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Impact of Thermal Reservoir on Entanglement and Squeezing Characteristics of a Three-Level Laser Driven by Coherent Light in an Open Cavity

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Abstract: This study investigates the entanglement and squeezing characteristics of light generated by a non-degenerate, coherently driven three-level laser in an open cavity, coupled to a thermal reservoir through a single-port mirror. By solving the master equation and Hamiltonian, we derived the steady-state solutions for the atomic operators, enabling us to calculate the average photon number, quadrature variance, entanglement, and normalized second-order correlation of the cavity radiation. Our results indicate that higher spontaneous emission rates significantly reduce the average photon number, while the mean of thermal light($\langle n_{th} \rangle$) enhance it. We found that the light is squeezed primarily in the minus quadrature, achieving maximum squeezing of 37.9% at $\langle n_{th} \rangle = 0$, and slightly decreasing with increased thermal effects. Spontaneous emission reduces photon number variance, while thermal reservoirs amplify it. Additionally, squeezing enhances entanglement, and both are interdependent, with higher squeezing producing more entanglement. These results have implications for quantum communication, computing, and sensing by improving secure communication, error correction, and measurement precision.

Keywords: Operator dynamics, Photon statistics, Quadrature squeezing, Photon entanglement, Second-order correlations.

1 Introduction

Entanglement is a fundamental building block of quantum protocols for information processing and communication. It can be created and manipulated for various applications, such as quantum cryptography, quantum error correction, quantum computation, quantum teleportation, and quantum dense coding [1]-[5]. The development of continuous-variable entanglement has recently garnered significant attention as a method for carrying out quantum information processing and managing quantum bits and discrete equivalents. Generally, when a system interacts with its environment, the degree of entanglement diminishes. However, the level of entanglement plays a crucial role in the efficiency of quantum information processing. Consequently, creating highly entangled states that are robust against external noise is essential. A two-mode squeezed state can be used to prepare Einstein?Podolsky?Rosen (EPR)-type entanglement, as it often violates classical inequalities due to the strong correlation between the cavity modes[6]. Steady-state entanglement has been studied in two cases involving a non-degenerate three-level laser: one where coherent light couples the upper and lower levels of three-level atoms injected into a cavity[16]-[21], and another where atomic coherence is induced by preparing atoms in a coherent superposition of the upper and lower levels [7]-[15]. The phenomenon of ”squeezing” in light, which has been extensively studied by various researchers [23], involves reducing noise in one part of the light while increasing fluctuations in another. According to the uncertainty principle, this results in reduced uncertainty in one quadrature while increasing uncertainty in the other. Squeezed light has potential applications in weak-signal detection and low-noise optical communication [6, 10]

Furthermore, Abebe[21] examined the light squeezing and entanglement attributes that are generated by a nondegenerate three level laser in an open cavity configuration driven coherently while it is coupled with two mode vacuum reservoir. A maximum quadrature squeezing of output light lowered by 50% compared to

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vacuum state level. Furthermore, Fesseha [10] has explored the squeezing and statistical properties of light produced by a three-level laser with atoms confined within a closed cavity and pumped with coherent light. According to his results, the largest quadrature squeezing corresponds to 43% less than that of the vacuum state.

In this study, we investigate the quantum properties of cavity light beams generated in an open cavity by a coherently driven nondegenerate three-level laser. The laser is coupled to a thermal reservoir via a single-port mirror. We begin by utilizing the Hamiltonian equation, which describes the coherently driven nondegenerate three-level atom, to derive the quantum Langevin equations for the cavity mode operators. Next, we derive the evolution equations for the expectation values of the atomic operators using the master equation and the long-time approximation method. By solving these equations, we obtain the steady-state solutions. Importantly, we account for noise from operators associated with the thermal reservoir to arbitrary order in our calculations. Using the steady-state solutions and the quantum Langevin equations, we compute the mean photon number, quadrature variance, quadrature squeezing, photon-cavity atom entanglement, and photon number correlations within the system.

2 Operator dynamics

This paper investigates a case where N three-level atoms are placed in an open cavity in a cascade configuration. The three levels that represent the atomic states are $|a\rangle_k$ for the top level, $|b\rangle_k$ for the middle level, and $|c\rangle_k$ for the lowest level. The light emanating from the middle state will be referred to as light mode b in the discussion that follows, and the light emanating from the higher state as light mode a. We assume that the frequencies of these modes may be the same or different. In addition, we presume that light modes a and b have frequencies that, according to electric-dipole selection criteria, correspond to transitions between two levels: $|a\rangle_k \rightarrow |b\rangle_k$ and $|b\rangle_k \rightarrow |c\rangle_k$; $|a\rangle_k$ and $|c\rangle_k$ are not connected by a direct transition.

The interaction between a three-level atom and cavity modes, as well as the coupling of the atom's top and bottom levels through coherent driving, can be mathematically characterized by the Hamiltonian[10,15]:

$$\hat{H} = ig\left[\hat{\sigma}_a^{\dagger k}\hat{a} - \hat{a}^\dagger\hat{\sigma}_a^k + \hat{\sigma}_b^{\dagger k}\hat{b} - \hat{b}^\dagger\hat{\sigma}_b^k\right] + i\frac{\Omega}{2}\left[\hat{\sigma}_c^{\dagger k} - \hat{\sigma}_c^k\right]. \tag{1}$$

Atomic lowering operators are denoted by $\sigma_a^k = |b\rangle kk\langle a|$, $\hat{\sigma}_b^k = |c\rangle kk\langle b|$, and $\sigma_c^k = |c\rangle kk\langle a|$. $\hat{a}$, $\hat{b}$, and $\hat{c}$ here stand for the cavity modes' annihilation operators. The coupling strength between the cavity modes and the atom is given by the parameter g. The coupling strength between the driving coherent light and the three-level atom is represented by the constant λ. Moreover, the positive real number $\Omega = 2\lambda\mu$ is proportional to the coupling coherent light's amplitude.

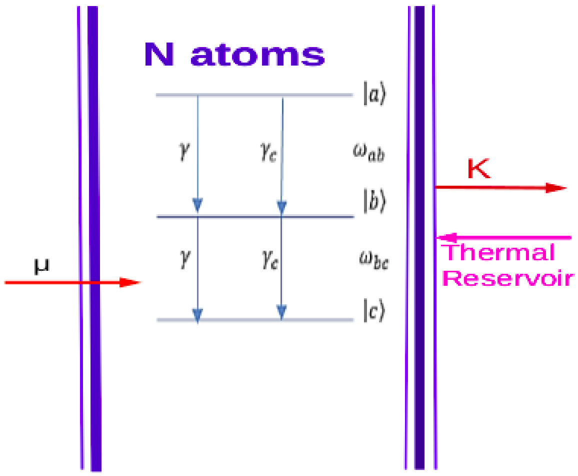

Fig. 1 A schematic representation of a non-degenerate three-level laser coupled to a thermal reservoir. In this setup, μ represents the amplitude of the coherent light driving the transition between the top and bottom atomic levels. The parameter κ denotes the cavity damping constant, which is considered equal for both transitions. The top, middle, and bottom levels of the three-level atom are labeled $|a\rangle$, $|b\rangle$, and $|c\rangle$, respectively Additionally, γ is assumed to be the same for both states $|a\rangle$ and $|b\rangle$, represents the spontaneous emission decay constant.

The quantum Langevin equations for the operators $\hat{a}$ and $\hat{b}$ are given by [9,10]

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - i[\hat{a},\hat{H}] + \hat{F}_a(t), \tag{2}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} - i[\hat{b},\hat{H}] + \hat{F}_b(t), \tag{3}$$

where κ is the cavity damping constant.

With the aid of Eqs. (1), (2), and (3), one can easily establish that

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} - g\hat{\sigma}_a^k + \hat{F}_a(t), \tag{4}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} - g\hat{\sigma}_b^k + \hat{F}_b(t). \tag{5}$$

Moreover, [10] provides the master equation for a three-level atom interacting with a thermal reservoir.

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= -i[\hat{H},\hat{\rho}] + \frac{\gamma}{2}\Big[2\hat{\sigma}_a^k\hat{\rho}\hat{\sigma}_a^{\dagger k} - \hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k\hat{\rho} - \hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{\sigma}_a^k \\ &+ 2\hat{\sigma}_b^k\hat{\rho}\hat{\sigma}_b^{\dagger k} - \hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^k\hat{\rho} - \hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{\sigma}_b^k\Big] \\ &+ \frac{\kappa}{2}(\langle n_{th}\rangle + 1)\Big[2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\Big] \\ &+ \frac{\kappa}{2}\langle n_{th}\rangle\Big[2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\Big],\end{aligned} \tag{6}$$

where γ is the spontaneous emission decay constant and is assumed to be the same for levels $|a\rangle$ and $|b\rangle$.

$$\hat{\eta}_a^k = |a\rangle_{kk}\langle a|, \tag{7}$$

$$\hat{\eta}_b^k = |b\rangle_{kk}\langle b|. \tag{8}$$

Using Eq. (1), we can put Eq. (6) in the form

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= \frac{\Omega}{2}\Big[\hat{\sigma}_c^{\dagger k}\hat{\rho} - \hat{\rho}\hat{\sigma}_c^{\dagger k} + \hat{\rho}\hat{\sigma}_c^k - \hat{\sigma}_c^k\hat{\rho}\Big] \\ &+ g[\hat{\sigma}_a^{\dagger k}\hat{a}\hat{\rho} - \hat{\rho}\hat{\sigma}_a^{\dagger k}\hat{a} + \hat{\sigma}_b^{\dagger k}\hat{b}\hat{\rho} - \hat{\rho}\hat{\sigma}_b^{\dagger k}\hat{b} - \hat{a}^\dagger\hat{\sigma}_a^k\hat{\rho} \\ &- \hat{b}^\dagger\hat{\sigma}_b^k\hat{\rho} + \hat{\rho}\hat{a}^\dagger\hat{\sigma}_a^k + \hat{\rho}\hat{b}^\dagger\hat{\sigma}_b^k] \\ &+ \frac{\gamma}{2}\Big[2\hat{\sigma}_a^k\hat{\rho}\hat{\sigma}_a^{\dagger k} - \hat{\eta}_a^k\hat{\rho} - \hat{\rho}\hat{\eta}_a^k + 2\hat{\sigma}_b^k\hat{\rho}\hat{\sigma}_b^{\dagger k} - \hat{\eta}_b^k\hat{\rho} - \hat{\rho}\hat{\eta}_b^k\Big] \\ &+ \frac{\kappa}{2}(\langle n_{th}\rangle + 1)\Big[2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\Big] \\ &+ \frac{\kappa}{2}\langle n_{th}\rangle\Big[2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\Big].\end{aligned} \tag{9}$$

Now applying the relation

$$\frac{d}{dt}\langle\hat{A}\rangle = Tr(\frac{d\hat{\rho}}{dt}\hat{A}) \tag{10}$$

along with Eq. (9), we can easily establish that

$$\frac{d}{dt}\langle\hat{\sigma}_a^k\rangle = -\gamma\langle\hat{\sigma}_a^k\rangle + g\Big[\langle\hat{\eta}_b^k\hat{a}\rangle - \langle\hat{\eta}_a^k\hat{a}\rangle + \langle\hat{b}^\dagger\hat{\sigma}_c^k\rangle\Big] + \frac{\Omega}{2}\langle\hat{\sigma}_b^{\dagger k}\rangle, \tag{11}$$

$$\frac{d}{dt}\langle\hat{\sigma}_b^k\rangle = -\frac{\gamma}{2}\langle\hat{\sigma}_b^k\rangle + g\Big[\langle\hat{\eta}_c^k\hat{b}\rangle - \langle\hat{\eta}_b^k\hat{b}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_c^k\rangle\Big] - \frac{\Omega}{2}\langle\hat{\sigma}_a^{\dagger k}\rangle, \tag{12}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c^k\rangle = -\frac{\gamma}{2}\langle\hat{\sigma}_c^k\rangle + g\Big[\langle\hat{\sigma}_b^k\hat{a}\rangle - \langle\hat{\sigma}_a^k\hat{b}\rangle\Big] + \frac{\Omega}{2}[\langle\hat{\eta}_c^k\rangle - \langle\hat{\eta}_a^k\rangle], \tag{13}$$

$$\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = -\gamma\langle\hat{\eta}_a^k\rangle + g\Big[\langle\hat{\sigma}_a^{\dagger k}\hat{a}\rangle + \langle\hat{a}^\dagger\hat{\sigma}_a^k\rangle\Big] + \frac{\Omega}{2}[\langle\hat{\sigma}_c^{\dagger k}\rangle + \langle\hat{\sigma}_c^k\rangle], \tag{14}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{\eta}_b^k\rangle = \ &\gamma\Big[\langle\hat{\eta}_a^k\rangle - \langle\hat{\eta}_b^k\rangle\Big] + g[\langle\hat{b}^\dagger\hat{\sigma}_b^k\rangle + \langle\hat{\sigma}_b^{\dagger k}\hat{b}\rangle - \langle\hat{\sigma}_a^{\dagger k}\hat{a}\rangle \\ &- \langle\hat{a}^\dagger\hat{\sigma}_a^k\rangle],\end{aligned} \tag{15}$$

$$\frac{d}{dt}\langle\hat{\eta}_c^k\rangle = \gamma\langle\hat{\eta}_b^k\rangle - g\Big[\langle\hat{b}^\dagger\hat{\sigma}_b^k\rangle + \langle\hat{\sigma}_b^{\dagger k}\hat{b}\rangle\Big] - \frac{\Omega}{2}[\langle\hat{\sigma}_c^{\dagger k}\rangle + \langle\hat{\sigma}_c^k\rangle], \tag{16}$$

where

$$\hat{\eta}_c^k = |c\rangle_{kk}\langle c|. \tag{17}$$

Since Eqs. (11)-(16) are nonlinear differential equations, it is impossible to precisely determine their time-dependent solutions. By using the large-time approximation, we hope to overcome this issue [13]. The roughly valid relations are then obtained from Eqs. (4) and (5) using this approximation approach.

$$\hat{a} = -\frac{2g}{\kappa}\hat{\sigma}_a^k + \frac{2}{\kappa}\hat{F}_a(t), \tag{18}$$

$$\hat{b} = -\frac{2g}{\kappa}\hat{\sigma}_b^k + \frac{2}{\kappa}\hat{F}_b(t). \tag{19}$$

These would obviously prove to be accurate relations in a stable state. Eqs. (18) and (19) are now combined with Eqs. (11)-(16) to obtain

$$\begin{aligned}\frac{d}{dt}\langle\hat{\sigma}_a^k\rangle = &-[\gamma + \gamma_c]\langle\hat{\sigma}_a^k\rangle + \frac{\Omega}{2}\langle\hat{\sigma}_b^{\dagger k}\rangle \\ &+ \frac{2g}{\kappa}\Big[\langle\hat{\eta}_b^k\hat{F}_a(t)\rangle - \langle\hat{\eta}_a^k\hat{F}_a(t)\rangle + \langle\hat{F}_b^\dagger(t)\hat{\sigma}_c^k\rangle\Big],\end{aligned} \tag{20}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{\sigma}_b^k\rangle = &-\Big[\frac{\gamma}{2} + \frac{\gamma_c}{2}\Big]\langle\hat{\sigma}_b^k\rangle - \frac{\Omega}{2}\langle\hat{\sigma}_a^{\dagger k}\rangle \\ &+ \frac{2g}{\kappa}\Big[\langle\hat{\eta}_c^k\hat{F}_b(t)\rangle - \langle\hat{\eta}_b^k\hat{F}_b(t)\rangle - \langle\hat{F}_a^\dagger(t)\hat{\sigma}_c^k\rangle\Big],\end{aligned} \tag{21}$$

$$\frac{d}{dt}\langle\hat{\sigma}_c^k\rangle = -\Big[\frac{\gamma}{2} + \frac{\gamma_c}{2}\Big]\langle\hat{\sigma}_c^k\rangle + \frac{2g}{\kappa}\Big[\langle\hat{\sigma}_b^k\hat{F}_a(t)\rangle - \langle\hat{\sigma}_a^k\hat{F}_b(t)\rangle\Big], \tag{22}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{\eta}_a^k\rangle = &-[\gamma + \gamma_c]\langle\hat{\eta}_a^k\rangle + \frac{\Omega}{2}[\langle\hat{\sigma}_c^{\dagger k}\rangle + \langle\hat{\sigma}_c^k\rangle] \\ &+ \frac{2g}{\kappa}\Big[\langle\hat{\sigma}_a^{\dagger k}\hat{F}_a(t)\rangle + \langle\hat{F}_a^\dagger(t)\hat{\sigma}_a^k\rangle\Big],\end{aligned} \tag{23}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{\eta}_b^k\rangle = &-[\gamma + \gamma_c]\langle\hat{\eta}_b^k\rangle + [\gamma + \gamma_c]\langle\hat{\eta}_a^k\rangle \\ &+ \frac{2g}{\kappa}[\langle\hat{F}_b^\dagger(t)\hat{\sigma}_b^k\rangle + \langle\hat{\sigma}_b^{\dagger k}\hat{F}_b(t)\rangle - \langle\hat{\sigma}_a^{\dagger k}\hat{F}_a(t)\rangle \\ &- \langle\hat{F}_a^\dagger(t)\hat{\sigma}_a^k\rangle],\end{aligned} \tag{24}$$

$$\begin{aligned}\frac{d}{dt}\langle\hat{\eta}_c^k\rangle = \ &[\gamma + \gamma_c]\langle\hat{\eta}_b^k\rangle - \frac{\Omega}{2}[\langle\hat{\sigma}_c^{\dagger k}\rangle + \langle\hat{\sigma}_c^k\rangle] \\ &- \frac{2g}{\kappa}\Big[\langle\hat{F}_b^\dagger(t)\hat{\sigma}_b^k\rangle + \langle\hat{\sigma}_b^{\dagger k}\hat{F}_b(t)\rangle\Big],\end{aligned} \tag{25}$$

where

$$\gamma_c = \frac{4g^2}{\kappa} \tag{26}$$

is the stimulated emission decay constant.

We next proceed to find the expectation value of the product involving a noise operator and an atomic operator that appears in Eqs. (20) - (25). To this end, after removing the angular brackets, Eq. (23) can be rewritten as

$$\frac{d}{dt}\hat{\eta}_a^k = -[\gamma+\gamma_c]\,\hat{\eta}_a^k + \frac{2g}{\kappa}\left[\hat{\sigma}_a^{\dagger k}\hat{F}_a(t) + \hat{F}_a^{\dagger}(t)\hat{\sigma}_a^k\right] + \frac{\Omega}{2}[\langle\hat{\sigma}_c^{\dagger k}\rangle + \langle\hat{\sigma}_c^k\rangle] + \hat{f}_a(t), \tag{27}$$

where the noise operator connected to $\hat{\eta}_a$ is denoted by $\hat{f}_a(t)$. This equation's formal solution is expressed as

$$\hat{\eta}_a^k(t) = \hat{\eta}_a^k(0)e^{-(\gamma+\gamma_c)t} + \int_0^t e^{-(\gamma+\gamma_c)(t-t')}\left[\frac{2g}{\kappa}[\hat{\sigma}_a^{\dagger k}(t')\hat{F}_a(t') + \hat{F}_a^{\dagger}(t')\hat{\sigma}_a^k(t')] + \frac{\Omega}{2}[\langle\hat{\sigma}_c^{\dagger k}\rangle + \langle\hat{\sigma}_c^k\rangle] + \hat{f}_a(t')\right]dt'. \tag{28}$$

After taking the expected value of the resultant equation and multiplying Eq. (28) on the right by $\hat{F}_a(t)$, we have

$$\langle\hat{\eta}_a^k(t)\hat{F}_a(t)\rangle = \langle\hat{\eta}_a^k(0)\hat{F}_a(t)\rangle e^{-(\gamma+\gamma_c)t} + \int_0^t e^{-(\gamma+\gamma_c)(t-t')}\left[\frac{2g}{\kappa}\left[\langle\hat{\sigma}_a^{\dagger k}(t')\hat{F}_a(t')\hat{F}_a(t)\rangle + \langle\hat{F}_a^{\dagger}(t')\hat{\sigma}_a^k(t')\hat{F}_a(t)\rangle\right] + \frac{\Omega}{2}[\langle\hat{\sigma}_c^{\dagger k}\rangle\hat{F}_a(t) + \langle\hat{\sigma}_c^k\rangle\hat{F}_a(t)] + \langle\hat{f}_a(t')\hat{F}_a(t)\rangle\right]dt'. \tag{29}$$

The approximately valid relations can be written by ignoring the noncommutativity of the atomic and noise operators as well as the correlation between $\hat{F}_a(t)$ and $\hat{\sigma}_a^k(t')$, which is assumed to be very tiny [6].

$$\langle\hat{\sigma}_a^{\dagger k}(t')\hat{F}_a(t')\hat{F}_a(t)\rangle = \langle\hat{\sigma}_a^{\dagger k}(t')\rangle\langle\hat{F}_a(t')\hat{F}_a(t)\rangle = 0, \tag{30}$$

$$\langle\hat{F}_a^{\dagger}(t')\hat{\sigma}_a^k(t')\hat{F}_a(t)\rangle = \langle\hat{\sigma}_a^k(t')\rangle\langle\hat{F}_a^{\dagger}(t')\hat{F}_a(t)\rangle = 0, \tag{31}$$

$$\langle\hat{f}_a(t')\hat{F}_a(t)\rangle = \langle\hat{f}_a(t')\rangle\langle\hat{F}_a(t)\rangle = 0. \tag{32}$$

Now, because of these roughly correct relations and the idea that a noise operator $\hat{F}$ at one point in time shouldn't have an impact on the atomic variable at a previous point in time, Eq. (29) has the following form.

$$\langle\hat{\eta}_a^k(t)\hat{F}_a(t)\rangle = 0. \tag{33}$$

Following a similar procedure, one can also check that

$$\langle\hat{\eta}_b^k(t)\hat{F}_a(t)\rangle = 0, \tag{34}$$

$$\langle\hat{\eta}_c^k(t)\hat{F}_b(t)\rangle = 0, \tag{35}$$

$$\langle\hat{\eta}_b^k(t)\hat{F}_b(t)\rangle = 0, \tag{36}$$

$$\langle\hat{F}_a^{\dagger}(t)\hat{\sigma}_a^k(t)\rangle = 0, \tag{37}$$

$$\langle\hat{F}_b^{\dagger}(t)\hat{\sigma}_b^k(t)\rangle = 0. \tag{38}$$

We also take

$$\langle\hat{F}_a^{\dagger}(t)\hat{\sigma}_c^k(t)\rangle = \langle\hat{F}_b^{\dagger}(t)\hat{\sigma}_c^k(t)\rangle = 0. \tag{39}$$

Using Eqs. (33)-(39) in to Eqs. (20) - (25) and summing over the N three-level atoms, so that

$$\frac{d}{dt}\langle\hat{m}_a\rangle = -[\gamma+\gamma_c]\,\langle\hat{m}_a\rangle + \frac{\Omega}{2}\langle\hat{m}_b^{\dagger}\rangle, \tag{40}$$

$$\frac{d}{dt}\langle\hat{m}_b\rangle = -[\frac{\gamma}{2}+\frac{\gamma_c}{2}]\langle\hat{m}_b\rangle - \frac{\Omega}{2}\langle\hat{m}_a^{\dagger}\rangle, \tag{41}$$

$$\frac{d}{dt}\langle\hat{m}_c\rangle = -[\frac{\gamma}{2}+\frac{\gamma_c}{2}]\langle\hat{m}_c\rangle + \frac{\Omega}{2}[\langle\hat{N}_c\rangle - \langle\hat{N}_a\rangle], \tag{42}$$

$$\frac{d}{dt}\langle\hat{N}_a\rangle = -[\gamma+\gamma_c]\,\langle\hat{N}_a\rangle + \frac{\Omega}{2}[\langle\hat{m}_c^{\dagger}\rangle + \langle\hat{m}_c\rangle], \tag{43}$$

$$\frac{d}{dt}\langle\hat{N}_b\rangle = -[\gamma+\gamma_c]\,\langle\hat{N}_b\rangle + [\gamma+\gamma_c]\,\langle\hat{N}_a\rangle, \tag{44}$$

$$\frac{d}{dt}\langle\hat{N}_c\rangle = [\gamma+\gamma_c]\,\langle\hat{N}_b\rangle - \frac{\Omega}{2}[\langle\hat{m}_c^{\dagger}\rangle + \langle\hat{m}_c\rangle]. \tag{45}$$

Furthermore, by defining $\sigma_a^k = |b\rangle kk\langle a|$ and setting $\hat{\sigma}_a^k = |b\rangle\langle a|$ for any k, we find $\hat{m}_a = N|b\rangle\langle a|$. Following the same procedure, it can be shown that $\hat{m}_b = N|c\rangle\langle b|$, $\hat{m}_c = N|c\rangle\langle a|$, $\hat{N}_a = N|a\rangle\langle a|$, $\hat{N}_b = N|b\rangle\langle b|$, and $\hat{N}_c = N|c\rangle\langle c|$.

Moreover, using the definition

$$\hat{m} = \hat{m}_a + \hat{m}_b \tag{46}$$

and considering Eqs. (46), we have

$$\hat{m}^{\dagger}\hat{m} = N(\hat{N}_a + \hat{N}_b), \tag{47}$$

$$\hat{m}\hat{m}^{\dagger} = N(\hat{N}_b + \hat{N}_c), \tag{48}$$

$$\hat{m}^2 = N\hat{m}_c. \tag{49}$$

Applying the large-time approximation scheme to Eq. (43), we find

$$\langle\hat{N}_b\rangle = \langle\hat{N}_a\rangle. \tag{50}$$

The steady-state solution of Eqs. (42) and (44) can be expressed as

$$\langle\hat{N}_b\rangle = \frac{\Omega}{[\gamma+\gamma_c]}\langle\hat{m}_c\rangle \tag{51}$$

$$\langle\hat{m}_c\rangle = (\frac{\Omega}{\gamma+\gamma_c})^2(\langle\hat{N}_c\rangle - \langle\hat{N}_a\rangle). \tag{52}$$

Subsequently, we find

$$\langle\hat{N}_a\rangle = \left[\frac{\Omega^2}{(\gamma+\gamma_c)^2 + 3\Omega^2}\right]N. \tag{53}$$

$$\langle\hat{N}_b\rangle = \left[\frac{\Omega^2}{(\gamma+\gamma_c)^2 + 3\Omega^2}\right]N. \tag{54}$$

$$\langle\hat{m}_c\rangle = \left[\frac{\Omega(\gamma+\gamma_c)}{(\gamma+\gamma_c)^2 + 3\Omega^2}\right]N. \tag{55}$$

$$\langle \hat{N}_c \rangle = \left[\frac{\Omega^2 + (\gamma+\gamma_c)^2}{(\gamma+\gamma_c)^2 + 3\Omega^2} \right] N. \tag{56}$$

Following the procedure outlined in [21,15], for a system of N atoms, we obtain the following differential equations:

$$\frac{d\hat{a}}{dt} = -\frac{\kappa}{2}\hat{a} + \frac{g}{\sqrt{N}}\hat{m}_a + \sqrt{N}\hat{F}_a(t). \tag{57}$$

$$\frac{d\hat{b}}{dt} = -\frac{\kappa}{2}\hat{b} + \frac{g}{\sqrt{N}}\hat{m}_b + \sqrt{N}\hat{F}_b(t). \tag{58}$$

Additionally, by combining Eqs. ((57) and (58)), we obtain

$$\frac{d\hat{c}}{dt} = -\frac{\kappa}{2}\hat{c} + \frac{g}{\sqrt{N}}\hat{m} + \sqrt{N}\hat{F}_c(t), \tag{59}$$

where

$$\hat{F}_c(t) = \hat{F}_a(t) + \hat{F}_b(t) \tag{60}$$

and

$$\hat{c} = \hat{a} + \hat{b}. \tag{61}$$

The noise operator $\langle \hat{F}_c(t) \rangle$ follows correlation properties[7]:

$$\langle \hat{F}_c(t) \rangle = 0, \tag{62}$$

$$\langle \hat{F}_c^\dagger(t)\hat{F}_c(t') \rangle = 2\kappa \langle n_{th} \rangle \delta(t-t'), \tag{63}$$

$$\langle \hat{F}_c^\dagger(t)\hat{F}_c^\dagger(t') \rangle = \langle \hat{F}_c(t)\hat{F}_c(t) \rangle = 0, \tag{64}$$

$$\langle \hat{F}_c(t)\hat{F}_c^\dagger(t') \rangle = 2\kappa(\langle n_{th} \rangle + 1)\delta(t-t'). \tag{65}$$

Thus, applying the large-time approximation scheme, the adjoint solution of Eq. (41) is found to be

$$\langle \hat{m}_b^\dagger \rangle = -\frac{\Omega}{(\gamma+\gamma_c)} \langle \hat{m}_a(t) \rangle. \tag{66}$$

Substituting Eq. (66) into Eq. (40), we find

$$\frac{d}{dt}\langle \hat{m}_a \rangle = -\frac{1}{2}\eta_0 \langle \hat{m}_a(t) \rangle, \tag{67}$$

where $\eta_0 = \frac{2(\gamma+\gamma_c)^2+\Omega^2}{(\gamma+\gamma_c)}$. By a similar procedure for Eq. (41), we obtain

$$\frac{d}{dt}\langle \hat{m}_b \rangle = -\frac{1}{4}\eta_0 \langle \hat{m}_b(t) \rangle. \tag{68}$$

Adding Eqs. (67) and (68), we get

$$\frac{d}{dt}\langle \hat{m}(t) \rangle = -\frac{1}{4}\eta_0 \langle \hat{m}(t) \rangle - \frac{1}{4}\eta_0 \langle \hat{m}_a(t) \rangle. \tag{69}$$

With the atoms initially in the bottom level, the solution of Eq. (67) is

$$\langle \hat{m}_a(t) \rangle = 0. \tag{70}$$

Similarly, the solution of Eq. (68) is

$$\langle \hat{m}_b(t) \rangle = 0. \tag{71}$$

Therefore, the expectation value of the solution of Eq. (59) is

$$\langle \hat{c}(t) \rangle = 0. \tag{72}$$

3 Photon statistics

Our goal in this part is to find the photon number mean and variance for the steady state two-mode cavity light.

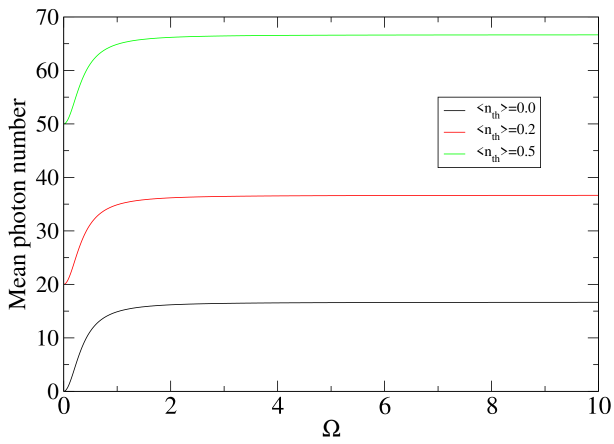

Fig. 2 Plots of the mean photon number for the two-mode cavity light at steady state [Eq. (88)] as a function of Ω for various parameter conditions. The plots are shown for $\kappa = 0.8$, $\gamma_c = 0.4$, $\gamma = 0.2$, and $N = 50$, with varying values of $\langle n_{th} \rangle$.

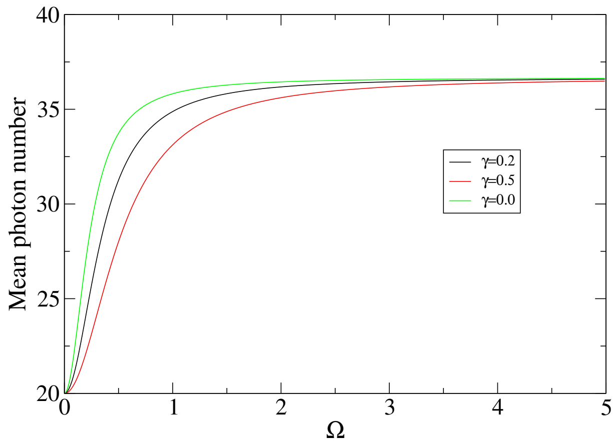

Fig. 3 Plots of the mean photon number for the two-mode cavity light at steady state [Eq. (88)] as a function of Ω for various parameter conditions. The plots are shown for $\kappa = 0.8$, $\gamma_c = 0.4$, $\langle n_{th} \rangle = 0.2$, and $N = 50$, with varying values of γ.

Starting with the relationship

$$\frac{d}{dt}\left\langle \hat{c}^\dagger(t)\hat{c}(t) \right\rangle = \left\langle \frac{d\hat{c}^\dagger(t)}{dt}\hat{c}(t) \right\rangle + \left\langle \hat{c}^\dagger(t)\frac{d\hat{c}(t)}{dt} \right\rangle, \tag{73}$$

along with Eq. (59), we find

$$\frac{d}{dt}\langle\hat{c}^{\dagger}t)\hat{c}(t)\rangle = -\kappa\langle\hat{c}^{\dagger}(t)\hat{c}(t)\rangle + \frac{g}{\sqrt{N}}\left[\langle\hat{c}^{\dagger}(t)\hat{m}(t)\rangle + \langle\hat{m}^{\dagger}(t)\hat{c}(t)\rangle\right] + \sqrt{N}\left[\langle\hat{F}_c^{\dagger}(t)\hat{c}(t)\rangle + \langle\hat{c}^{\dagger}(t)\hat{F}_c(t)\rangle\right]. \tag{74}$$

Next, we aim to evaluate $\langle\hat{c}^{\dagger}(t)\hat{m}(t)\rangle$. By applying the large-time approximation to Eq. (59), we obtain the approximately valid relation

$$\hat{c}(t) = \frac{2g}{\kappa\sqrt{N}}\hat{m} + \frac{2\sqrt{N}}{\kappa}\hat{F}_c(t). \tag{75}$$

Multiplying the adjoint of Eq. (75) on the right by $\hat{m}(t)$ and taking the expectation value of the resulting expression, we find

$$\langle\hat{c}^{\dagger}(t)\hat{m}(t)\rangle = \frac{2g\sqrt{N}}{\kappa}[\langle\hat{N}_a(t)\rangle + \langle\hat{N}_b(t)\rangle] + \frac{2\sqrt{N}}{\kappa}\langle\hat{F}_c^{\dagger}(t)\hat{m}(t)\rangle. \tag{76}$$

We will now calculate $\langle\hat{F}_c^{\dagger}(t)\hat{m}(t)\rangle$. Starting with the formal solution of Eq. (69):

$$\hat{m}(t) = \hat{m}(0)e^{-\frac{\eta_0}{2}t} + \int_0^t e^{-\frac{\eta_0}{2}(t-t')}[-\frac{\eta_0}{2}\hat{m}_a(t') + \hat{F}_c(t')]dt'. \tag{77}$$

Multiplying Eq. (77) on the left by $\hat{F}_c^{\dagger}(t)$ and taking the expectation value of the resulting expression, we get:

$$\langle\hat{F}_c^{\dagger}(t)\hat{m}(t)\rangle = \langle\hat{F}_c^{\dagger}(t)\hat{m}(0)\rangle e^{-\frac{\eta_0}{2}t} + \int_0^t e^{-\frac{\eta_0}{2}(t-t')}\left[-\frac{\eta_0}{2}\langle\hat{F}_c^{\dagger}(t)\hat{m}_a(t')\rangle + \langle\hat{F}_c^{\dagger}(t)\hat{F}_c(t')\rangle\right]dt'. \tag{78}$$

Fig. 2 illustrates the mean photon number of two-mode cavity light under various conditions of the mean thermal light. The results show that the mean photon number in the two-mode cavity increases as the mean thermal light ($\langle n_{th}\rangle$) rises. Conversely, Fig. 3 depicts the effect of the spontaneous emission rate on the mean photon number of the two-mode cavity light. The results indicate that the mean photon number decreases as the spontaneous emission rate increases, suggesting that a higher spontaneous emission rate negatively impacts the photon count, leading to a reduction in the mean photon number within the two-mode cavity.

Fig. 4 illustrates the plots of the mean photon number for the two-mode cavity light at steady state versus γ under various parameter conditions, with $\kappa = 0.8$, $\gamma_c = 0.4$, and $\Omega = 0.3$, for varying values of $\langle n_{th}\rangle$. As the mean photon number increases, the mean of thermal light also increases for small values of the spontaneous emission rate. Assuming that the cavity mode and atomic mode operators are uncorrelated and that a noise operator $\hat{F}$ at one point in time shouldn't have an impact on the atomic variable at a previous time, we obtain

$$\langle\hat{F}_c^{\dagger}(t)\hat{m}(t)\rangle = 0. \tag{79}$$

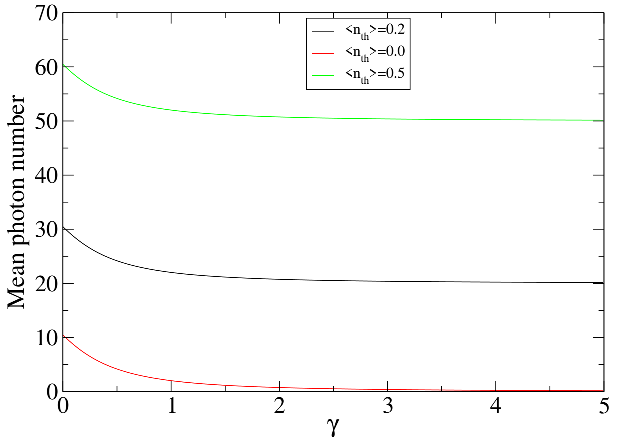

Fig. 4 Plots of the mean photon number for the two-mode cavity light at steady state [Eq. (88)] as a function of γ for various parameter conditions. The plots are shown for $\kappa = 0.8$, $\gamma_c = 0.4$, $\Omega = 0.3$, and $N = 50$, with varying values of $\langle n_{th}\rangle$.

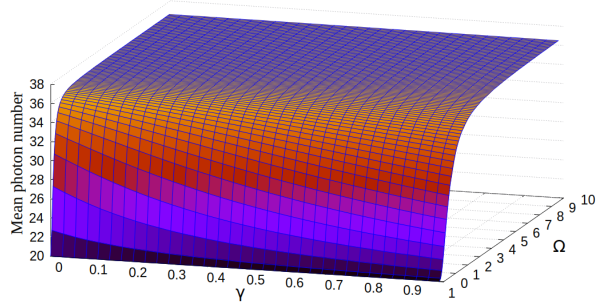

Fig. 5 Plots of the mean photon number for the two-mode cavity light at steady state [Eq. (88)] for different parameter variations. Mean photon number is shown as a function of Ω and γ for $\kappa = 0.8$, $\gamma_c = 0.4$, $\langle n_{th}\rangle = 0.02$, and $N = 50$.

On account of this result, Eq. (76) takes the form

$$\langle\hat{c}^{\dagger}(t)\hat{m}(t)\rangle = \frac{2g\sqrt{N}}{\kappa}[\langle\hat{N}_a(t)\rangle + \langle\hat{N}_b(t)\rangle]. \tag{80}$$

We next seek to evaluate $\langle\hat{F}_c^{\dagger}(t)\hat{c}(t)\rangle$. To this end, a formal solution of Eq. (59) can be written as

$$\hat{c}(t) = \hat{c}(0)e^{-\frac{\kappa}{2}t} + \int_0^t e^{-\frac{\kappa}{2}(t-t')}\left[\frac{g}{\sqrt{N}}\hat{m}(t') + \sqrt{N}\hat{F}_c(t')\right]dt'. \tag{81}$$

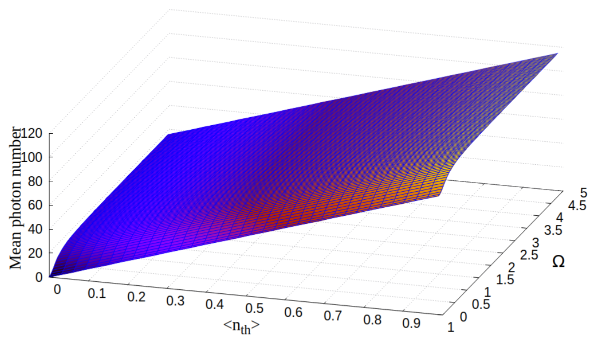

Fig. 6 Plots of the mean photon number for the two-mode cavity light at steady state [Eq. (88)] for different parameter variations. Mean photon number is shown as a function of Ω and $\langle n_{th}\rangle$ for $\gamma = 0.2$, $\kappa = 0.8$, $\gamma_c = 0.4$, and $N = 50$.

Multiplying Eq. (81) on the left by $\hat{F}_c^\dagger(t)$ and taking the expectation value of the resulting expression, we get

$$\begin{aligned}\langle \hat{F}_c^\dagger(t)\hat{c}(t)\rangle =\ & \langle \hat{F}_c^\dagger(t)\hat{c}(0)\rangle e^{-\frac{\kappa}{2}t} \\ & + \int_0^t e^{-\frac{\kappa}{2}(t-t')}\left[\frac{g}{\sqrt{N}}\langle \hat{F}_c^\dagger(t)\hat{m}(t')\rangle \right. \\ & \left. + \sqrt{N}\langle \hat{F}_c^\dagger(t)\hat{F}_c(t')\rangle\right] dt'.\end{aligned} \tag{82}$$

Eq. (82) becomes because of Eqs. (63) and (79) as well as the idea that a noise operator $\hat{F}$ at one point in time shouldn't have an impact on the atomic variable at a previous time.

$$\langle \hat{F}_c^\dagger(t)\hat{c}(t)\rangle = \sqrt{N}\langle n_{th}\rangle\kappa. \tag{83}$$

Now on account of Eqs. (80) and (83) along with their complex conjugates, we can rewrite Eq. (74) as

$$\begin{aligned}\frac{d}{dt}\left\langle \hat{c}^\dagger(t)\hat{c}(t)\right\rangle =\ & -\kappa\left\langle \hat{c}^\dagger(t)\hat{c}(t)\right\rangle + \frac{4g^2}{\kappa}\left[\langle \hat{N}_a(t)\rangle \right. \\ & \left. + \langle \hat{N}_b(t)\rangle\right] + 2\sqrt{N}\langle n_{th}\rangle\kappa.\end{aligned} \tag{84}$$

The steady-state solution of this equation is expressible as

$$\langle \hat{c}^\dagger\hat{c}\rangle = \frac{\gamma_c}{\kappa}\left[\langle \hat{N}_a\rangle + \langle \hat{N}_b\rangle\right] + 2N\langle n_{th}\rangle. \tag{85}$$

Following a similar procedure, one can establish that

$$\langle \hat{c}\hat{c}^\dagger\rangle = \frac{\gamma_c}{\kappa}\left[\langle \hat{N}_c\rangle + \langle \hat{N}_b\rangle\right] + 2N(\langle n_{th}\rangle + 1), \tag{86}$$

$$\langle \hat{c}^2\rangle = \frac{\gamma_c}{\kappa}\langle \hat{m}_c\rangle. \tag{87}$$

Considering Eqs. (50), (53), and (56), Eqs. (85) and (86) can be expressed as

$$\langle \hat{c}^\dagger\hat{c}\rangle = \frac{\gamma_c N}{\kappa}\left[\frac{2\Omega^2}{(\gamma+\gamma_c)^2 + 3\Omega^2}\right] + 2N\langle n_{th}\rangle, \tag{88}$$

$$\langle \hat{c}\hat{c}^\dagger\rangle = \frac{\gamma_c N}{\kappa}\left[\frac{2\Omega^2 + (\gamma+\gamma_c)^2}{(\gamma+\gamma_c)^2 + 3\Omega^2}\right] + 2N(\langle n_{th}\rangle + 1). \tag{89}$$

The mean photon number of a cavity mode can therefore be interpreted as the total of the mean photon number of laser light and the mean photon number of the thermal reservoir, as shown by Eq. (88). The shape of the mean photon number of the two-mode cavity light is as follows for $(\langle n\rangle_{th} = 0)$.

$$\bar{n}_c = \frac{\gamma_c N}{\kappa}\left[\frac{2\Omega^2}{(\gamma+\gamma_c)^2 + 3\Omega^2}\right] \tag{90}$$

Figs. 5 and 6 provide a detailed examination of how various factors influence the average number of photons in the two-mode cavity light. Fig. 5 illustrates the relationship between the mean photon number and two critical parameters: the amplitude of the driving coherent light and the spontaneous emission rate. The graph clearly shows that as the amplitude of the driving coherent light increases from 0 to 10, the average photon number rises accordingly. However, this trend reverses with an increase in the spontaneous emission rate, which causes a decrease in the photon number. This behavior highlights the role of external driving forces in regulating the photon population within the cavity, as well as the dampening effect of spontaneous emission.

Fig. 6 further investigates the impact of the thermal reservoir on the average photon number in the cavity. The mean photon number of the thermal reservoir, $\langle n_{th}\rangle$, has a pronounced effect on the cavity's photon population. In the absence of thermal photons ($\langle n_{th}\rangle = 0$), the average photon number in the reservoir is zero, establishing a baseline for the system. This condition is essential for understanding how thermal photons contribute to the overall dynamics. As the thermal reservoir is introduced with $\langle n_{th}\rangle = 1$, the average number of photons in the cavity significantly increases. This increase demonstrates how even a modest presence of thermal photons can substantially enhance the total photon population within the cavity modes. The thermal reservoir essentially acts as an additional energy source, boosting the overall photon count.

Furthermore, the variance of the photon number for the two-mode cavity light is expressible as

$$(\Delta n)^2 = \langle(\hat{c}^\dagger\hat{c})^2\rangle - \langle \hat{c}^\dagger\hat{c}\rangle^2. \tag{91}$$

Using the fact that $\hat{c}$ is a Gaussian variable with zero mean, we readily get

$$(\Delta n)^2 = \langle \hat{c}^\dagger\hat{c}\rangle\langle \hat{c}\hat{c}^\dagger\rangle + \langle \hat{c}^{2\dagger}\rangle\langle \hat{c}^2\rangle. \tag{92}$$

Considering Eqs. (85), (86), and (87), Eq. (92) can be rewritten as

$$(\Delta n)^2 = \left[\frac{\gamma_c}{\kappa}\left(\langle\hat{N}_a\rangle + \langle\hat{N}_b\rangle\right) + 2N\langle n\rangle_{th}\right] \left[\frac{\gamma_c}{\kappa}\left(\langle\hat{N}_b\rangle + \langle\hat{N}_c\rangle\right) + 2N(\langle n_{th}\rangle + 1)\right] + \left(\frac{\gamma_c}{\kappa}\langle\hat{m}_c\rangle\right)^2. \tag{93}$$

Finally, on account of Eqs. (50), (53), (55), and (85) along with Eq. (93), we arrive at

$$(\Delta n)^2 = \left(\frac{\gamma_c}{\kappa}\right)^2\left[\left(\frac{2\eta^2 N}{1+3\eta^2}\right)^2 + 3\left(\frac{\eta N}{1+3\eta^2}\right)^2\right] + 4\frac{\gamma_c}{\kappa}\left[\left(\frac{(\eta N)^2}{1+3\eta^2}\right)\langle n_{th}\rangle + \frac{(\eta N)^2}{1+3\eta^2}\right] + 2\frac{\gamma_c}{\kappa}\left(\frac{2\eta^2+1}{1+3\eta^2}\right)N^2\langle n_{th}\rangle + 4N\langle n_{th}\rangle^2 + 4N^2\langle n_{th}\rangle. \tag{94}$$

For $\langle n_{th}\rangle$=0, we readily find

$$(\Delta n)^2 = \left(\frac{\gamma_c}{\kappa}\right)^2\left[\left(\frac{2\eta^2 N}{1+3\eta^2}\right)^2 + 3\left(\frac{\eta N}{1+3\eta^2}\right)^2\right] + 4\frac{\gamma_c}{\kappa}\frac{(\eta N)^2}{1+3\eta^2}, \tag{95}$$

where

$$\eta = \frac{\Omega}{\gamma + \gamma_c}. \tag{96}$$

Our result shows that the photon number variance of the two-mode cavity light is greater than the one obtained by Tamirat [15]. This must be due to the reservoir noise operators.

4 Quadrature squeezing

We now proceed to calculate the quadrature squeezing of the two-mode cavity light in the entire frequency interval. To this end, the squeezing properties of the two-mode cavity light are described by two quadrature operators defined by

$$\hat{c}_+ = \hat{c}^\dagger + \hat{c} \tag{97}$$

and

$$\hat{c}_- = i(\hat{c}^\dagger - \hat{c}). \tag{98}$$

It can be readily established that [14, 15]

$$[\hat{c}_-, \hat{c}_+] = 2i\frac{\gamma_c}{\kappa}(\langle\hat{N}_a\rangle - \langle\hat{N}_c\rangle) - 4Ni. \tag{99}$$

It then follows that [16, 17]

$$\Delta c_+ \Delta c_- \geq \frac{\gamma_c}{\kappa}(\langle\hat{N}_c\rangle - \langle\hat{N}_a\rangle) + 2N. \tag{100}$$

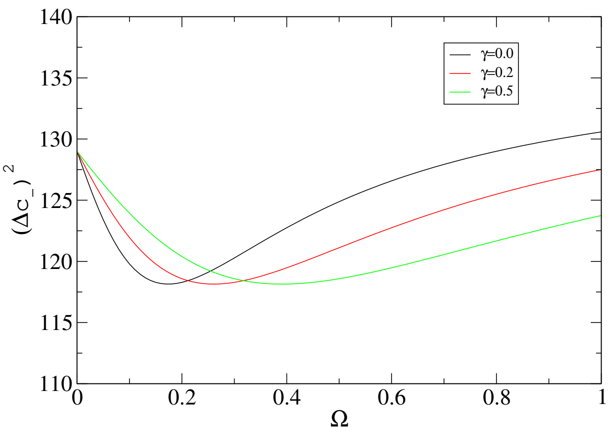

Fig. 7 Plots of the minus quadrature for the two-mode cavity light at steady state [Eq. (105)] for different parameter variations. Minus quadrature is shown as a function of Ω for $\kappa = 0.8$, $\gamma_c = 0.4$, $\langle n_{th}\rangle = 0.02$, $N = 50$, and varying values of γ.

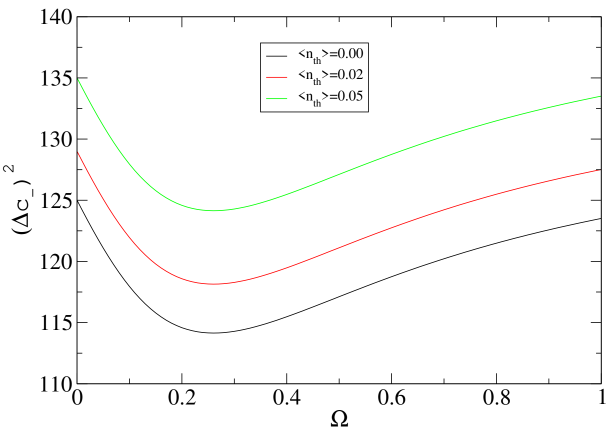

Fig. 8 Plots of the minus quadrature for the two-mode cavity light at steady state [Eq. (105)] for different parameter variations. Minus quadrature is shown as a function of Ω for $\gamma = 0.2$, $\kappa = 0.8$, $\gamma_c = 0.4$, $N = 50$, and varying values of $\langle n_{th}\rangle$.

Upon setting $\Omega = 0$, we see that

$$\Delta c_+ \Delta c_- \geq \frac{\gamma_c}{\kappa}N + 2N. \tag{101}$$

This represents the quadrature variance for two-mode vacuum state. The variance of the quadrature operator is expressible as

$$(\Delta c_\pm)^2 = \pm\langle(\hat{c}^\dagger \pm \hat{c})^2\rangle \mp [\langle\hat{c}^\dagger + \hat{c}\rangle]^2, \tag{102}$$

so that on account of Eq. (73), we have

$$(\Delta c_\pm)^2 = \langle\hat{c}^\dagger\hat{c}\rangle + \langle\hat{c}\hat{c}^\dagger\rangle \pm \langle\hat{c}^{\dagger 2}\rangle \pm \langle\hat{c}^2\rangle. \tag{103}$$

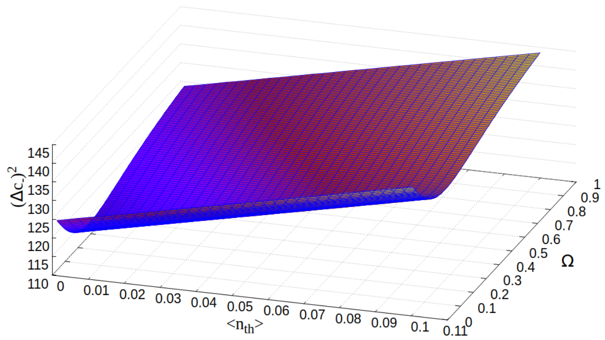

Fig. 9 Plots of the minus quadrature for the two-mode cavity light at steady state [Eq. (105)] for different parameter variations. Minus quadrature is shown as a function of Ω and $\langle n_{th}\rangle$ for $\kappa = 0.8$, $\gamma_c = 0.4$, $\gamma = 0.2$, and $N = 50$.

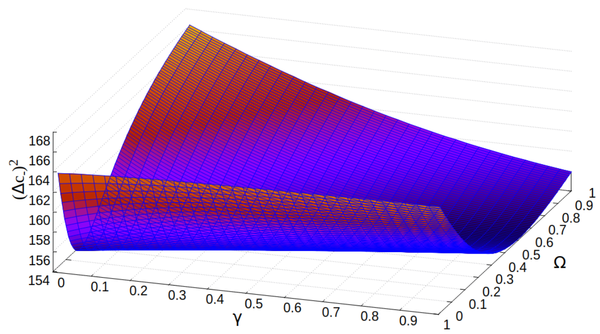

Fig. 10 Plots of the minus quadrature for the two-mode cavity light at steady state [Eq. (105)] for different parameter variations. Minus quadrature versus Ω and γ for $\langle n_{th}\rangle = 0.02$, $\kappa = 0.8$, $\gamma_c = 0.4$, and $N = 50$.

Now employing Eqs. (85), (86), and (87), we arrive at

$$(\Delta c_\pm)^2 = \frac{\gamma_c}{\kappa}\left(3\langle\hat{N}_a\rangle + \langle\hat{N}_c\rangle \pm 2\langle\hat{m}_c\rangle\right) + 4N\langle n_{th}\rangle + 2N, \tag{104}$$

Considering Eqs. (53), (55), (56), and Eq. (104) can be rewritten as

$$(\Delta c_\pm)^2 = \frac{\gamma_c N}{\kappa}\left(\frac{4\eta^2 \pm 2\eta + 1}{1+3\eta^2}\right) + 4N\langle n_{th}\rangle + 2N, \tag{105}$$

Furthermore, when we substitute $\eta = 0$ and $\langle n_{th}\rangle = 0$ into Eqs. (106), we obtain

$$(\Delta c_+)_v^2 = (\Delta c_-)_v^2 = \frac{\gamma_c}{\kappa}N + 2N. \tag{106}$$

$$S = \frac{(\Delta c_-)_v^2 - (\Delta c_-)^2}{(\Delta c_-)_v^2}. \tag{107}$$

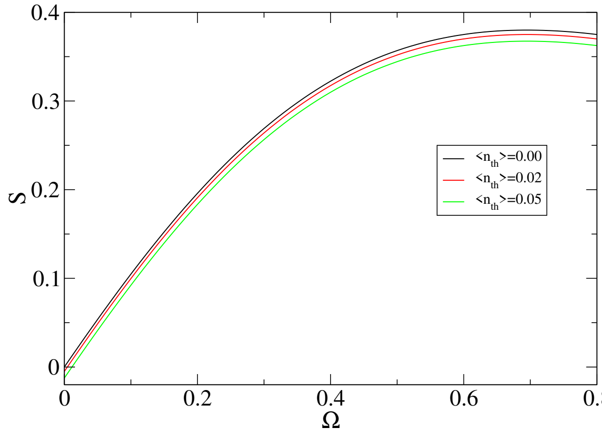

Fig. 11 Plots of the quadrature squeezing for the two-mode cavity light at steady state [Eq. (108)] for different parameter variations. Squeezing is shown as a function of Ω for $\kappa = 0.1$, $\gamma_c = 1.2$, $\gamma = 0.2$, and varying values of $\langle n_{th}\rangle$.

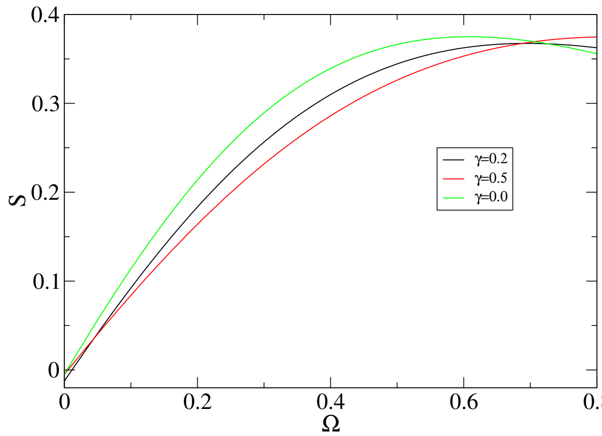

Fig. 12 Plots of the quadrature squeezing for the two-mode cavity light at steady state [Eq. (108)] for different parameter variations. Squeezing as a function of Ω is shown for $\kappa = 0.1$, $\gamma_c = 1.2$, $\langle n_{th}\rangle = 0.02$, and varying values of γ.

Using equations (106) and (107), Eq. (108) can be expressed as

$$S = \frac{\gamma_c}{\gamma_c + 2\kappa}\left(\frac{2\eta - \eta^2}{1+3\eta^2}\right) - \frac{4\langle n_{th}\rangle\kappa}{\gamma_c + 2\kappa}. \tag{108}$$

We find that the degree of squeezing in the quadrature is independent of the number of atoms, unlike the mean photon number. This suggests that the quadrature squeezing of the light inside the cavity is independent of the number of atoms. Moreover, we find that the quadrature squeezing is diminished by the cavity mode's interaction with the thermal reservoir. The plots in Fig. 11 show that the maximum quadrature squeezing is 37.9%

for $\langle n \rangle_{th} = 0$, 37.5% for $\langle n \rangle_{th} = 0.02$, and 37% for $\langle n \rangle_{th} = 0.05$. As the mean thermal photon number increases, the squeezing decreases. Thermal reservoir noise accounts for the below results as compared to [10, 22]. Similarly, Fig. 12 demonstrates that the squeezing decreases with increasing spontaneous emission rate, especially at low values of the driving coherent light amplitude.

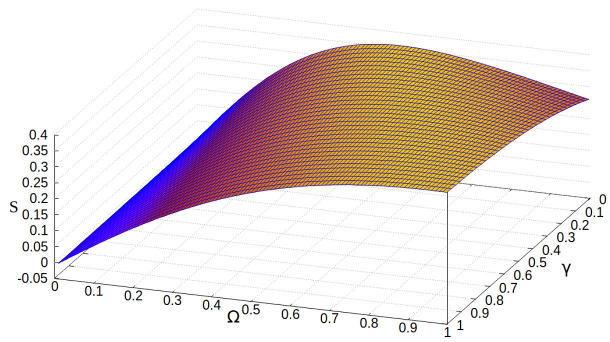

Fig. 13 Plots of the quadrature squeezing for the two-mode cavity light at steady state [Eq. (108)] versus Ω and γ for $\kappa = 0.1$, $\gamma_c = 1.2$ and $\langle n_{th} \rangle = 0.02$.

5 Entanglement Properties of the Two-Mode Light

In other words, if the density operator for the combined state cannot be expressed as a product of the parts' density operators,

$$\hat{\rho} \neq \sum_j P_j \hat{\rho}_j^1 \bigotimes \hat{\rho}_j^2, \tag{109}$$

where $\sum_j P_j = 1$ and $P_j \geq 0$ are chosen to guarantee that the total density of state is normalized. Many criteria have been devised in the modern period to quantify, identify, and control the entanglement produced by different quantum optical systems.

A system's quantum state is considered entangled, according to DGCZ, if the inequality is satisfied by the sum of the variances of the EPR-like quadrature operators, $\hat{u}$ and $\hat{v}$.

$$(\Delta \hat{u})^2 + (\Delta \hat{v})^2 < 2N, \tag{110}$$

where

$$\hat{u} = \hat{x}_a - \hat{x}_b, \tag{111}$$

$$\hat{v} = \hat{p}_a + \hat{p}_b, \tag{112}$$

where $\hat{x}_a = (\hat{a}^\dagger + \hat{a})/\sqrt{2}$, $\hat{x}_b = (\hat{b}^\dagger + \hat{b})/\sqrt{2}$ $\hat{p}_a = i(\hat{a}^\dagger - \hat{a})/\sqrt{2}$, $\hat{p}_b = i(\hat{b}^\dagger - \hat{b})/\sqrt{2}$, are quadrature operators for modes $\hat{a}$ and $\hat{b}$. Taking into account (112) and (113), (111) yields

$$(\Delta \hat{u})^2 + (\Delta \hat{v})^2 = 2\frac{\gamma_c}{\kappa}[N + \langle \hat{N}_b \rangle - \langle \hat{m}_c \rangle]. \tag{113}$$

Thus, in view of equation (104) together with (113) the sum of the variances of $\hat{u}$ and $\hat{v}$ can be expressed as

$$(\Delta \hat{u})^2 + (\Delta \hat{v})^2 = 2\Delta c_-^2, \tag{114}$$

where Δc_-^2 given by (105). This result clearly shows that the degree of entanglement is proportional to the degree of two-mode light squeezing. This relationship suggests that entanglement will always exist in the system when there is two-mode squeezing. It's crucial to remember that the entanglement vanishes along with the squeezing. This is due to the fact that equation Eq. (104) states that the squeezing directly affects the existence of entanglement. It also implies that the degree of entanglement is dependent on the number of atoms, just like the mean photon number and quadrature variance. We obtain that there is a large entanglement between the states of the light generated in the cavity with the aid of criterion Eq. (110). This is the outcome of a strong correlation found in the radiation that is released when atoms move from an upper energy level to a lower energy level via an intermediate level. To clearly show the available entanglement for different values of the spontaneous emission rate, γ, the variances of a pair of EPR-type operators, $\Delta \hat{u}^2 + \Delta \hat{v}^2$, are plotted against the driving coherent light in the accompanying figure.

5.1 Cavity Atomic-States Entanglement

A necessary condition for identifying entangled quantum states is the notion of quantum entanglement between the two cavity modes a and b, as put out by Duan, Giedke, Cirac, and Zoller (DGCZ)[23]. A quantum state is considered entangled in the DGCZ framework if the sum of variances of the EPR-like quadrature operators $\hat{u}$ and $\hat{v}$ satisfies the following inequality: If the inequality is satisfied by the total variances of the EPR-like quadrature operators $\hat{u}$ and $\hat{v}$.

$$(\Delta \hat{u})^2 + (\Delta \hat{v})^2 < 2N^2, \tag{115}$$

The system's quantum states are thought to be entangled. Similarly, the cavity atomic-states are entangled if the sum of the variances of a pair of EPR-like operators is provided by

$$\hat{u} = \hat{x}_a^{'} - \hat{x}_b^{'}, \tag{116}$$

$$\hat{v} = \hat{p}_a^{'} + \hat{p}_b^{'}, \tag{117}$$

where the atomic quadrature operators are defined as $\hat{x}_a^{'} = \frac{\hat{m}_a^\dagger + \hat{m}_a}{\sqrt{2}}$, $\hat{x}_b^{'} = \frac{\hat{m}_b^\dagger + \hat{m}_b}{\sqrt{2}}$, $\hat{p}_a^{'} = i\frac{\hat{m}_a^\dagger - \hat{m}_a}{\sqrt{2}}$, and $\hat{p}_b^{'} = i\frac{\hat{m}_b^\dagger - \hat{m}_b}{\sqrt{2}}$.

Given that $\hat{m}_a$ and $\hat{m}_b$ are Gaussian variables with zero means, it can be verified that:

$$(\Delta\hat{u})^2 + (\Delta\hat{v})^2 = \langle\hat{m}_a^\dagger\hat{m}_a\rangle + \langle\hat{m}_a\hat{m}_a^\dagger\rangle + \langle\hat{m}_b^\dagger\hat{m}_b\rangle + \langle\hat{m}_b\hat{m}_b^\dagger\rangle - \langle\hat{m}_b^\dagger\hat{m}_a^\dagger\rangle - \langle\hat{m}_a\hat{m}_b\rangle, \tag{118}$$

leading to the conclusion that:

$$(\Delta\hat{u})^2 + (\Delta\hat{v})^2 = N[N + \langle\hat{N}_a\rangle - 2\langle\hat{m}_c\rangle]. \tag{119}$$

Considering Eq. (53) and (55), the steady-state entanglement of the cavity atomic-states of the two-mode cavity light is expressed as:

$$(\Delta\hat{u})^2 + (\Delta\hat{v})^2 = N^2\left[\frac{(\gamma+\gamma_c)^2 + 4\Omega^2 - 2\Omega(\gamma+\gamma_c)}{(\gamma+\gamma_c)^2 + 3\Omega^2}\right]. \tag{120}$$

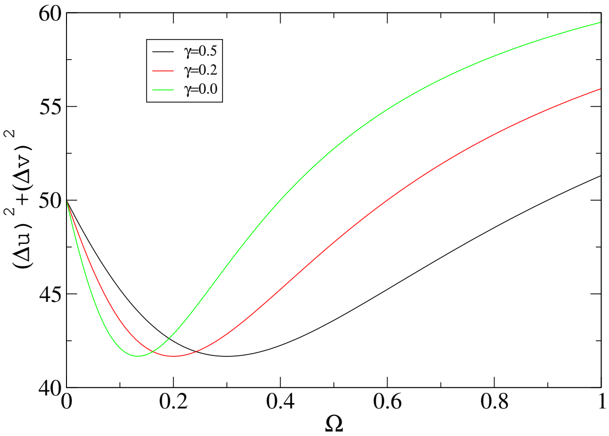

Fig. 14 Plots of $\Delta u^2 + \Delta v^2$ of the two-mode cavity light [Eq. (114)] versus Ω for $\gamma_c = 0.4, \kappa = 0.8$, and $N = 50$ for different γ.

From the plots in Figs. 14 and 15, we observe that as the spontaneous emission decay constant increases, the atom entanglement decreases at the same value of Ω. Similarly, the data in Fig. 16 indicate that atom entanglement decreases with an increasing spontaneous emission decay constant, γ. From these plots, we see values of $\kappa = 0.8$, $\gamma_c = 0.4$, and $N = 50$. For large amplitudes of driving coherent light, entanglement decreases as spontaneous emission increases. Unlike squeezing, entanglement is independent of the mean of thermal light.

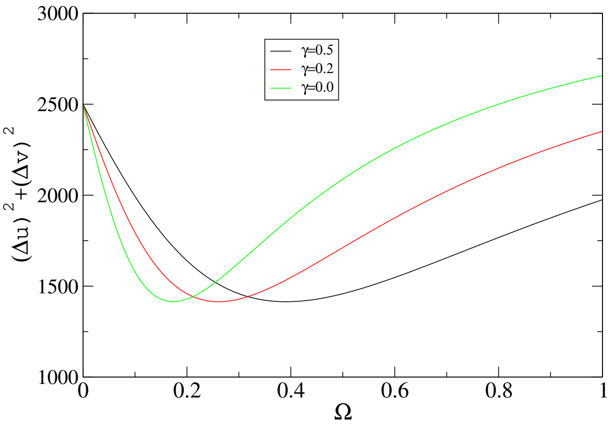

Fig. 15 Plots of $\Delta u^2 + \Delta v^2$ of the two-mode cavity light [Eq. (120)] versus Ω for $\gamma_c = 0.4$, and $N = 50$ for different γ.

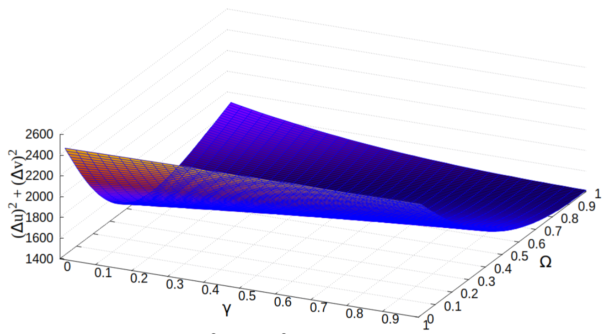

Fig. 16 Plots of $\Delta u^2 + \Delta v^2$ of the two-mode cavity light [Eq. (120)] versus γ and Ω for $\gamma_c = 1.2$, and $N = 50$.

6 Normalized Second-Order Correlation Functions

It is also possible to examine the second-order correlation function for the superposition of the two cavity radiation modes at the same time by utilizing [18]-[21]:

$$g^{(2)}_{(a,b)}(0) = \frac{\langle\hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b}\rangle}{\langle\hat{a}^\dagger\hat{a}\rangle\langle\hat{b}^\dagger\hat{b}\rangle}. \tag{121}$$

At the steady-state, the normalized second-order correlation function for the two-mode light adopts the following form since $\hat{a}$ and $\hat{b}$ are Gaussian variables with vanishing means.

$$g^{(2)}_{(a,b)}(0) = 1 + \frac{\langle\hat{b}\hat{a}\rangle\langle\hat{a}^\dagger\hat{b}^\dagger\rangle}{\langle\hat{a}^\dagger\hat{a}\rangle\langle\hat{b}^\dagger\hat{b}\rangle}. \tag{122}$$

It then follows that

$$g^{(2)}_{(a,b)}(0) = 1 + \frac{\langle\hat{m}_c\rangle^2}{\langle\hat{N}_a\rangle\langle\hat{N}_b\rangle}. \tag{123}$$

In view of (50), (53),(55) and (118), we obtain

$$g^{(2)}_{(a,b)}(0) = 1 + \left[\frac{\gamma+\gamma_c}{\Omega}\right]^2. \tag{124}$$

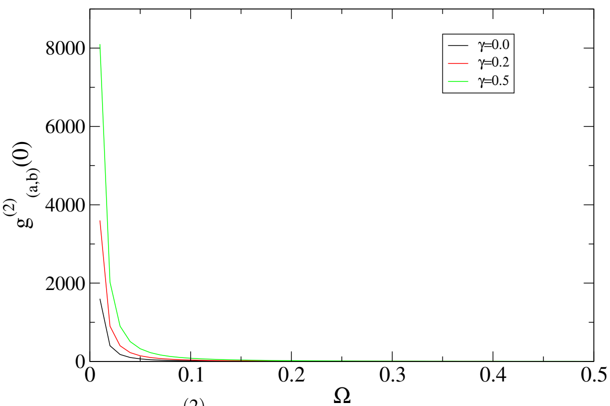

Fig. 17 Plots of $g^{(2)}_{(a,b)}(0)$ of the two-mode cavity light versus [Eq. (124)] Ω for $\gamma_c = 0.4$, for varying value of γ.

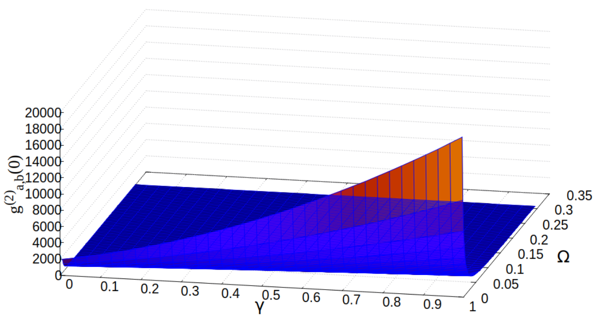

Fig. 18 Plots of $g^{(2)}_{(a,b)}(0)$ of the two-mode cavity light [Eq. (124)] versus r and Ω for $\gamma_c = 0.4$

This result shows that the number of atoms has no effect on the two-mode light's second-order correlation function.

The second-order correlation function for the two-mode light as a function of Ω in the cases where spontaneous emission is present ($\gamma \neq 0$) and absent ($\gamma = 0$) is depicted in Figs. 17 and 18.

These graphs unequivocally demonstrate that in both cases, the value of $g^{(2)}_{a,b}(0)$ falls as Ω increases. In particular, Figs. 12 shows that when $\Omega < 0.01$, the second-order correlation function becomes unimportant. Furthermore, the influence on the second-order correlation function is amplified in the presence of spontaneous emission.

7 Conclusion

In this work, we studied the properties of light created by three-level atoms in an open cavity powered by coherent light, including its squeezing and entanglement. We got the steady-state solutions of the evolution equations for the expectation values of atomic operators and the quantum Langevin equations for cavity mode operators using the large-time approximation method. We calculated the quadrature variance, quadrature squeezing, and entanglement for the steady-state two-mode cavity light using these answers. Furthermore, we obtained the superposition of the two modes' normalized second-order correlation function. Our findings indicate that intracavity quadrature squeezing is improved by spontaneous emission. We discovered a significant correlation between squeezing and entanglement, suggesting that higher squeezing levels are associated with higher entanglement levels. This relationship demonstrates that entanglement always exists in the two-mode light when squeezing takes place. In addition, our analysis shows that there is a strong correlation between photons inside the laser cavity, and that correlation gets stronger as the spontaneous emission decay constant, γ, increases. Consequently, increased entanglement, squeezing, and photon number correlation are caused by spontaneous emission. On the other hand, quadrature squeezing is decreased and the mean and variance of photon counts are increased in the presence of a thermal reservoir. The findings of this study have several potential practical applications. The ability to achieve significant squeezing and control over entanglement in cavity light can be highly beneficial for quantum information processing and quantum communication technologies. Specifically, the results could enhance the performance of quantum key distribution (QKD) systems by providing more reliable entangled photon sources. Furthermore, the insights gained from this study could contribute to the development of high-precision measurement devices, such as quantum sensors and interferometers, which rely on squeezed light to achieve superior sensitivity. Although this work offers valuable insights into the quantum properties of light in a three-level atomic system, there are still a number of unexplored research areas. Studying how squeezed vacuum reservoirs affect the system is one possible path, and exploring how a parametric amplifier might be included into the cavity is another. Moreover, our study recognizes specific limitations, such as maintaining the stability of the open cavity system and precise control of the coupling between the thermal reservoir and the cavity mode.

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