solution when $h=-1$ is

$$y_1(x)=\frac{1}{98}x^7-\frac{1}{78}x^6+x^4-x^3,$$

$$y_2(x)=-\frac{1}{16660}x^{10}+\frac{1}{11232}x^9-\frac{1}{98}x^7+\frac{1}{78}x^6,$$

$$y_3(x)=\frac{1}{4331600}x^{13}-\frac{1}{2560896}x^{12}+\frac{1}{16660}x^{10}-\frac{1}{11232}x^9,$$

$$y_4(x)=-\frac{1}{1594028800}x^{16}+\frac{1}{845095680}x^{15}-\frac{1}{4331600}x^{13}+\frac{1}{2560896}x^{12},$$
$$\vdots$$

Thus, the approximate solution in a series form is given by

$$y(x)=x^4-x^3+\text{noise terms} \tag{26}$$

This series has the closed form as $n\to\infty$

$$y(x)=x^4-x^3,$$

which is exactly the exact solution (23) compatible with VIM, ADM and HPM. Notice that the noise terms that appear between various components vanish.

In Table 6 show a comparison of the numerical results applying the HAM $(n=5)$, Iteration of the Integral Equation (IIE) (25) and the numerical solution of (25) with the Simpson rule (SIMP) with the exact solution (23). Twenty points have been used in the Simpson rule. In Table 7, we list the MAE, $\|\cdot\|_2$, MRE and MRR obtained by HAM with the exact solution on $[0,1]$. The EOC at the point x_i and the GEOC on $[0,1]$ are given in Table 8.

As it is shown in [17], a necessary condition for the convergence of the method is that $\|y_{n+1}\|_2<\|y_n\|_2$ for all n. In Figure 2 we represent the plot of $\dfrac{\|y_{n+1}\|_2}{\|y_n\|_2}$ for $n=0,\ldots,8$.

Using GA by the same procedure as in example 1, we get the optimal solution is found after 51 generation to converge to the exact solution, where $x=0.75$. After execute the Eq. (26) many times by using GA as in Table 9 we found the optimal solution

Example 3. Finally, let us consider the following nonlinear homogeneous Lane-Emden equation

$$y''+\frac{2}{x}y'+y^3=0, \tag{27}$$

subject to initial conditions

$$y(0)=1,\ y'(0)=0. \tag{28}$$

Table 6. Numerical results for example 2

x	$y_{Exact}(x)$	HAM	IIE	SIMP
0.0	0.0	0.0	0.0	0.0
0.1	-0.0009	-0.000899	-0.0009	-0.0009
0.2	-0.0064	-0.006399	-0.0064	-0.0064
0.3	-0.0189	-0.018899	-0.0189	-0.0189
0.4	-0.0384	-0.038399	-0.0384	-0.0384
0.5	-0.0625	-0.062499	-0.0625	-0.0625
0.6	-0.0864	-0.086399	-0.0864	-0.0864
0.7	-0.1029	-0.102899	-0.1029	-0.1029
0.8	-0.1024	-0.102399	-0.1024	-0.1024
0.9	-0.0729	-0.072899	-0.0729	-0.0729
1.0	0.0	$5.55E-10$	$2.24E-10$	$2.24E-10$

Table 7. MAE, $\|\cdot\|_2$, MRE and MRR for example 2

n	MAE	$\|\cdot\|_2$	MRE	MRR
2	$2.616E-03$	$9.467E-04$	$2.651E-02$	$8.289E-02$
3	$2.900E-05$	$7.372E-06$	$1.860E-04$	$2.645E-03$
4	$1.596E-07$	$3.374E-08$	$7.078E-07$	$2.916E-05$
5	$5.559E-10$	$1.034E-10$	$1.747E-09$	$1.601E-07$
6	$1.359E-12$	$2.290E-13$	$3.060E-12$	$5.573E-10$
7	$2.481E-15$	$3.853E-16$	$4.023E-15$	$1.362E-12$
8	$3.522E-18$	$5.103E-19$	$4.126E-18$	$2.485E-15$

Table 8. EOC and GEOC for example 2

x	EOC, $n=5$
0.1	1.0345
0.2	1.0421
0.3	1.0486
0.4	1.0548
0.5	1.0612
0.6	1.0681
0.7	1.0760
0.8	1.0857
0.9	1.0987
1.0	1.1185
GEOC	1.1090

Table 9. Optimal solution of Genetic Algorithm for example 2

x	$y_{Exact}(x)$	GA	HAM	SIMP
0.726	-0.1048480	-0.1048509	-0.1048480	-0.1048480
0.735	-0.1052224	-0.1052224	-0.1052223	-0.1052223
0.742	-0.1053977	-0.1054036	-0.1053977	-0.1053977
0.749	-0.1054676	-0.1054681	-0.1054676	-0.1054676
0.750	-0.1054687	-0.1054687	-0.1054687	-0.1054687

To solve Eqs. (27)–(28) by means of the standard HAM, we choose the initial approximation

$$y_0(x) = 1,$$

and the linear operator

$$L[\phi(x;q)] = \frac{\partial^2\phi(x;q)}{\partial x^2} + \frac{2}{x}\frac{\partial\phi(x;q)}{\partial x},$$

with the property

$$L\left[-\frac{c_1}{x} + c_2\right] = 0,$$

where $c_i \quad (i = 1,2)$ are constants of integration. Furthermore, Eq .(27) suggests that we define the nonlinear operator as

$$N[\phi(x;q)] = \frac{\partial^2\phi(x;q)}{\partial x^2} + \frac{2}{x}\frac{\partial\phi(x;q)}{\partial x} + \phi(x;q)^3,$$

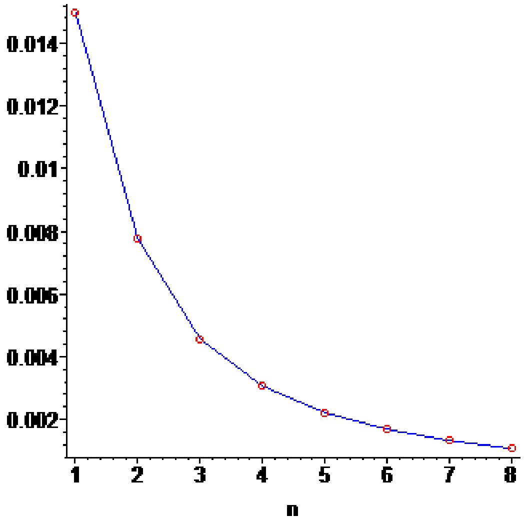

Fig. 2: Plot of $\frac{\|y_{n+1}\|_2}{\|y_n\|_2}$ for $n = 0, 1, \ldots, 8$.

Using the above definition, we construct the zeroth-order deformation equation as in (5) and (6) and the mth-order deformation equation for $m \geq 1$ is

$$L\left[y_m(x) - \chi_m y_{m-1}(x)\right] = hR_m\left(\overrightarrow{y}_{m-1}\right), \qquad (29)$$

with the initial conditions

$$y_m(0) = 0, \;\; y_m'(0) = 0$$

where

$$R_m\left(\overrightarrow{y}_{m-1}\right) = y''_{m-1} + \frac{2}{x} y'_{m-1} + y^3_{m-1}.$$

Now, the solution of the mth-order deformation Eq. (29) for $m \geq 1$ is

$$y_m(x) = \chi_m y_{m-1}(x) + h\int_0^x x^{-2}\int_0^x x^2 R_m\left(\overrightarrow{y}_{m-1}\right). \qquad (30)$$

We now successively obtain

$$y_1(x) = \frac{h}{6}x^2,$$

$$y_2(x) = \frac{h^2}{40}x^4 + \frac{1}{6}\left(h + h^2\right)x^2,$$

$$y_3(x) = \frac{19}{5040}h^3x^6 + \frac{1}{20}\left(h^2 + h^3\right)x^4 + \frac{1}{6}\left(h + 2h^2 + h^3\right)x^2,$$

$$y_4(x) = \frac{619}{1088640}h^4x^8 + \frac{19}{1680}\left(h^3 + h^4\right)x^6 + \frac{3}{40}\left(h^2 + 2h^3 + h^4\right)x^4 + \frac{1}{6}\left(h + 3h^2 + 3h^3 + h^4\right)x^2,$$

$$\vdots$$

and so on, in this manner the rest of the iterations can be obtained. Thus, the approximate solution in a series form is given by

$$y(x) = y_0(x) + \sum_{m=1}^{+\infty} y_m(x),$$

Hence, the series solution when $h = -1$ is

$$y_1(x) = -\frac{1}{6}x^2,$$

$$y_2(x) = \frac{1}{40}x^4,$$

$$y_3(x) = -\frac{19}{5040}x^6,$$

$$y_4(x) = \frac{619}{1088640}x^8,$$

$$y_4(x) = -\frac{17117}{199584000}x^{10},$$

$$\vdots$$

Thus, the approximate solution in a series form is given by

$$y(x) = 1 - \frac{1}{6}x^2 + \frac{1}{40}x^4 - \frac{19}{5040}x^6 + \frac{619}{1088640}x^8 - \frac{17117}{199584000}x^{10} + \cdots. \qquad (31)$$

In Table 10 show a comparison of the numerical results applying the HAM $(n = 5)$, Padé approximants (PA) of order $[4,4]$, Iteration of the Integral Equation (IIE) (30) and the numerical solution of (30) with the Simpson rule (SIMP). Twenty points have been used in the Simpson rule. The HAM which we have designed to solve this problem is called IIE, that gives identical values for $n = 5$, which are used as exact solution.

As it is shown in [17], a necessary condition for the convergence of the method is that $\|y_{n+1}\|_2 < \|y_n\|_2$ for all n. In Figure 3 we represent the plot of $\frac{\|y_{n+1}\|_2}{\|y_n\|_2}$ for $n = 0, \ldots, 7$.

Using GA by the same procedure as in example 1, we otain the optimal solution is found after 51 generation to converge to the exact solution, where $x = 1.0$. After execute the Eq. (31) many times by using GA as in Table 11 we found the optimal solution

5 Conclusion

To our best knowledge this is the first result on the application of the HAM with GA to Lane-Emden type singular IVPs for ordinary differential equations. The HAM with GA have been successfully applied to solve

Table 10. Numerical results for example 3

x	IIE	HAM	PA $[4,4]$	SIMP
0.0	1.0	1.0	1.0	1.0
0.1	0.998335829	0.998335829	0.998335829	0.998335829
0.2	0.993373093	0.993373093	0.993373093	0.993373093
0.3	0.985199789	0.985199789	0.985199788	0.985199789
0.4	0.973958264	0.973958264	0.973958255	0.973958264
0.5	0.959839146	0.959839150	0.959839069	0.959839146
0.6	0.943073637	0.943073664	0.943073172	0.943073637
0.7	0.923924947	0.923925093	0.923922838	0.923924947
0.8	0.902679862	0.902680487	0.902672089	0.902679862
0.9	0.879641571	0.879643815	0.879617167	0.879641571
1.0	0.855125080	0.855132091	0.855057569	0.855125080

Table 11. Optimal solution of Genetic Algorithm for example 3

x	IIE	GA	HAM	PA $[4,4]$	SIMP
0.948	0.8680383	0.8679326	0.8680422	0.8679980	0.8680383
0.960	0.8650884	0.8652159	0.8650929	0.8650429	0.8650884
0.973	0.8618718	0.8619235	0.8618771	0.8618200	0.8618718
0.992	0.8571331	0.8571995	0.8571396	0.8570707	0.8571331
1.000	0.8551250	0.8551320	0.8551320	0.8550575	0.8551250

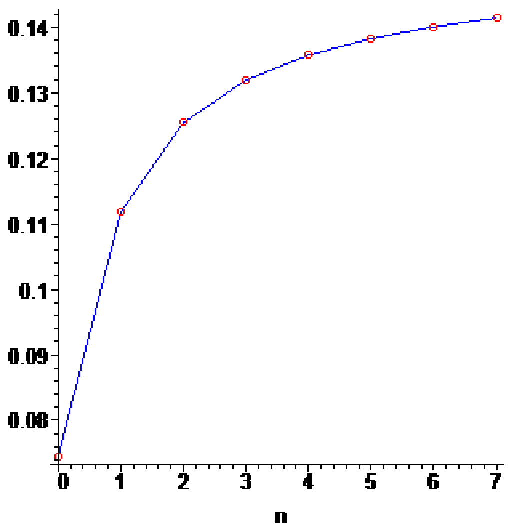

Fig. 3: Plot of $\frac{\|y_{n+1}\|_2}{\|y_n\|_2}$ for $n = 0, 1, \ldots, 7$.

models of Lane-Emden type singular IVPs. The HAM with GA have worked effectively to handle these models giving it a wider applicability. The proposed scheme of HAM has been applied directly without any need for transformation formulae or restrictive assumptions. The solution process of HAM is compatible with those methods in the literature providing analytical approximation such as VIM, ADM and HPM.

The approach of HAM has been tested by employing the method to obtain approximate-exact solutions of three examples. The results obtained in all cases demonstrate the reliability and the efficiency of this method.

Acknowledgements

We would like to thank the referees for their careful review of our manuscript.

References

[1] H.T. Davis, Introduction to Nonlinear Differential and Integral Equations, Dover, New York, 1962.
[2] S. Chandrasekhar, Introduction to the Study of Stellar Structure, Dover, New York, 1967.
[3] O. U. Richardson, The Emission of Electricity from Hot Bodies, Longman's Green and Company, London, 1921.
[4] A. Yıldırım, T. Özis, Nonlinear Anal. 70, 2480-2484, (2009).
[5] M. Dehghan, F. Shakeri, , New Astron. 13, 53-59, (2008).
[6] N. Shawagfeh, J. Math. Phys. 34 (9), 4364-4369, (1993).
[7] A.M. Wazwaz, Appl. Math. Comput. 118, 287-310, (2001).
[8] A.M Wazwaz, Appl. Math. Comput. 128, 45-57, (2001).
[9] A-M. Wazwaz, R. Rach, Jun-Sheng Duan, Appl. Math. Comput. 219, 5004-5019, (2013).
[10] J. Ramos, Chaos Soliton Fract. 38 (2), 400-408, (2008).
[11] A. Yıldırım, T. Özis, Phys. Lett. A 369, 70-76, (2007).
[12] M. Chowdhury, I. Hashim, Phys. Lett. A. 365 (5-6), 439-447, (2007).

[13] S.J. Liao, Beyond Perturbation: Introduction to the Homotopy Analysis Method, Chapman and Hall, CRC Press, Boca Raton, 2003.
[14] J.H. He. Appl. Math. Comput. 135 (1), 73–79, (2003).
[15] Mitsuo Gen, Runwei Cheng, Genetic Algorithms and Engineering Design, John Wiley & Sons, 1997.
[16] Mitchell, Melanie. An introduction to genetic algorithms. MIT press, 1998.
[17] Y. Cherruault, G. Adomian, Math. Comput. Model. 18 (12), 103-106, (1993).

Waleed Al-Hayani is an Assistant Professor in the Department of Mathematics, College of Computer Science and Mathematics, Mosul University, Iraq. He was an Assistant Professor, Departamento de Matemáticas, Escuela Politécnica Superior, Universidad Carlos III de Madrid, Spain (2007-2012). He received his Ph.D. degree from the Universidad Politécnica de Madrid, Spain. Head of Department of Mathematics 2015-. His fields of research include Adomian Decomposition Method, Homotopy Perturbation Method, Homotopy Analysis Method,Variational Iteration Method, Laplace Transform related to before topics and Fractional Calculus related to befor topics.

Laheeb Alzubaidy is an Assistant Professor in the Department of Computer Science, College of Computer Science and Mathematics, Mosul University, Iraq. She received her B.Sc. in 1987, M.Sc. in 1992 and Ph.D. in 2002, in Computer Science, Department of Computer Science, University of Mosul, Iraq. Head of Department of Computer Science 2003-2004. Visiting lecturer at Isra private University, Jordan in 2004. Head of Department of Software Engineering in 2007-2009. Visiting lecturer at USM University, NAV6 Center, Malaysia in 2009. Interested research fields are in Artificial Intelligent technique, network security, image processing, pattern recognition and software Engineering.

Ahmed Entesar is a Lecturer in the Department of Mathematics, College of Computer Science and Mathematics, Mosul University, Iraq. He received his Master degree in Mathematics in 2008. His fields of research include Numerical Analysis, Adomian Decomposition Method, Finite Difference, Homotopy Analysis Method and Homotopy Perturbation Method. Now, he is a student of Ph.D..